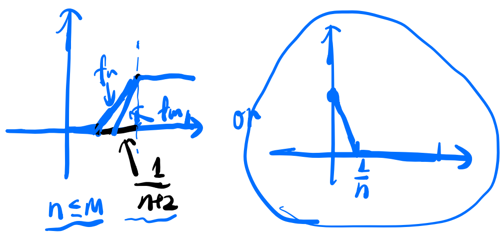
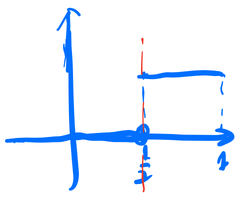
Ex. $(C[0,1], d_{L^1})$ is not complete metric space.

$$f_n(x) = \begin{cases} 0, & 0 \le x \le \frac{n}{2(2+n)} \\ (n+2)x - \frac{n}{2}, & \frac{n}{2(2+n)} \le x \le \frac{1}{2} \\ 1, & \frac{1}{2} \le x \le 1 \end{cases}$$





$$g(x) = \begin{cases} 0, & 0 \le x < \frac{1}{2} \\ 1, & \frac{1}{2} \le x \le 1 \end{cases}$$

Note that when $m > n > N$, where $N \in \mathbb{N}$ large that $\frac{1}{2\varepsilon}$, then

$$d(f_n, f_m) = \int_0^1 |f_n(x) - f_m(x)|\,dx = \int_0^1 (f_n(x) - f_m(x))\,dx$$

$$= \frac{1}{2(n+2)} - \frac{1}{2(m+2)}$$

$$< \frac{1}{2(n+2)} < \varepsilon.$$

$$d(f_n, g) = \int_0^1 |f_n(x) - g(x)|\,dx = \int_0^1 f_n(x)\,dx - \int_0^1 g(x)\,dx$$

$$= \frac{1}{2} + \frac{1}{2(n+2)} - \frac{1}{2} = \frac{1}{2(n+2)} \to 0.$$

Cauchy sequence $\{f_n\}$ doesn't converge in $(C[0,1], d_{L^1})$.

The Contraction Mapping Theorem

Def. Let (X,d) be a metric space. Then $f: X \to X$ is a contraction if there exists $0 \le L < 1$ such that

$$d(f(x), f(y)) \le L\, d(x,y), \quad \forall x, y \in X.$$

(Contraction Mapping Theorem) Suppose (X,d) is a complete metric space. If $f: (X,d) \to (X,d)$ is a contraction, then f has a unique fixed point.

- $$f: [1,+\infty) \to [1,+\infty), \qquad x \mapsto x + \frac{1}{x}.$$

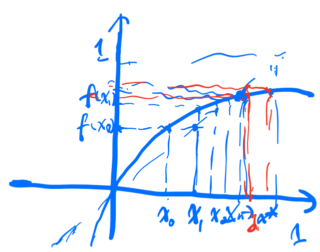
$$|f(x) - f(y)| = \left|\left(x + \frac{1}{x}\right) - \left(y + \frac{1}{y}\right)\right| = \left|\frac{x^2y + y - xy^2 - x}{xy}\right|$$

$$= |x-y| \left|\frac{xy-1}{xy}\right| < |x-y|$$

$f(x) = x \iff x + \frac{1}{x} = x \iff \frac{1}{x} = 0$, contradiction.

Pf. Let

$$x_n = f^n(x) := f(f \cdots f(x)).$$



$$(x_n) = (x, f(x), f^2(x), \dots, f^n(x), \dots)$$

$$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \le L\, d(x_n, x_{n-1})$$
$$\le L(L\, d(x_{n-1}, x_{n-2}))$$
$$= L^2 d(x_{n-1}, x_{n-2})$$
$$\vdots$$
$$= L^n d(x_1, x_0)$$

$m > n$

$$\Rightarrow \quad d(x_n, x_m) \le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{m-1}, x_m)$$
$$= L^n d(x_1, x_0) + L^{n+1} d(x_1, x_0) + \dots + L^{m-1} d(x_1, x_0)$$
$$= (L^n + L^{n+1} + \dots + L^{m-1})\, d(x_1, x_0)$$
$$= \frac{L^n(1 - L^{m-n})}{1-L} d(x_1, x_0) = \frac{L^n - L^m}{1-L} d(x_1, x_0) < \frac{L^n}{1-L} d(x_1, x_0)$$

$\forall \varepsilon > 0$, choosing $N \in \mathbb{N}$ satisfies $\frac{L^N}{1-L} d(x_1, x_0) < \varepsilon$, $\forall n > m > N$

$$d(x_n, x_m) < \varepsilon.$$

By (X, d) is complete, then (x_n) has a limit in X. $\lim_{n\to\infty} x_n = p$.

$$x_{n+1} = f(x_n)$$

$$\Rightarrow \quad p = \lim_{n\to\infty} x_{n+1} = f(\lim_{n\to\infty} x_n) = f(p).$$

Suppose that x, y are fixed points of f. Then

$$d(x,y) = d(f(x), f(y)) \le L\, d(x,y)$$

$$\Rightarrow \quad d(x,y) = 0 \quad \Rightarrow \quad x = y.$$

Ex $\phi : (C[0,1], d_{L^\infty}) \to (C[0,1], d_{L^\infty})$.

$$f \mapsto \phi(f)$$

where
$$\phi(f)(x) = x + \frac{1}{5}\left(f(x) + f\left(\frac{e^x-1}{e-1}\right)\right),$$

Prove that ϕ has a fixed point.

