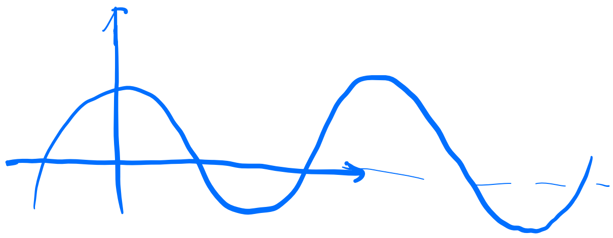
§1 Introduction

$$\psi(x) = a\cos(\underset{\omega}{kx} + \phi)$$



a : amplitude ϕ : phase

k : wavenumber $\lambda = \frac{2\pi}{k}$: wave length.

ω : angular frequency. $T = \frac{2\pi}{\omega}$: period.

$\boxed{f(x) = \cos 2x. \quad \text{period} : \pi}$

§2. Introduce to the Dirac delta function.

Def.

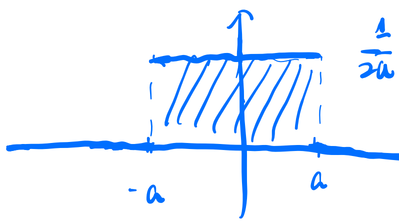
$$\delta(x-d) = \begin{cases} 0, & x \neq d, \\ +\infty, & x = d. \end{cases}$$

- distribution

- $$\int_{-\infty}^{+\infty} \underset{\infty\cdot 0}{\delta(x-d)\, dx} = 1. \qquad \left(\int_{d-R}^{d+R} \delta(x-d)\, dx = 1 \right).$$

- Top-hat function

$$\Pi_a(x) = \begin{cases} \frac{1}{2a} & , \ |x| < a \\ 0 & , \ |x| \geq a \end{cases}$$



$$\int_{-a}^{a} \Pi_a(x)\, dx = 1.$$

$$\delta(x) = \lim_{a \to 0} \Pi_a(x)$$

$$\Rightarrow \quad \int_{-\infty}^{+\infty} \delta(x)\, dx = \lim_{a \to 0} \int_{-a}^{a} \Pi_a(x)\, dx = 1.$$

Sifting property.

$$\int_{-\infty}^{+\infty} \delta(x-d) f(x)\, dx = f(d).$$

Compare:

$$\delta_{mn} = \begin{cases} 1 & , \ m = n \\ 0 & , \ m \neq n. \end{cases}$$

$$\sum_{n=0}^{\infty} A_n \delta_{mn} = A_m$$

Proof of the sifting property: $d = 0$

$$\int_{-\infty}^{+\infty} \delta(x) f(x)\, dx = \int_{-\infty}^{+\infty} \lim_{a\to 0} \Pi_a(x) f(x)\, dx$$

$$= \lim_{a\to 0} \int_{-\infty}^{+\infty} \Pi_a(x) f(x)\, dx$$

$$= \lim_{a\to 0} \frac{1}{2a} \int_{-a}^{a} f(x)\, dx$$

$$= f(0).$$

§3. Fourier Series.

$$\sum_{n=0}^{\infty} a_n x^n$$

3.1. Overview

$$f(x) = \sum_i a_i \cos(k_i x + \phi_i) = \sum_i \left(\tilde{a}_i \cos k_i x + \tilde{b}_i \sin k_i x\right)$$

- $\cos(A+B) = \cos A \cos B - \sin A \sin B$.

- The sines and cosines are said to complete set.

3.2 Periodic function.

Def.

$$f(x+T) = f(x), \quad \forall x \in \mathbb{R}.$$

T is the period. nT is also a period.

- The smallest period is called the <u>fundamental period</u>.

<u>Note</u>: Dirichlet function

$$D(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

<u>Exercise</u>. Show that if $f(x)$ and $g(x)$ are periodic with period T, then so are $af(x)+bg(x)$ and $f(x)g(x)$.

<u>Exam</u>. • $\sin x$, $\cos x$. $T = 2\pi$.

• $\sin(\frac{n\pi}{L}x)$. $T = 2\pi \cdot \frac{L}{n\pi} = \frac{2L}{n}$.

