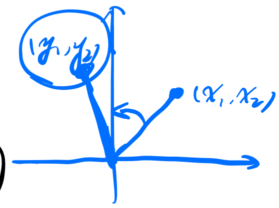
Isometry.

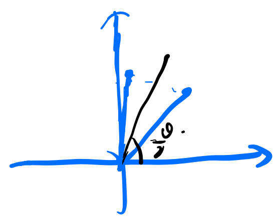
Ex. $(\mathbb{R}^2, d_2)$



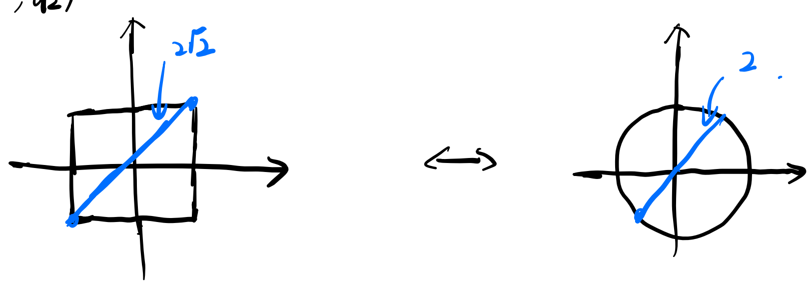
$$f(x_1, x_2) = ((\cos\theta)x_1 - (\sin\theta)x_2 \,,\, (\sin\theta)\, x_1 + (\cos\theta)\, x_2)$$

$$= A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{where} \quad A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

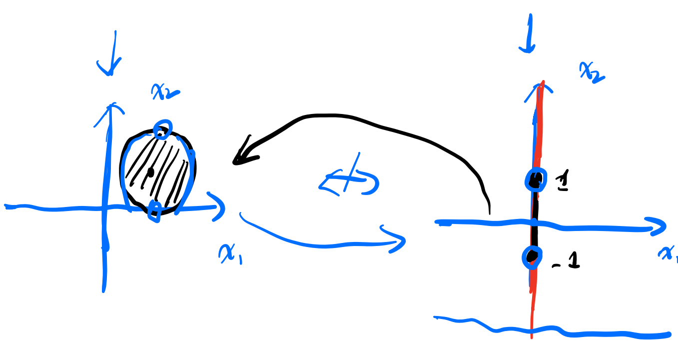
$$f(x_1, x_2) = ((\cos\theta)x_1 + (\sin\theta)x_2 \,,\, (\sin\theta)\, x_1 - (\cos\theta)\, x_2)$$



Ex $(\mathbb{R}^2, d_2)$



Ex. $(\mathbb{R}^2, d_2)$, $(\{(x_1, x_2) : x_1 = 0\}, d_2)$



If there exists an isometry $\phi : \mathbb{R}^2 \to \{(x_1, x_2) : x_1 = 0\}$. ϕ is injective.

Contradicts.

§2. Continuity and Convergence in metric spaces.

Convergence.

Recall. (x_n) is a sequence converges to $\ell \in \mathbb{R}$ if for $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, $\forall n > N$, $|x_n - \ell| < \varepsilon$. $\Leftrightarrow$ $d_1(x_n, \ell) < \varepsilon$.

Def. A sequence of elements of a set X is a function $x: \mathbb{N} \to X$, $x = (x_n)_{n \in \mathbb{N}}$, or (x_n), or $\{x_n\}$.

Def. Let (X, d) be a metric space, and $\{x_n\}$ is a sequence in X, $\ell \in X$. Then we say a sequence $\{x_n\}$ converges to ℓ in X if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, $\forall n > N$, $d(x_n, \ell) < \varepsilon$. $\Leftrightarrow$ $x_n \in B_\varepsilon(\ell)$.

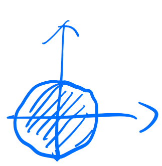
- $\lim\limits_{n \to \infty} x_n = \ell$. $\quad x_n \xrightarrow{d} \ell$, $n \to \infty$. $\quad x_n \to \ell$ in (X, d).

