$\underline{f(x) = x^2}$, $\quad -L \leq x \leq L$.

Fourier expansion.

$$a_m = \frac{1}{L} \int_{-L}^{L} f(x) \underline{\cos \frac{m\pi x}{L}}\, dx$$

$$b_m = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{m\pi x}{L}\, dx .$$

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}\right) .$$

Since $f(x) = x^2$ is even in $[-L, L]$, then $b_n = 0$.

$$a_0 = \frac{1}{L} \int_{-L}^{L} x^2\, dx = \frac{2L^2}{3} .$$

$$a_m = \frac{1}{L} \int_{-L}^{L} x^2 \cos \frac{m\pi x}{L}\, dx$$

$$= \frac{2}{L} \int_{0}^{L} x^2 \cos \boxed{\frac{m\pi x}{L}}\, dx$$

$$\Uparrow$$

Let $\quad y = \frac{m\pi x}{L}$, $\quad \underline{x = \frac{L}{m\pi} y}$. then

$$\underline{dx = \frac{L}{m\pi}\, dy}$$

$$= \frac{2}{L} \int_{0}^{m\pi} \left(\frac{L}{m\pi} y\right)^2 \cos y \cdot \frac{L}{m\pi}\, dy$$

$$= \frac{2L^2}{m^3\pi^3} \int_{0}^{m\pi} y^2 \cos y\, dy$$

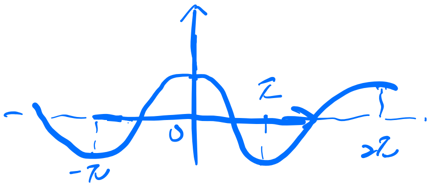
$$\boxed{\int u \frac{dv}{dx}\, dx = uv - \int v \frac{du}{dx}\, dx}$$

$$\int y^2 \cos y \, dy = y^2 \sin y - 2\underline{\int y \sin y \, dy}$$

$$= y^2 \sin y + 2y \cos y - 2 \sin y + C$$

$$= \frac{2L^2}{m^3\pi^3}\left[y^2 \underline{\sin y} + 2y \underline{\cos y} - 2\cancel{\sin y} \Big|_0^{m\pi} \right]$$

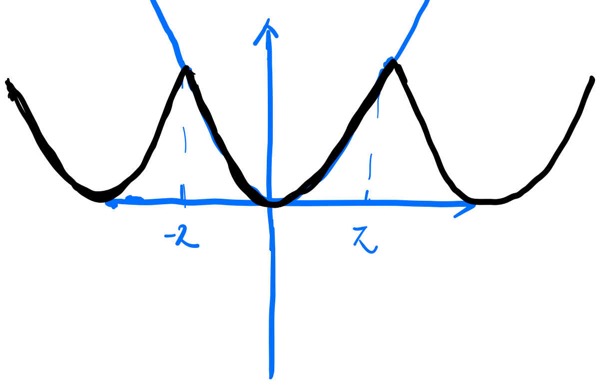
$$= \frac{2L^2}{m^{\cancel{3}}\pi^3} \cdot 2m\cancel{\pi}\, \underline{\cos m\pi}^{(-1)^m}$$

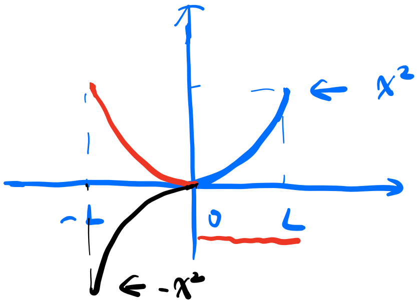


$$= \frac{\textcircled{4} L^2 (-1)^m}{m^2\pi^2}$$

$\Rightarrow$ Fourier series is

$$f(x) = \underline{\frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\frac{n\pi x}{L}}, \qquad -L \le x \le L.$$





$$f(x) = \begin{cases} x^2 & 0 \le x \le L \\ -x^2 & -L \le x < 0 \end{cases}.$$

Since $f(x)$ is odd, then $a_n = 0$.

$$b_n = \frac{2}{L}\int_0^L x^2 \sin\frac{n\pi x}{L}\, dx \quad (\text{Exercise})$$

$$= \frac{2L^2}{m^3\pi^3}\left[(-1)^{n+1} n^2\pi^2 + 2(-1)^n - 2\right]$$

even: $f(x) = \underline{\frac{L^2}{3}} - \frac{4L^2}{\pi^2}\cos\left(\frac{\pi x}{L}\right) + \cdots$

odd: $f(x) = \underline{\frac{2L^2}{\pi}\sin\frac{\pi x}{L}} + \cdots$

3.8 Complex Fourier Series.

$f(x)$ in $[-L, L]$.

