§1 Introduction

<u>Def</u>. Let X be a set. A distance on X is a function

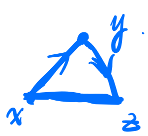
$$d : X \times X \to \mathbb{R}_{\geq 0}$$

satisfies

(M1) $d(x,y) = 0 \iff x = y$.

(M2) $d(x,y) = d(y,x)$, $\forall x, y \in X$.

(M3) $d(x,z) \leq d(x,y) + d(y,z)$, $\forall x, y, z \in X$. (Triangle inequality)



<u>Def</u>. A metric space is a pair (X, d).

<u>Ex</u>. $X = \mathbb{R}$, define

$$d_1(x,y) = |x-y|.$$

$(\mathbb{R}, d_1)$ metric space

- (M1) $\left(\underbrace{d_1(x,y) = 0}^{d_1 \geq 0} \iff x = y \right)$

  Suppose $x = y$. $d_1(x,y) = |x-y| = |0| = 0$.

  On the other hand, $d_1(x,y) = 0$, then $|x-y| = 0$ i.e. $x = y$.

- (M2) $(d_1(x,y) = d_1(y,x))$

$\forall x, y \in \mathbb{R}$. $\quad d_1(x,y) = |x-y| = |y-x| = d_1(y,x)$.

- (M_3) $\quad \left(d_1(x,z) \le d_1(x,y) + d_1(y,z) \right)$

$$|x-z| \le |x-y| + |y-z|.$$

Note that $\quad x-z = \underbrace{(x-y)}_{a} + \underbrace{(y-z)}_{b}$

$$\boxed{|a+b| \le |a| + |b|}$$

Lem. For $a, b, c, d \in \mathbb{R}$, we have

$$\max\{a+b,\ c+d\} \le \max\{a, c\} + \max\{b, d\}$$

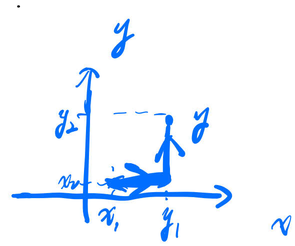
$$\begin{aligned}
d_1(x,z) = |x-z| &= \max\{x-z,\ z-x\}. \\
&= \max\{(x-y) + (y-z),\ (z-y) + (y-x)\}. \\
&\le \max\{x-y,\ y-x\} + \max\{y-z,\ z-y\}. \\
&= |x-y| + |y-z| = d_1(x,y) + d_2(y,z).
\end{aligned}$$

Ex. $X = \mathbb{R}^2$. $\quad x, y \in \mathbb{R}^2$, $\quad x = (x_1, x_2)$, $\quad y = (y_1, y_2)$

$$d_1(x,y) = |x_1 - y_1| + |x_2 - y_2|.$$

$(\mathbb{R}^2, d_1)$ metric space.

$d_1 \ge 0$

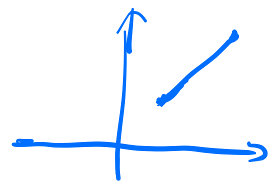


$\underline{Ex}$. $X = \mathbb{R}^2$,

$$d_2(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$$

$(\mathbb{R}^2, d_2)$ metric space.

$d_2 \geq 0$



- (M_3) $d_2(x,z) \leq d_2(x,y) + d_2(y,z)$

$$\Leftrightarrow \sqrt{(x_1-z_1)^2 + (x_2-z_2)^2} \leq \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2} + \sqrt{(y_1-z_1)^2 + (y_2-z_2)^2}$$

$a_1 = x_1 - y_1$, $b_1 = y_1 - z_2$

$$\Leftrightarrow \sqrt{(a_1+b_1)^2 + (a_2+b_2)^2} \leq \sqrt{a_1^2 + a_2^2} + \sqrt{b_1^2 + b_2^2}.$$

$$\Leftrightarrow (a_1+b_1)^2 + (a_2+b_2)^2 \leq a_1^2 + a_2^2 + b_1^2 + b_2^2 + 2\sqrt{a_1^2+a_2^2}\sqrt{b_1^2+b_2^2}$$

$$\Leftrightarrow \cancel{2}a_1b_1 + \cancel{2}a_2b_2 \leq \cancel{2}\sqrt{a_1^2+a_2^2}\sqrt{b_1^2+b_2^2}$$ (Cauchy-Schwarz)

$(\mathbb{R}^2$, $d_p(x,y) = \left(|x_1-y_1|^p + |x_2-y_2|^p\right)^{1/p}$ $)$.

Similarly, $X = \mathbb{R}^n$

$$d_p(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{\frac{1}{p}}. \quad (p = \infty)$$

$$d_\infty(x,y) = \max_{1 \leq i \leq n} \{|x_i - y_i|\}.$$

$n=1$, $d_p = d_1$.