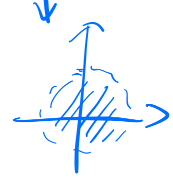
Def. Open ball of radius R centered at $p \in X$ is defined by

$$B_R(p) = \{x \in X : d(x, p) < R\}.$$

$$\bar{B}_R(p) = \{x \in X : d(x, p) \le R\}.$$

$B_R^d(p)$





Def. Let (X, d) is a metric space, then we say that $\{x_n\}$ is convergent to $l \in X$ if for $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, $\forall n > N$, $x_n \in B_\varepsilon(l)$.

x_n : $\exists l \in X$, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, $\forall n > N$, $x_n \in B_\varepsilon(l)$

x_n doesn't converge : $\forall l \in X$, $\exists \varepsilon_0 > 0$, $\forall N \in \mathbb{N}$, $\exists n > N$, $x_n \notin B_{\varepsilon_0}(l)$.

Ex. $(C[0,1], d_{L^\infty})$ Recall. $d_{L^\infty}(f, g) = \max_{x \in [0,1]} |f(x) - g(x)|$.

$f_n \in C[0,1]$, $\boxed{f_n(x) = \frac{x}{n+1}}$.

$$d_{L^\infty}(f_n, 0) < \varepsilon$$

$\forall \varepsilon > 0$, Taking $N > \frac{1}{\varepsilon} - 1$, $\forall n > N$

$$d_{L^\infty}(f_n, 0) = \max_{x \in [0,1]} \left| \frac{x}{n+1} \right| = \boxed{\frac{1}{n+1} < \varepsilon}.$$

$$\Updownarrow$$

$$\frac{1}{\varepsilon} < n+1.$$

$$\Updownarrow$$

$$\frac{1}{\varepsilon} - 1 < n.$$

Recall. $A \in \ell^p$, $A = (A_n) = \{A_0, A_1, A_2, \dots, A_n, \dots\}$.

(A_k) is a sequence of elements of ℓ^p.

$$A_k = (A_{k,0}, A_{k,1}, \cdots, A_{k,n}, \cdots)$$

Ex. $(A_k)_{k\in\mathbb{N}} = \left(\left(\frac{(-1)^n}{k+1}\right)_{n\in\mathbb{N}} \right)_{k\in\mathbb{N}}$ of (ℓ^∞, d_∞).

$A_k \xrightarrow{d_\infty} B$, $k\to\infty$.

$A_0 = (1, -1, 1, -1, \cdots, (-1)^n, \cdots)$

$A_1 = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \cdots\cdots, \frac{(-1)^n}{2}, \cdots\right)$

$\vdots$

$A_k = \left(\frac{1}{k+1}, \frac{-1}{k+1}, \cdots, \frac{(-1)^n}{k+1}, \cdots\right)$

$B = (B_0, B_1, \cdots, B_n, \cdots)$

$= (0, 0, \cdots, 0, \cdots)$

Pf. $\boxed{\begin{array}{l} \forall \varepsilon>0, \text{ choosing an integer } N > \frac{1}{\varepsilon}-1, \ \forall k>N, \\ d_\infty(A_k, B) = \sup\limits_{n\in\mathbb{N}} |A_{k,n} - B_n| = \sup\limits_{n\in\mathbb{N}} \left|\frac{(-1)^n}{k+1}\right| \\ \qquad = \frac{1}{k+1} < \varepsilon \end{array}}$

$\Updownarrow$

$\frac{1}{\varepsilon} < k+1 \iff k > \frac{1}{\varepsilon} - 1$

Ex. $(A_k)_{k\in\mathbb{N}} = \left(\left(\frac{(-1)^k}{n+1}\right)_{n\in\mathbb{N}} \right)_{k\in\mathbb{N}}$.

$$A_0 = (1, \tfrac{1}{2}, \cdots, \tfrac{1}{n+1}, \cdots)$$

$$A_1 = (-1, -\tfrac{1}{2}, \cdots, -\tfrac{1}{n+1}, \cdots)$$

$$\vdots$$

$$A_k = ((-1)^k, \tfrac{(-1)^k}{2}, \cdots, \tfrac{(-1)^k}{n+1}, \cdots)$$

(A_k) doesn't converge.

