2023 Summer School on Differential Geometry

Introduction to second order linear elliptic PDEs

Exam[1]

Question 1. *Let u be a non-negative harmonic function in $B_R(0)$. Prove the Harnack inequality: for any $x \in B_R(0)$,*

$$\left(\frac{R}{R+|x|}\right)^{n-2} \frac{R-|x|}{R+|x|} u(0) \leqslant u(x) \leqslant \left(\frac{R}{R-|x|}\right)^{n-2} \frac{R+|x|}{R-|x|} u(0).$$

Proof. By the Poisson integral formula, we have

$$u(x) = \frac{R^2 - |x|^2}{n\alpha(n)R} \int_{\partial B_R} \frac{u(y)}{|x-y|^n} \, \mathrm{d}S$$

where $\alpha(n)$ is the volume of unit sphere. Denote $r = |x|$, since $R - r \le |x-y| \le R + r$ with $|y| = R$, we have

$$\begin{aligned} \frac{1}{n\alpha(n)R} \cdot \frac{R-|x|}{R+|x|} \left(\frac{1}{R+|x|}\right)^{n-2} \int_{\partial B_R} u(y) \, \mathrm{d}S &\le u(x) \\ &\le \frac{1}{n\alpha(n)R} \cdot \frac{R+|x|}{R-|x|} \left(\frac{1}{R-|x|}\right)^{n-2} \int_{\partial B_R} u(y) \, \mathrm{d}S. \end{aligned}$$

Then the Mean Value Property gives us that

$$u(0) = \frac{1}{n\alpha(n)R^{n-1}} \int_{\partial B_R} u(y) \, \mathrm{d}S,$$

which completes the proof. □

Remark. *As a corollary, we shall have a Liouville theorem for bounded harmonic functions in $\mathbb{R}^n$. Indeed, We may assume $u \ge 0$ in $\mathbb{R}^n$. Taking any point $x \in \mathbb{R}^n$ and applying the Harnack inequality to any ball $B_R(0)$ with $R > |x|$, we obtain*

$$\left(\frac{R}{R+|x|}\right)^{n-2} \frac{R-|x|}{R+|x|} u(0) \le u(x) \le \left(\frac{R}{R-|x|}\right)^{n-2} \frac{R+|x|}{R-|x|} u(0),$$

which yields $u(x) = u(0)$ by letting $R \to +\infty$.

Question 2. *Let Ω be a bounded domain in $\mathbb{R}^n$ and L be a linear operator given by*

$$L = \sum_{i,j=1}^{n} a_{ij}\partial_{ij} + \sum_{i=1}^{n} b_i \partial_i + c$$

[1] Solutions are given by Ling Wang (lingwang@stu.pku.edu.cn)

where $a_{ij}, b_i, c \in L^\infty(\Omega) \cap C(\Omega)$ and $a_{ij} = a_{ji}$. Suppose that L is strictly elliptic and there exists a function $v \in C(\bar{\Omega}) \cap C^2(\Omega)$ such that $v > 0$ in $\bar{\Omega}$ and $Lv \leqslant 0$ in Ω. Prove that if $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfies $Lu \geqslant 0$ in Ω and $u \leqslant 0$ on $\partial\Omega$, then $u \leqslant 0$ in Ω.

Proof. Let $w = \frac{u}{v}$, we have

$$\begin{aligned}
\partial_i w &= \frac{\partial_i u}{v} - \frac{\partial_i v\, u}{v^2}, \\
\partial_{ij} w &= \frac{\partial_{ij} u}{v} - \frac{\partial_j v \partial_i u}{v^2} - \frac{\partial_i v \partial_j u}{v^2} - \frac{\partial_{ij} v\, u}{v^2} + 2\frac{\partial_i v \partial_j v\, u}{v^3}.
\end{aligned}$$

Hence, there is

$$\begin{aligned}
\sum_{i,j=1}^{n} a_{ij}\partial_{ij} w + \sum_{i=1}^{n} b_i \partial_i w &= \frac{1}{v} Lu - cw - \frac{u}{v^2} Lv + cw \\
&\quad - \frac{2}{v^2} \sum_{i,j=1}^{n} a_{ij} \partial_i u \partial_j v + \frac{2u}{v^3} \sum_{i,j=1}^{n} a_{ij} \partial_i v \partial_j v \\
&= \frac{1}{v} Lu - \frac{Lv}{v} w - \frac{2}{v} \sum_{i,j=1}^{n} a_{ij} \partial_j v \partial_i w \\
&\geq -\frac{Lv}{v} w - \frac{2}{v} \sum_{i,j=1}^{n} a_{ij} \partial_j v \partial_i w,
\end{aligned}$$

i.e.

$$\sum_{i,j=1}^{n} a_{ij}\partial_{ij} w + \sum_{i=1}^{n} \left(b_i + \frac{2}{v} \sum_{j=1}^{n} a_{ij} \partial_j v \right) \partial_i w + \frac{Lv}{v} w \geq 0.$$

Since $Lv \leq 0$ and $v > 0$, we know that $\frac{Lv}{v} \leq 0$. Note that $w \leq 0$ on $\partial\Omega$, then the classical maximum principle of second order elliptic PDEs gives us that $w \leq 0$ in Ω. Hence we have $u = wv \leq 0$ in Ω. □