Pf. $\forall f, g \in C[0,1]$, then

$$d(\phi(f), \phi(g)) = \max_{0\le x\le 1} \left| \left(\cancel{x} + \frac{1}{5}\left(f(x) + f\left(\frac{e^x-1}{e-1}\right)\right)\right) - \left(\cancel{x} + \frac{1}{5}\left(g(x) + g\left(\frac{e^x-1}{e-1}\right)\right)\right)\right|$$

$$= \frac{1}{5} \max_{0\le x\le 1} \left| \underline{f(x) - g(x)} + f\left(\frac{e^x-1}{e-1}\right) - g\left(\frac{e^x-1}{e-1}\right)\right|$$

$$\le \frac{1}{5} d_{L^\infty}(f,g) + \frac{1}{5} \max_{0\le x\le 1} \underline{\left| f\left(\frac{e^x-1}{e-1}\right) - g\left(\frac{e^x-1}{e-1}\right)\right|}$$

$$= \frac{2}{5} d_{L^\infty}(f,g)$$

$\Rightarrow$ ϕ is contraction. $\underline{\text{CMT} \Rightarrow \text{fixed point.}}$

Compactness.

Def. We say that a metric space (X, d) is (sequential) compact if for every sequence in X, there exists a subsequence that converges in (X, d).

open covering: $X \subset \bigcup_{\lambda \in \Lambda} A_\lambda$, A_λ is open.

X is compact $\Leftrightarrow$ Every open covering has a finite open subcoving.

$$[0,1] \subset \bigcup_{n=1}^{\infty} \left(-\frac{1}{n}, 1+\frac{1}{n}\right)$$

$$\Rightarrow [0,1] \subset (-1, 2)$$

Thm. (Key Result). If $f: (X, d_X) \to (Y, d_Y)$ is a continuous map, and X is compact, then $(f(X), d_Y)$ is also compact.

$\forall (y_n) \subset f(Y)$. $\exists (x_n) \subset X$, s.t. $f(x_n) = y_n$.

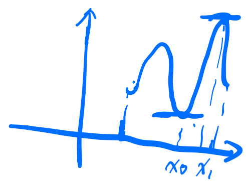
$\exists$ subsequence of (x_n), say (x_{n_k}), converges to ℓ.

$$y_{n_k} = f(x_{n_k}) \xrightarrow{d_Y} f(\ell), \quad k \to \infty.$$

$\Rightarrow (f(Y), d_Y)$ is compact.

Cor. If (X, d_X) is homeomorphic to (Y, d_Y), then

(X, d_X) is compact $\Leftrightarrow$ (Y, d_Y) is compact.



Thm. Suppose (X, d_X) is compact, and $f: X \to \mathbb{R}$ is a continuous function. Then f admits its maximum and minimum.

Lem. compact $\Rightarrow$ closed.

- closed $\not\Rightarrow$ compact. Ex $(\mathbb{R}, d_1)$ is not compact.

  $x_n = n$.

Thm. (Bolzano - Weierstrass Theorem) Let $X \subset (\mathbb{R}^n, d_p)$. Then (X, d_p) is compact $\Leftrightarrow$ X is closed and bounded.

bounded : $X \subset B_R^{d_p}(0)$, $\exists R > 0$.

Thm. A compact metric space is complete.

(X, d). $\{x_n\} \subset X$. $x_{n_k} \xrightarrow{d} l$, $k \to \infty$.

Lem. Let (X, d) be a metric space. Suppose $\{x_n\}$ is a Cauchy sequence and a subsequence $\{x_{n_k}\}$ converges to ℓ. Then $\{x_n\}$ is convergent to ℓ.

$\forall k > N$, $d(x_{n_k}, \ell) < \varepsilon/2$.

$\forall m, n > N$ $d(x_n, x_m) < \varepsilon/2$.

$\Rightarrow$ $d(x_n, \ell) \leq d(x_n, x_{n_k}) + d(x_{n_k}, \ell) < \varepsilon$.

Ex. $\bar{B}_1 = \{x : d_2(x, 0) \leq 1\}$ in $(\mathbb{R}^n, d_2)$ is compact $\Rightarrow$ complete.

Ex. (ℓ^p, d_p), $\bar{B}_1(0)$. closed and bounded.

$(\bar{B}_1, d_p)$ complete.

We show $\bar{B}_1(0)$ is not compact.

Pf. $x_1 = (1, 0, 0, \cdots)$

$x_2 = (0, 1, 0, \cdots)$

$\vdots$

$x_n = (0, \cdots, 1, \cdots)$

$x_n \in \bar{B}_1(0)$. However, $d(x_m, x_n) = 2^{\frac{1}{p}}$, $\{x_n\}$ does not have a subsequence which is convergent.

§5. Sapces of continuous functions.

$(C[0,1], d_{L^\infty})$.

Uniform convergence.

Def. (f_n) converges uniformly to f if for $\forall \varepsilon>0$, $\exists N \in \mathbb{N}$ depending only on ε, $\forall n>N$, s.t

$$d_{L^\infty}(f_n, f) = \max_{x\in[0,1]} |f_n(x) - f(x)| < \varepsilon.$$

- (f_n) converges pointwise to f if for $\forall \varepsilon>0$, $\exists \underbrace{N \in \mathbb{N}}_{N(\varepsilon, x)}$, $\forall n>N$, s.t

$$|f_n(x) - f(x)| < \varepsilon.$$

Ex. $f_n : [0,1] \to \mathbb{R}$

$\qquad x \mapsto x^n$

$\downarrow$ pointwise but uniformly.

$$f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases}$$

~~$\sup_{x\in[0,1]} |f_n(x) - f(x)| = 1, \quad \forall n \in \mathbb{N}.$~~

Thm. If (X,d) is a compact metric space, then the metric space $(C(X), d_{L^\infty})$ is complete.