Pf. If $A_k \xrightarrow{d_\infty} B$. Fix $\varepsilon_0 = \frac{1}{2}$ $\exists N \in \mathbb{N}$, $\forall k > N$

$$\frac{1}{2} > d_\infty(A_k, B) = \sup_{n \in \mathbb{N}} |A_{k,n} - B_n|$$

$$\geq |A_{k,0} - B_0|$$

$$\Rightarrow \quad -\frac{1}{2} + A_{k,0} < B_0 < \frac{1}{2} + A_{k,0}$$

$$\Rightarrow \quad \frac{1}{2} < B_0 < \frac{3}{2} \text{ and } -\frac{3}{2} < B_0 < -\frac{1}{2}, \quad \text{contradiction.}$$

Continuity.

Recall. f is continuous at x_0 if $\forall \varepsilon > 0$, $\exists \delta > 0$, $\forall x$: $|x - x_0| < \delta$ (d_X), $|f(x) - f(x_0)| < \varepsilon$. ($d_Y$).

Def. Let (X, d_X), (Y, d_Y) be two metric spaces. Let $f: X \to Y$ be a function. Then f is continuous at $x_0 \in X$ if $\forall \varepsilon > 0$, $\exists \delta > 0$,

$d_Y(f(x), f(x_0)) < \varepsilon$ whence $0 < d_X(x, x_0) < \delta$.

- $\lim_{x \to x_0} f(x) = f(x_0)$.

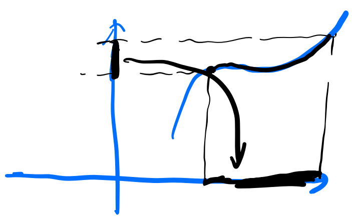
Direct image.

$$f(A) = \{ f(x) : x \in X \}.$$

Inverse image (preimage)

$$f^{-1}(A) = \{ x \in X : f(x) \in A \}.$$

$$f^{-1}(y_0) = \{ x_1, x_2, \cdots x_n \}.$$



Lem. f is continuous at x_0 $\Leftrightarrow$ $\forall \varepsilon > 0, \exists \delta > 0,$ $d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \varepsilon$

$\Leftrightarrow$ $\forall \varepsilon > 0, \exists \delta > 0,$ $x \in B_\delta^{d_X}(x_0) \Rightarrow f(x) \in B_\varepsilon^{d_Y}(f(x_0))$

$\Leftrightarrow$ $\forall \varepsilon > 0, \exists \delta > 0,$ $f(B_\delta^{d_X}(x_0)) \subset B_\varepsilon^{d_Y}(f(x_0))$.

$\Leftrightarrow$ $\forall \varepsilon > 0, \exists \delta > 0,$ $B_\delta^{d_X}(x_0) \subset f^{-1}(B_\varepsilon^{d_Y}(f(x_0)))$.

Lem f is continuous at p $\Leftrightarrow$ $\forall \{x_n\}, x_n \xrightarrow{d_X} p \Rightarrow f(x_n) \xrightarrow{d_Y} f(p)$.

Ex. $(X, d) = ([0,1], d_{L^1})$.

$$\phi: C[0,1] \to \mathbb{R}$$
$$f \mapsto \max_{x\in[0,1]} f$$

Prove that ϕ is not continuous at $f=0$.

Pf. $f_n(x) = x^n$.

$$d_{L^1}(f_n, 0) = \int_0^1 x^n\,dx = \frac{1}{n+1} \to 0, \quad n\to\infty$$

$$\phi(f_n) = \max_{x\in[0,1]} (x^n) = 1.$$

$$\Rightarrow \; 0 = \phi(\lim_{n\to\infty} f_n) \neq \lim_{n\to\infty} \phi(f_n) = 1$$

• If $(X,d) = (C[0,1], d_{L^\infty})$. $\phi: C[0,1] \to \mathbb{R}$
$$f \mapsto \max_{x\in[0,1]} f(x).$$

Pf. $\forall \varepsilon>0$, choosing $\delta = \varepsilon$, then

$$d_{L^\infty}(f,g) < \delta \Rightarrow |\phi(f) - \phi(g)| = \left|\max_{x\in[0,1]} f(x) - \max_{x\in[0,1]} g(x)\right|$$
$$\le \max_{x\in[0,1]} |f(x) - g(x)| < \varepsilon.$$

Ex. $(X,d) = (C[0,1], d_{L^\infty})$. $(f_n) \subset C[0,1]$.