$$f(x) = \sum_{n=-\infty}^{+\infty} C_n \phi_n(x), \quad \text{where } \phi_n(x) = \boxed{e^{\frac{in\pi x}{L}}}.$$

Euler formula: $e^{i\theta} = \cos\theta + i\sin\theta$.

$$\int_{-L}^{L} \phi_n(x)\phi_m^*(x)\,dx = \int_{-L}^{L} e^{\frac{in\pi x}{L}} e^{-\frac{im\pi x}{L}}\,dx = \begin{cases} 2L & n = m \\ 0 & n \ne m \end{cases}.$$

$$\Rightarrow \quad C_n = \frac{1}{2L}\int_{-L}^{L} f(x)\,\phi_n^*(x)\,dx$$

$$= \frac{1}{2L}\int_{-L}^{L} f(x)\,e^{-\frac{in\pi x}{L}}\,dx.$$

Example. $f(x) = x^2$ in $[-L, L]$. Complex Fourier series.

$$C_0 = \frac{1}{2L}\int_{-L}^{L} x^2\,dx = \frac{L^2}{3}$$

$$C_n = \frac{1}{2L}\int_{-L}^{L} f(x)\,e^{-\frac{in\pi x}{L}}\,dx$$

$$= \frac{1}{2L}\int_{-L}^{L} \underline{x^2 e^{-\boxed{\frac{in\pi x}{L}}}}\,dx$$

$$y = \frac{n\pi x}{L} \quad \Rightarrow \quad x = \frac{L}{n\pi}\,y \quad \Rightarrow \quad dx = \frac{L}{n\pi}\,dy.$$

$$= \frac{1}{2L}\int_{-n\pi}^{n\pi} \frac{L^2}{(n\pi)^2}\,y^2\,e^{-iy}\,\frac{L}{n\pi}\,dy$$

$$= \frac{L^2}{2n^3\pi^3}\,\underline{\int_{-n\pi}^{n\pi} y^2 e^{-iy}}\,dy$$

$$\cdot \quad \int y^2 e^{-iy}\,dy = \underline{\frac{1}{-i}}\,y^2 e^{-iy} + \frac{2}{i}\,\underline{\int y\,e^{-iy}\,dy}$$

$$= iy^2 e^{-iy} + \frac{2}{i}\left(-\frac{1}{i}\,y e^{-iy} + \underline{\frac{1}{i}\cdot -\frac{1}{i}}\,e^{-iy}\right)$$

$$= iy^2 e^{-iy} + 2y\,e^{-iy} - 2i\,e^{-iy}$$

$$= \frac{L^2}{2n^3\pi^3}\left[e^{-iy}\left(iy^2+2y-2i\right)\right]\Big|_{\underline{-n\pi}}^{\underline{n\pi}}$$

$$= \frac{2L^2(-1)^n}{n^2\pi^2}$$

$$\Rightarrow \quad f(x) = \frac{L^2}{3} + \sum_{n\neq 0} \boxed{\frac{2L^2(-1)^n}{\underline{n^2\pi^2}}}\, e^{\frac{in\pi x}{L}}$$

Comparing real and complex Fourier expansions

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, \underline{e^{\boxed{-}\frac{in\pi x}{L}}}\, dx$$

$$= \frac{1}{2L}\int_{-L}^{L} \underline{f(x)\left(\cos\frac{n\pi x}{L}\right.} - i\sin\frac{n\pi x}{L}\Big)\, dx$$

$$= \frac{1}{2}\left(a_n - \underline{ib_n}\right).$$

$$c_n^* = \frac{1}{2}\left(a_n + ib_n\right) = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{\boxed{\frac{in\pi x}{L}}}\, dx.$$

$$\Rightarrow \quad \underline{a_n = c_n + c_n^*, \qquad b_n = i\left(c_n - c_n^*\right).}$$

2.9. Differentiating and integrating Fourier series.

$$\underline{f(x)} = \sum_{n=-\infty}^{\infty} \boxed{c_n}\, \underline{e^{ik_n x}}, \qquad k_n = \boxed{\frac{n\pi}{L}}$$

$$\underline{\frac{df}{dx}} = \sum_{n=-\infty}^{\infty} \boxed{ik_n c_n}\, e^{ik_n x}$$

$$\int f(x)\, dx = \sum_{n=-\infty}^{+\infty} c_n (ik_n)^{-1} e^{ik_n x} + \text{const}.$$

$\underline{3.10}$. Fourier series and series expansions.

