Open sets and Closed sets.

Lem. Let (X, d_X) be a metric space.

(i) ϕ, X are open.

(ii) An arbitrary union of open sets is open.

(iii) A finite intersection of open sets is open.

- $A_k = (-\frac{1}{k}, \frac{1}{k})$ in $(\mathbb{R}, d_1)$

$$\bigcap_{k=1}^{\infty} A_k = \{0\}.$$

$\forall k$, $0 \in (-\frac{1}{k}, \frac{1}{k}) \Rightarrow 0 \in A_k$, $\forall k$, $\Rightarrow 0 \in \bigcap_{k=1}^{\infty} A_k$

$\Rightarrow \{0\} \subset \bigcap_{k=1}^{\infty} A_k$.

$\forall x \in \bigcap_{k=1}^{\infty} A_k$, if $x \neq 0$, without loss of generality, we assume $x > 0$. $\exists k \in \mathbb{N}$, s.t. $\frac{1}{k} < x$. $\Rightarrow x \notin (-\frac{1}{k}, \frac{1}{k})$, contract.

- $\forall k$, $x \in (-\frac{1}{k}, \frac{1}{k})$, Let $k \to \infty$, $x = 0$.

Pf of (iii). $A_1, \dots, A_N$ open sets.

$p \in \bigcap_{i=1}^{N} A_i$. $p \in A_i$, $i = 1, 2, \dots, N$.

$\exists\, \varepsilon_i > 0$, s.t. $B_{\varepsilon_i}(p) \subset A_i$, $i = 1, 2, \dots, N$. Taking $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_N\}$.

$\nexists \inf \{\varepsilon_1, \dots, \varepsilon_n, \dots\}$.

Cor. (i) ϕ, X are closed.

(ii) The arbitrary intersection of closed sets is closed.

(iii) A finite union of closed sets is closed.

Lem. The subset A is closed $\Leftrightarrow$ For every sequence (x_n) of A, if (x_n) converges to ℓ, then $\ell \in A$.

Ex. Let $p<q$, $\ell^p \subsetneq \ell^q$ and ℓ^p is not closed in (ℓ^q, d_q).

Taking $B = (B_n) = \left(\frac{1}{n^p}\right) \in \ell^q$, but $B \notin \ell^p$.

$$A_{k,n} = \begin{cases} \frac{1}{n^{1/p}}, & n<k \\ 0, & n \geq k. \end{cases}$$

$(A_1) = (A_{1,1}, A_{1,2}, \dots, A_{1,n}, \dots) = (0, 0, \dots, 0, \dots)$

$(A_2) = (A_{2,1}, A_{2,2}, \dots A_{2,n}, \dots) = (1, 0, \dots\dots, 0)$

$\vdots$

$(\dots, 0, 0, \dots, 0)$

Then for every k, $(A_k) \in \ell^p$.

$d_q((A_k), B) \to 0$, $k \to \infty$ $(\varepsilon - N)$

$\Rightarrow$ ℓ^p is not closed. $\ell^p \subset (\ell^q, d_q)$ (ℓ^p, d_p)

Introduce to topology

Def. A topogy $\mathcal{U}$ on X is a collection of subsets $U_i \subset X$, called the open sets, satisfing

(i) $\phi, X \in \mathcal{U}$,

(ii) If $U_i \in \mathcal{U}$, $\bigcup_{i \in I} U_i \in \mathcal{U}$.

(iii) If $U_1, \dots, U_N \in \mathcal{U}$, then $\bigcap_{i=1}^{N} U_i \in \mathcal{U}$.

$(X, \mathcal{U})$ is topological space.

Ex. (X, d_X).

$$\mathcal{U} = \{ A \subset X \text{ is open in } (X, d_X) \}.$$

$(X, \mathcal{U})$ is a topological space.

Ex. $X = \{A, B\}$.

$\mathcal{U}_1 = \{\phi, X\}$. Trival pogology

$\mathcal{U}_2 = \{\phi, \{A\}, \{B\}, X\}$ discrete topology.

Remark. $\mathcal{P}(X) = \{ A : A \subset X \}$.

Def. A sequence (x_n) in a topological space converges to limit ℓ if for every open subset $U \subset X$ with $\ell \in U$, there exists $N \in \mathbb{N}$

$\forall n > N, \ x_n \in U.$

A function f of topological space is continuous if the preimage of every open set is open.

<u>Equivalent distances</u>.

<u>Def</u>. Let X be a set. Let d, d' be two different distance on X. Then we say that d and d' are equivalent whenever the open sets of (X, d) coincide with those of (X, d'). $d \sim d'$

<u>Cor</u>. (X, d), (X, d'), $d \sim d'$.

$$x_n \xrightarrow{d} l \iff x_n \xrightarrow{d'} l$$

<u>Cor</u>. (X, d_X), (X, d_X'), $d_X \sim d_X'$, (Y, d_Y), (Y, d_Y'), $d_Y \sim d_Y'$

$f: (X, d_X) \to (Y, d_Y)$ is continuous $\iff$ $f: (X, d_X') \to (Y, d_Y')$ is continuous.