Question 3. *Let Ω be a bounded domain in $\mathbb{R}^n$ and L be a linear operator given by*

$$L = \sum_{i,j=1}^{n} a_{ij}\partial_{ij} + \sum_{i=1}^{n} b_i \partial_i + c$$

where $a_{ij}, b_i, c \in L^\infty(\Omega) \cap C(\Omega), a_{ij} = a_{ji}$ and $c \leqslant 0$. Suppose that L is strictly elliptic and Ω satisfies the exterior sphere condition at $x_0 \in \partial\Omega$ (i.e. there exists a ball $B_R(y_0)$ such that $\Omega \cap B_R(y_0) = \emptyset$ and $\bar{\Omega} \cap \overline{B_R(y_0)} = \{x_0\}\big)$. Prove that there exists a function $w_{x_0} \in C(\bar{\Omega}) \cap C^2(\Omega)$ such that $w_{x_0}(x_0) = 0, w_{x_0}(x) > 0$ for any $x \in \partial\Omega \backslash \{x_0\}$ and $Lw_{x_0} \leqslant -1$ in Ω.

Proof. Since L is strictly elliptic and $a_{ij}, b_i, c \in L^\infty(\Omega) \cap C(\Omega)$, we know that there are positive constants λ and Λ such that

$$\sum_{i,j=1}^{n} a_{ij} x_i x_j \geq \lambda |x|^2, \ \forall\, x \in \mathbb{R}^n, \quad |a_{ij}|, |b_i|, |c| \leq \Lambda.$$

Define

$$w_{x_0}(x) = e^{\alpha d} - e^{\alpha(d+R) - \alpha|x-y_0|}, \quad \forall\, x \in \overline{\Omega},$$

where $d = \operatorname{diam}(\Omega)$ and $\alpha > 0$ is a constant to be determined later. It is clear that $w_{x_0} \in C(\overline{\Omega}) \cap C^2(\Omega)$ and $w_{x_0}(x_0) = 0, w_{x_0}(x) > 0$ for any $x \in \partial\Omega \setminus \{x_0\}$. Next, we show $Lw_{x_0} \leqslant -1$ in Ω. Indeed, a direct calculating yields

$$\begin{aligned}
\partial_i w_{x_0}(x) &= \alpha \, \frac{x_i - y_{0,i}}{|x-y_0|} \, e^{\alpha(d+R)-\alpha|x-y_0|}, \\
\partial_{ij} w_{x_0}(x) &= \left(\alpha \, \frac{\delta_{ij}}{|x-y_0|} - \alpha \, \frac{(x_i - y_{0,i})(x_j - y_{0,j})}{|x-y_0|^3} \right) e^{\alpha(d+R)-\alpha|x-y_0|} \\
&\quad - \alpha^2 \frac{(x_i - y_{0,i})(x_j - y_{0,j})}{|x-y_0|^2} \, e^{\alpha(d+R)-\alpha|x-y_0|}.
\end{aligned}$$

Then there is

$$\begin{aligned}
Lw_{x_0}(x) &= \Bigg[\alpha \, \frac{a_{ii}}{|x-y_0|} - \alpha \, \frac{a_{ij}(x_i - y_{0,i})(x_j - y_{0,j})}{|x-y_0|^3} - \alpha^2 \frac{a_{ij}(x_i - y_{0,i})(x_j - y_{0,j})}{|x-y_0|^2} \\
&\qquad + \alpha \frac{b_i(x_i - y_{0,i})}{|x-y_0|} - c \Bigg] e^{\alpha(d+R)-\alpha|x-y_0|} + c e^{\alpha d} \\
&\leq \left(\frac{n\Lambda}{|x-y_0|} - \frac{\lambda\alpha}{|x-y_0|} - \alpha^2 \lambda + \Lambda\alpha + \Lambda \right) e^{\alpha(d+R)-\alpha|x-y_0|} \\
&\leq \left(-\alpha^2 \lambda + \big((n\Lambda - \lambda)/R + \Lambda \big) \alpha + \Lambda \right) e^{\alpha(d+R)-\alpha|x-y_0|}.
\end{aligned}$$

Since $\lim\limits_{\alpha \to +\infty} \left(-\alpha^2 \lambda + \big((n\Lambda - \lambda)/R + \Lambda \big) \alpha + \Lambda \right) = -\infty$, we may choose a sufficiently large constant $\alpha > 0$ such that $-\alpha^2 \lambda + \big((n\Lambda - \lambda)/R + \Lambda \big) \alpha + \Lambda \leq -1$. Then

$$\begin{aligned}
Lw_{x_0}(x) &\leq \left(-\alpha^2 \lambda + \big((n\Lambda - \lambda)/R + \Lambda \big) \alpha + \Lambda \right) e^{\alpha(d+R)-\alpha|x-y_0|} \\
&\leq -e^{\alpha(d+R)-\alpha|x-y_0|} \\
&\leq -1.
\end{aligned}$$

□

Remark. *Under the conditions of Question 3, we can use this barrier function w_{x_0} to show that if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a solution of $Lu = f$ in Ω and $u = \varphi$ on $\partial\Omega$ for some $\varphi \in C^2(\partial\Omega)$, then u satisfies a Lipschitz conditions at x_0, i.e.*

$$|u(x) - u(x_0)| \leq C|x - x_0|, \quad \forall\, x \in \Omega,$$

where $C = C(\lambda, \Lambda, R, \Omega, \sup|f|, \|\varphi\|_{C^2(\partial\Omega)})$. (The detail is leave to the reader.)