$\forall n \in \mathbb{N}$. $\underline{2L}$ is the period.

<u>3.3</u>. The Fourier expansion.

- $-L \le x \le L$.

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

a_n, b_n are <u>Fourier coefficients</u>.

(What about $n<0$? P5 - P6).

- $x = \sum_{i=1}^{n} a_i e_i$ in $\mathbb{R}^n$

$a_i = \langle x, e_i \rangle$

3.4 Orthogonality

$f(x), g(x)$ in $L^2(-L, L)$.

$$\langle f, g\rangle = \int_{-L}^{L} f(x) g(x)\, dx.$$

Prop.

① $$\int_{-L}^{L} \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L}\, dx = \begin{cases} 0, & m \neq n. \\ L, & m = n. \end{cases}$$

② $$\int_{-L}^{L} \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\, dx = \begin{cases} 0, & m \neq n \\ L, & m = n. \end{cases}$$

③ $$\int_{-L}^{L} \cos\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\, dx = 0.$$

Pf. Recall:

$$\boxed{\begin{aligned} 2\cos A \cos B &= \cos(A+B) + \cos(A-B) \\ 2\sin A \cos B &= \sin(A+B) + \sin(A-B) \\ 2\sin A \sin B &= -\cos(A+B) + \cos(A-B) \\ 2\cos A \sin B &= \sin(A+B) - \sin(A-B). \end{aligned}}$$

①: $$\int_{-L}^{L} \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L}\, dx = \frac{1}{2}\int_{-L}^{L} \cos\left(\frac{n+m}{L}\pi x\right) + \cos\left(\frac{n-m}{L}\pi x\right) dx$$

$$= \begin{cases} 0, & m \neq n \\ L, & m = n. \end{cases}$$

2. 3: Exercise

3.5 Calculating the Fourier components.

To get a_m:

$$\int_{-L}^{L} f(x) \cos \frac{m\pi x}{L} \, dx$$

$$= \frac{1}{2} a_0 \int_{-L}^{L} \cos \frac{m\pi x}{L} \, dx + \sum_{n=1}^{\infty} \left(a_n \int_{-L}^{L} \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \, dx \right.$$

$$\left. + \, b_n \underbrace{\int_{-L}^{L} \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \, dx}_{0} \right)$$

$$= a_0 L \delta_{m0} + \sum_{n=1}^{\infty} a_n L \delta_{mn}$$

$$= L a_m .$$

$$\Rightarrow \quad a_m = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{m\pi x}{L} \, dx .$$

Similarly,

$$b_m = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{m\pi x}{L} \, dx .$$

3.6 Even and Odd expansion.

- $f(x)$ is even:

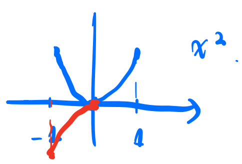
$$b_n = 0 . \qquad a_n = \frac{2}{L} \int_{0}^{L} f(x) \cos \frac{n\pi x}{L} \, dx .$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}.$$

- $f(x)$ is odd:

$$a_n = 0. \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}.$$



<u>Exam</u>. $f(x) = e^{-|x|}$. $|x| < 1$. Fundamental period 2

$2L$

$b_n = 0$.

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

$$= 2\int_0^1 e^{-x}\cos(n\pi x)\,dx$$

Euler formula: $e^{i\theta} = \cos\theta + i\sin\theta$

$$= 2\int_0^1 e^{-x}\cdot\frac{1}{2}\left(e^{in\pi x} + e^{-in\pi x}\right)dx.$$

$$= \left.\frac{e^{(in\pi-1)x}}{in\pi-1} + \frac{e^{-(in\pi+1)x}}{-(in\pi+1)}\right|_0^1 = \frac{2\left(1-(-1)^n e^{-1}\right)}{1+n^2\pi^2}$$

$$\Rightarrow \quad f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$

$$= (1 - e^{-1}) + 2 \sum_{n=1}^{\infty} \frac{1 - (-1)^n e^{-1}}{1 + n^2 \pi^2} \cos(n\pi x)$$

Exercise. ① $f(x) = x^2$, $|x| < 1$, $T = 2$.

② $f(x) = x$, $|x| < 1$, $T = 2$.