- $n=2$, $d_1(x,y) = |x_1-y_1| + |x_2-y_2|$

  $d_\infty(x,y) = \max\{|x_1-y_1|, |x_2-y_2|\}$

$\Rightarrow$ $\frac{1}{2} d_1(x,y) \le d_\infty(x,y) \le d_1(x,y)$ (n, $\frac{1}{n} d_1 \le d_\infty \le d_1$).

- $p \ge q$ $\boxed{d_p(x,y) \le d_q(x,y)}$, $\forall x,y \in \mathbb{R}^n$.

Pf. $x=y$. Trivial.

$x \ne y$. Let $A_i = \frac{|x_i-y_i|}{(\sum_{i=1}^n |x_i-y_i|^p)^{1/p}} \le 1$. $f(t) = \sum_{i=1}^n A_i^t$.

$t=p \Rightarrow f(p) = 1 \le f(q) = \sum_{i=1}^n A_i^q = \frac{\sum_{i=1}^n |x_i-y_i|^q}{(\sum_{i=1}^n |x_i-y_i|^p)^{\frac{q}{p}}}$

$f'(t) = \sum_{i=1}^n A_i^t \ln A_i \le 0$

$(\sum_{i=1}^n |x_i-y_i|^p)^{1/p} \ge |x_i-y_i|$.

Ex. X set. function

$$d_{discr}(x,y) = \begin{cases} 0, & x=y \\ 1, & x\ne y. \end{cases}$$

$d(x,z) \le d(x,y) + d(y,z)$

Space of sequences.

$$A : \mathbb{N} \to \mathbb{R}$$

$$A = (A_0, A_1, A_2, \cdots, A_n, \cdots)$$

Ex. $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$. $x_n = \frac{1}{n}$

Def. For $p \geq 1$, we let

$$\ell^p := \{ (A_n) \text{ real sequences such that } \sum_{n=0}^{\infty} |A_n|^p < \infty \}.$$

$$\ell^\infty := \{ (A_n) \text{ real sequences such that } \sup_{n \in \mathbb{N}} |A_n| < \infty \}.$$

$$d_p(A, B) = \left(\sum_{n=0}^{\infty} |A_n - B_n|^p \right)^{1/p}. \quad (p = \infty) \ldots$$

(ℓ^p, d_p) metric space.

Prop. For $p \leq q$, the inclusion $\ell^p \subseteq \ell^q$ holds.

Pf. $(\forall \{A_n\} \in \ell^p \Rightarrow \{A_n\} \in \ell^q.)$ $\quad \ell^1 \subseteq \ell^2 \subseteq \ell^3 \subseteq \cdots \subseteq \ell^\infty.$

- $\sum_{n=0}^{\infty} |A_n|^q < \infty.$

Note that

$$\sum_{n=0}^{\infty} |A_n|^p < \infty,$$

which implies

$|a_n| \to 0, \quad n \to \infty.$

$\exists\, N \in \mathbb{N}, \quad \forall n > N \quad |a_n| < 1. \quad \Rightarrow \quad |a_n|^2 \le |a_n|^p, \quad \forall n > N.$

$$\Rightarrow \quad \sum_{n=N+1}^{\infty} |a_n|^2 \le \sum_{n=N+1}^{\infty} |a_n|^p < \infty.$$

$$\Rightarrow \quad \sum_{n=0}^{\infty} |a_n|^2 < \infty \quad \Rightarrow \quad \{a_n\} \in \ell^2. \qquad \#.$$

Space of functions.

Def.

$$C[0,1] = \{ f : [0,1] \to \mathbb{R} \text{ such that } f \text{ is continuous} \}.$$

$f(x) = x \in C[0,1].$

$$f(x) = \begin{cases} \frac{1}{2}, & x \in [0, \frac{1}{2}) \\ 1, & x \in [\frac{1}{2}, 1] \end{cases} \notin C[0,1].$$

Def. We define a distance d_{L^1} in $C[0,1]$ by

$$d_{L^1}(f,g) = \int_0^1 |f(x) - g(x)|\, dx.$$

$$d_{L^\infty}(f,g) = \max_{x \in [0,1]} |f(x) - g(x)|.$$

- (M_3) $\quad d_{L^1}(f,g) \le d_{L^1}(f,h) + d_{L^1}(h,g)$

$$\Leftrightarrow \quad \int_0^1 |f(x)-g(x)|\,dx \le \int_0^1 |f(x)-h(x)|\,dx + \int_0^1 |f(x)-g(x)|\,dx$$

$$\Leftarrow \quad |f(x)-g(x)| \le |f(x)-h(x)| + |h(x)-g(x)|. \qquad \#.$$

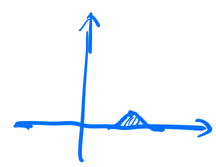
Remark (M_1). $\int_0^1 |f(x)-g(x)|\,dx = 0 \Leftrightarrow f=g$.

$\Rightarrow$

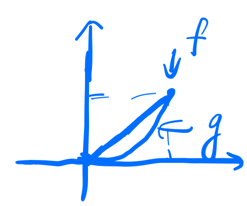
Lem. Let $h: [0,1] \to \mathbb{R}$ be continuous. Then

$$\int_0^1 |h(x)|\,dx = 0$$

implies $h \equiv 0$.



