2023 SUMMER SCHOOL ON DIFFERENTIAL GEOMETRY (RIEMANNIAN GEOMETRY)

Q1. ($10'$) Let M^n be a differentiable n-manifold with differential structure $\mathcal{C} = \{(U_\alpha, \varphi_\alpha)\}$. Please construct a differential structure for

$$T^{(1,1)}M := \{(p, v, \omega) : p \in M, v \in T_pM, \omega \in T_p^*M\}.$$

Q2. ($30'$) Let

$$\mathbb{S}^n := \left\{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1\right\}$$

be the unit n-sphere in $(\mathbb{R}^{n+1}, g_{\mathbb{E}})$ with the induced metric $g_{\mathbb{S}}$. Please

(1) compute the metric expression of $g_{\mathbb{S}}$ under stereographic projection

$$\phi : \mathbb{R}^n \to \mathbb{S}^n,\ (y_1, y_2, \ldots, y_n) \mapsto \left(\frac{2y_1}{1+|y|^2}, \ldots, \frac{2y_n}{1+|y|^2}, \frac{|y|^2-1}{|y|^2+1}\right),$$

where $|y|^2 = \sum_{i=1}^n y_i^2$.

(2) compute the sectional curvature of $g_{\mathbb{S}}$ with above metric expression.

Q3. ($15'$) Consider the upper plane model $(\mathbb{R}^n_+, g_{\mathbb{H}})$ of hyperbolic space, where

$$\mathbb{R}^n_+ = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\} \text{ and } g_{\mathbb{H}} = \frac{1}{x_n^2} g_{\mathbb{E}}.$$

Please verify that: every geodesic in $(\mathbb{R}^n_+, g_{\mathbb{H}})$ is either a ray or a semi-circle perpendicular to the plane $\{x_n = 0\}$.

Q4. ($10'$) Let (M, g) be a complete Riemannian manifold and p be a point in M. Please verify that: $(d\exp_p)_v$ is not injective for $v \in T_pM$ if and only if $\gamma(1)$ is a conjugate point of $\gamma(0)$ along $\gamma(s) = \exp(sv) : [0, 1] \to (M, g)$.

Q5. ($15'$) Let (M, g) be $(\mathbb{S}^n, g_{\mathbb{S}})$ and p be some point in $\mathbb{S}^n$. Please verify that: the injective region $M_{inj}(p) = \mathbb{S}^n \setminus \{-p\}$ and the cut locus $C(p) = -p$. (Hint: use the fact that geodesics in $(\mathbb{S}^n, g_{\mathbb{S}})$ are great circles)

Q6. ($20'$) Let (M, g) be a closed Riemannian manifold with $\mathrm{Ric}(g) \geq 0$ and $\mathrm{diam}(M, g) \leq D_0$. By *$\epsilon$-separation* we mean a subset S of (M, g) such that $d_g(x, y) \geq \epsilon$ for every pair of points $x, y \in S$.

Please prove: for every $\epsilon > 0$ there is an integer $N = N(\epsilon)$ such that each ϵ-separation S satisfies $\#S \leq N$. (Hint: use volume comparison theorem)