$f(x) = x^2$ in $[-\pi, \pi]$.

$$\underline{x^2} = \underline{\frac{\pi^2}{3}} + 4 \underline{\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \overset{1}{\boxed{\cos nx}}} \quad \text{in } [-\pi, \pi].$$

Let $\underline{x=0}$.

$$0 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\Rightarrow \quad \underline{\frac{\pi^2}{12}} = \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}} .$$

$$\sum_{n=1}^{N} \frac{(-1)^{n-1}}{n^2}$$

$$\Rightarrow \quad \pi_N = \sqrt{12 \sum_{n=1}^{N} \frac{(-1)^{n-1}}{n^2}} .$$

$N = 1$, $\pi_1 = 3.464101\ldots$

⋮

$N = 1000$, $\pi_{1000} = \underline{3.14159}16986\ldots$

⋮

$N = 100000$, $\pi_{100000} = 3.1415926535\ldots$

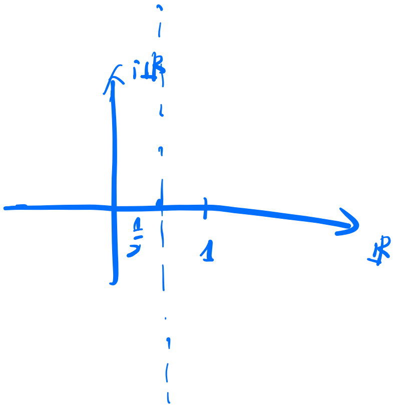
Let $x = \pi$,

$$\pi^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \underline{\overset{(-1)^n}{\cos n\pi}}$$

$$= \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \qquad \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2).$$

$$\boxed{\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}}$$



3.10 Parseval's theorem.

Thm. (Parseval's formula).

Real:
$$\frac{1}{2L} \int_{-L}^{L} |f(x)|^2 \, dx = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2} \sum_{n=1}^{\infty} \left(|a_n|^2 + |b_n|^2\right)$$

Complex:
$$\frac{1}{2L} \int_{-L}^{L} |f(x)|^2 \, dx = \sum_{n=-\infty}^{+\infty} |c_n|^2 .$$

Pf.
$$|f(x)|^2 = f(x) f^*(x) = \sum_{n=-\infty}^{\infty} c_n \phi_n(x) \sum_{n=-\infty}^{\infty} c_n^* \phi_n^*(x)$$

$$= \sum_{m,n=-\infty}^{+\infty} c_m c_n^* \phi_m(x) \phi_n^*(x)$$

$$\int_{-L}^{L} |f(x)|^2 dx = \sum_{m,n=-\infty}^{\infty} c_m c_n^* \overset{2L\,\delta_{mn}.}{\underline{\int_{-L}^{L} \phi_m(x) \phi_n^*(x) \, dx}}$$

$$= 2L \sum_{m,n=-\infty}^{+\infty} \underline{c_m c_n^* \delta_{mn}} = 2L \sum_{n=-\infty}^{\infty} |c_n|^2 .$$

Summing series via Parseval formula.

$f(x) = x^2$ in $[-L, L]$.

$$f(x) = \frac{L^2}{3} + \sum_{n\neq 0} \boxed{\frac{2L^2(-1)^n}{n^2\pi^2}} e^{\frac{in\pi x}{L}}$$

So by Parserval's formula, there is

$$\sum_{n=-\infty}^{+\infty} |c_n|^2 = |c_0|^2 + \sum_{n\neq 0} |c_n|^2$$

$$= \left(\frac{L^2}{3}\right)^2 + 2\sum_{n=1}^{\infty} \left(\frac{2L^2}{n^2\pi^2}\right)^2$$

$$= \frac{L^4}{9} + \sum_{n=1}^{\infty} \frac{8L^4}{n^4\pi^4} .$$

$$\frac{1}{2L}\int_{-L}^{L} (x^2)^2 dx = \frac{1}{5} L^4 .$$

$$\Rightarrow \quad \frac{1}{5} L^4 = \frac{L^4}{9} + \sum_{n=1}^{\infty} \frac{8L^4}{n^4\pi^4}$$

$$\Rightarrow \quad \frac{1}{5} = \frac{1}{9} + \sum_{n=1}^{\infty} \frac{8}{n^4\pi^4}$$

$$\Rightarrow \quad \underline{\sum_{n=1}^{\infty} \frac{1}{n^4}} = \frac{\pi^4}{90} . \qquad \underline{\zeta(4)} .$$