Ex. $f(x) = x$, $g(x) = x^2 \in C[0,1]$. $\quad d_{L^1}$, d_{L^∞}.



$$d_{L^1}(f,g) = \int_0^1 |f(x)-g(x)|\,dx$$

$$= \int_0^1 |x-x^2|\,dx.$$

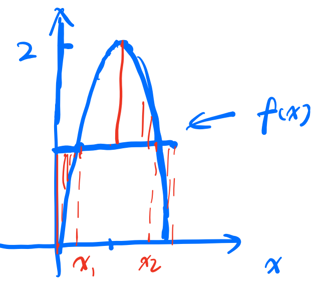
$$= \int_0^1 (x-x^2)\,dx.$$

$$= \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 \Big|_0^1 = \tfrac{1}{6}.$$

$$d_{L^\infty}(f,g) = \max_{x\in[0,1]} |x-x^2| = \max_{x\in[0,1]} (x-x^2) = \tfrac{1}{4}.$$

Ex. $f(x)=1$, $g(x)=2\sin \pi x$ $\in C[0,1]$. d_{L^1}, d_{L^∞}.

$$d_{L^1}(f,g) = \int_0^1 |f(x)-g(x)|\,dx$$



$$= \int_0^{x_1}(f(x)-g(x))\,dx + \int_{x_1}^{x_2}(g(x)-f(x))\,dx + \int_{x_2}^{1}(f(x)-g(x))\,dx$$

$f(x)=g(x)$

$\Longleftrightarrow$ $2\sin \pi x = 1$

$\Longleftrightarrow$ $\sin \pi x = \frac{1}{2}$

$x_1=\frac{1}{6}$, $x_2=\frac{5}{6}$

$$= \int_0^{\frac{1}{6}}(1-2\sin \pi x)\,dx + \int_{\frac{1}{6}}^{\frac{5}{6}}(2\sin \pi x-1)\,dx + \int_{\frac{5}{6}}^{1}(1-2\sin \pi x)\,dx$$

$$d_{L^\infty}(f,g) = \max_{x\in[0,1]} |f(x)-g(x)| = \max_{x\in[0,1]} |\underline{1-2\sin \pi x}| = 1.$$

Product metrics.

- A and B are two sets, the cartesian product, $A\times B$, of A, B is defined by

$$\underline{A\times B} := \{(x,y) : x\in A,\ y\in B\}.$$

- $\#A$ the number of the elements in A.

$$\#(A\times B) = \underline{\#A}\times\underline{\#B}$$

$A=\underline{\{1,2,3\}}$

$B=\underline{\{1,2\}}$

$A\times B = \{\underline{(1,1),\ (1,2),\ (2,1)}$

$(2,2), (3,1), (3,2)\}$

$\mathbb{R}$, $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, $\mathbb{R}^n = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n}$

(X, d_X) , (Y, d_Y) metric spaces.

$(x_1, y_1), (x_2, y_2) \in X \times Y$.

$$D_1((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2).$$

$$D_2((x_1, y_1), (x_2, y_2)) = \sqrt{(d_X(x_1, x_2))^2 + (d_Y(y_1, y_2))^2}$$

$\vdots$

$$D_\infty((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}.$$

Thm 1.5.1 D_2 is distance.

Ex. $(X, d_X) = (\mathbb{R}^2, d_2)$, $(Y, d_Y) = (\mathbb{R}, d_{discr})$

$x = (\underbrace{(1,2)}_{\in \mathbb{R}^2}, \underbrace{3}_{\in \mathbb{R}})$, $y = ((4,5), 6)$. $D_\infty(x, y) = ?$

$$D_\infty(x, y) = \max(d_2((1,2), (4,5)), d_{discr}(3, 6))$$

$$= \max\{\sqrt{3^2+3^2}, 1\} = 3\sqrt{2}.$$

Isometries

Def. An isometry from (X, d_X) to (Y, d_Y) is a function $\phi: X \to Y$ satisfying

(I_1) For $\forall x_1, x_2 \in X$, $d_Y(\phi(x_1), \phi(x_2)) = d_X(x_1, x_2)$.

(I_2) ϕ is surjective.

Lem. An isometry is injective.

Def. Two metric spaces (X, d_X), (Y, d_Y) are isometry if there exists an isometry $\phi: X \to Y$.

Ex. $([0,1], d_1)$, $([2,3], d_1)$

$$\phi: [0,1] \to [2,3]$$
$$x \mapsto x+2.$$

Ex. $([0,2], d_1)$ and $([0,1], d_1)$ are not isometric.

If there exists an isometry $\phi: [0,2] \to [0,1]$.

$$d_1(\phi(0), \phi(2)) = d_1(0,2) = 2$$

However, $\phi([0,2]) \subset [0,1]$, no two points in $[0,1]$ have a distance greater than 1. Contradiction.