$f_n(x) = (1+\frac{1}{n})e^x$. $\quad g(x) = e^x$

Pf. $\forall \varepsilon>0$, choosing an integer $N > \frac{e}{\varepsilon}$, $\forall n > N$.

$$d_{L^\infty}(f_n, g) = \max_{x\in[0,1]} \left|\left(1+\frac{1}{n}\right)e^x - e^x\right| = \max_{x\in[0,1]} \left|\frac{1}{n}e^x\right|$$

$$= \frac{e}{n} < \varepsilon$$

$$\Updownarrow$$

$$\frac{e}{\varepsilon} < n$$

Ex. $(X,d) = (C[0,1], d_{L^\infty})$. $f_n(x) = x^n$, doesn't converge.

$$\to \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1. \end{cases}$$

- $(X,d) = (C[0,1], d_{L^1})$ $f_n(x) = x^n$ converges to $0 \in C[0,1]$.

The topology of metric space.

Open sets and closed sets.

Def. Let (X,d) be a metric space, and $A \subset X$ be a subset. Then we say that A is open if and only if $\forall p \in A$, there exists an $\varepsilon > 0$, such that $B_\varepsilon(p) \subset A$. If $B \subset X$, then B is closed if $X \setminus B$ is open.

Ex. 1. $(\mathbb{R}, d_1)$ open interval is open set.

2. $(\mathbb{R}^n, d_2)$ $B_R(p)$ is an open set.

Ex. $[0,1)$ is not open in $(\mathbb{R}, d_1)$. But $[0,1)$ is open in $([0,+\infty), d_1)$

Ex. $(X,d) = (C[0,1], d_{L^\infty})$.

$$A = \{ f \in X : f(\tfrac{1}{3}) > 1 \}.$$

Prove that A is open in X.

Pf. $\forall f_0 \in A$. $f_0(\tfrac{1}{3}) > 1$

$d_{L^\infty}(f, f_0) < \varepsilon$

$B_\varepsilon(f_0) \subset A$ $\qquad B_\varepsilon(f_0) = \{ f \in X : \max_{x \in [0,1]} |f(x) - f_0(x)| < \varepsilon \}$.

$\forall f \in B_\varepsilon(f_0) \Rightarrow f \in A$. $\qquad f(\tfrac{1}{3}) > 1$.

$\Rightarrow \qquad |f(\tfrac{1}{3}) - f_0(\tfrac{1}{3})| < \varepsilon \iff f_0(\tfrac{1}{3}) - \varepsilon < f(\tfrac{1}{3}) < f_0(\tfrac{1}{3}) + \varepsilon$

$f_0(\tfrac{1}{3}) - \varepsilon \geq 1 \iff \varepsilon \leq f_0(\tfrac{1}{3}) - 1 > 0$

Taking $\varepsilon = f_0(\tfrac{1}{3}) - 1$, then

$B_\varepsilon(f_0) \subset A$.

Remark. ϕ, X are both open and closed.

Ex. Let $(X, d) = (X, d_{discr})$, $A \subset X$.

$$d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$$

Prove A is open.

Pf. $\forall x_0 \in A$

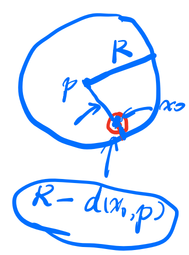
$$B_\varepsilon(x_0) = \{x \in X : d_{discr}(x, x_0) < \varepsilon\}.$$

$\varepsilon < 1$ $\quad B_\varepsilon(x_0) = \{x_0\} \subset A$. $\quad B_{1/2}(x_0) = \{x_0\} \subset A$.

Ex. $B_R^d(p)$ is an open set in (X, d). $\bar{B}_R^d(p)$ is closed.

Pf. $\forall x_0 \in B_R(p)$, choosing $0 < \varepsilon < R - d(x_0, p)$, then

$B_\varepsilon(x_0) \subset B_R(p)$.



Indeed, $\forall x \in B_\varepsilon(x_0)$, i.e. $d(x, x_0) < \varepsilon$. Then

$$\begin{aligned} d(x,p) &\le d(x, x_0) + d(x_0, p) \\ &< \varepsilon + d(x_0, p) < R - d(x_0, p) + d(x_0, p) = R. \end{aligned}$$

$\Rightarrow$ $x \in B_R(p)$.

Since $\bar{B}_R(p) = \{x : d(x,p) \le R\} = X \setminus \{x : d(x,p) > R\}$,

$\bar{B}_R(p)$ is closed.