<u>Ex</u>. $(C[0,1], d_{L^1})$, $(C[0,1], d_{L^\infty})$

$$x^n \xrightarrow{d_{L^1}} 0, \qquad x^n \not\xrightarrow{d_{L^\infty}} 0.$$

$(\mathbb{R}, d_1)$, $(\mathbb{R}, d_{discr})$

Lem. d, d' on X. There exist $c, c' > 0$, s.t.

$$d(x,y) \le c\, d'(x,y)$$

and

$$d'(x,y) \le c'\, d(x,y)$$

Then $d \sim d'$.

Ex. $\mathbb{R}^n$, d_p, d_q are equivalent.

$$\underline{d_\infty(x,y) \le d_p(x,y) \le d_q(x,y) \le d_1(x,y) \le n\, d_\infty(x,y)}$$

Lem. d, d' ...

$$d(x,y) \le c\, d'(x,y)$$

The $U \subset (X,d)$ is open implies that $U \subset (X,d')$ is open.

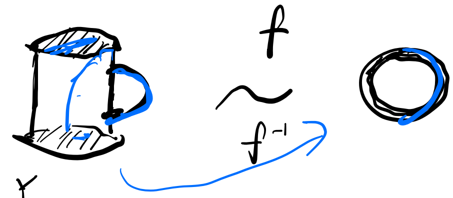
Ex. $(C[0,1], d_{L^1})$, $(C[0,1], d_{L^\infty})$

$$d_{L^1}(f,g) = \int_0^1 |f(x)-g(x)|\, dx \le \max_{x\in[0,1]} |f(x)-g(x)| \int_0^1 1\, dx$$

$$= d_{L^\infty}(f,g).$$

Exam 3.3.13 , 3.3.14

Homeomorphisms

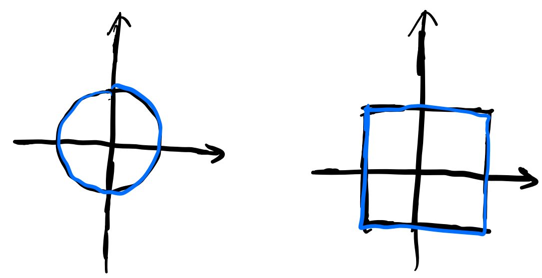


Def. Let $(X, \mathcal{U}_X)$, $(Y, \mathcal{U}_Y)$ be two topological spaces. We say that $f: X \to Y$ is a homeomorphism when

(i) f is bijective and

(ii) both f and f^{-1} are continuous.

- $(\mathbb{R}^2, d_2)$, $(\mathbb{R}^2, d_\infty)$



Ex. Let $f: (X, d_X) \to (Y, d_Y)$ is an isometry. Prove f is homeomor.

(i) f is bijective. ✓

(ii) $d_Y(f(x), f(x_0)) = d_X(x, x_0) < \varepsilon$. Taking $\delta = \varepsilon$.

$$d_Y(y, y_0) = d_X(f^{-1}(y), f^{-1}(y_0))$$

Ex. The interval $(-1,1)$ and $(\mathbb{R}, d_1)$ are homeomorphism.

$$f: (-1,1) \to \mathbb{R}$$
$$x \mapsto \tan(\tfrac{\pi}{2}x)$$

§4. Completness and compactness.

Cauchy convergence and completness.

Def. Let (X,d) be a metric space and let $\{x_n\}$ be a sequence of X. We say that it is Cauchy (convergent) if for $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, $\forall m,n > N$, $d(x_m, x_n) < \varepsilon$.

Lem. If $\{x_n\}$ is convergent, then $\{x_n\}$ is Cauchy.

$x_n \xrightarrow{d} \ell$

$d(x_n, x_m) \le d(x_n, \ell) + d(x_m, \ell) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$

Ex. 1. $(X,d) = (\mathbb{Q}, d_1)$, π $3.1415926535\cdots \notin \mathbb{Q}$.

(x_n) $x_n \in \mathbb{Q}$. $d(x_n, x_m) \le \frac{2}{10^N}$, $\forall n,m > N$.

2. $((0,1), d_1)$ $x_n = \frac{1}{n+1} \to 0 \notin (0,1)$.

Def. A metric space (X,d) is complete if every Cauchy sequence converges.

Ex. 1. $(\mathbb{R}^n, d_p)$, $p = 1, \dots, \infty$.

$d_{L^2}(f,g) = \left(\int_0^1 |f-g|^2 dx\right)^{1/2}$

2. $(C[0,1], d_{L^\infty})$ is complete. But $(C[0,1], d_{L^1})$ is not complete.

3. Every discrete metric space (X, d_{disr}) is complete.

Thm. (X, d) metric space. $A \subset X$. $d_A = d|_A$.

1. If (A, d_A) is complete, then A is closed in (X, d)

2. If (X, d) is complete, and A is closed, then (A, d_A) complete

Pf. 1. $\{x_n\} \subset A$, $x_n \xrightarrow{d} \ell$ in X since $\{x_n\}$ Cauchy

(A, d_A) complete $\Rightarrow$ $\{x_n\}$ converges in (A, d_A) $\Rightarrow$ $\ell \in A \Rightarrow A$ is closed.

2. ...

Remark. $(\mathbb{R}, d_1)$, $((-1, 1), d_1)$ homeomorphism.

complete   Not complete.