<u>Thm</u>. If (X, d) is a compact metric space, then the metric space $(C(X), d_{L^\infty})$ is complete.

<u>Pf</u>. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $C(X)$, i.e. $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, $\forall m, n > N$, such that

$$d_{L^\infty}(f_n, f_m) < \varepsilon.$$

Since $d_{L^\infty}(f_n, f_m) = \sup_{x\in X} |f_n(x) - f_m(x)|$, we have for any $x \in X$,

$$|f_n(x) - f_m(x)| < \varepsilon.$$

Then $\{f_n(x)\}_{n\in\mathbb{N}}$ is also a Cauchy sequence of $(\mathbb{R}, d_{L^1})$. By completeness we know for any fixed $x \in X$, there exists a function $f: X \to \mathbb{R}$ satisfying

$$\lim_{n\to\infty} f_n(x) = f(x). \qquad \text{(pointwise)}.$$

Since $\{f_n\}$ is Cauchy, i.e. $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, $\forall m, n > N$, there is

$$|f_n(x) - f_m(x)| < \varepsilon, \quad \forall x \in X.$$

Let $m \to \infty$, there is

$$|f_n(x) - f(x)| \le \varepsilon, \quad \forall x \in X,$$

i.e.

$$\sup_{x\in X} |f_n(x) - f(x)| \le \varepsilon.$$

i.e.

$$d_{L^\infty}(f_n, f) \le \varepsilon.$$

It suffices to show f is continuous. Indeed, since $\{f_n\}$ converges uniformly to f in X. $\forall \varepsilon>0$ $\exists N \in \mathbb{N}$, $\forall n \geq N$, there is

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3}, \quad \forall x \in X.$$

Specially, choosing $n=N$, there is

$$|f_N(x) - f(x)| < \frac{\varepsilon}{3}.$$

Since $f_N(x)$ is a continuous function at any point $x_0 \in X$, then $\exists \delta>0$, $\forall x \in B_\delta(x_0)$, there

$$|f_N(x) - f_N(x_0)| < \frac{\varepsilon}{3}.$$

Therefore, there is

$$|f(x) - f(x_0)| \leq \overset{①}{|f(x) - f_N(x)|} + \overset{②}{|f_N(x) - f_N(x_0)|} + \overset{③}{|f_N(x_0) - f(x_0)|}$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Hence f is continuous at x_0, i.e. f is continuous in X. #

- $(C[0,1], d_{L^\infty})$ is complete.

$$f_n(x) = x^n, \quad x \in [0,1]$$

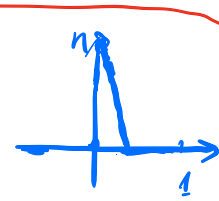
has no convergence subsequence.

Thm. Let $f_n : [a,b] \to \mathbb{R}$ define a sequence of continuous functions. Assume $\{f_n\}$ converges uniformly to $f : [a,b] \to \mathbb{R}$. Then for all $x_1, x_2 \in [a,b]$, there is

$$\lim_{n\to\infty} \int_{x_1}^{x_2} f_n(x)\,dx = \int_{x_1}^{x_2} \lim_{n\to\infty} f_n(x)\,dx.$$

Remark.

$$f_n(x) = \begin{cases} 0, & x \geq \frac{1}{n} \text{ or } x = 0 \\ n - n^2 x, & 0 < x < \frac{1}{n} \end{cases}$$



Pf. $I : (C[x_1, x_2], d_{L^\infty}) \to (\mathbb{R}, d_1)$.

$$f \mapsto \int_{x_1}^{x_2} f(x)\,dx.$$

$\forall \varepsilon > 0$, choosing $\delta = \frac{\varepsilon}{(x_2 - x_1)}$, $\forall\ d_{L^\infty}(f,g) < \delta$, there is

$$|I(f) - I(g)| = \left| \int_{x_1}^{x_2} f(x) - g(x)\,dx \right|$$

$$\leq \int_{x_1}^{x_2} |f(x) - g(x)|\,dx.$$

$$\leq d_{L^\infty}(f,g)\,(x_2 - x_2) < \varepsilon.$$

$\Rightarrow$ I is continuous. $\Rightarrow$ $\lim_{n\to\infty} I(f_n) = I(\lim_{n\to\infty} f_n)$. #

Thm. Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of continuous functions and assume $\{f_n\}$ converges uniformly to $f : [a,b] \to \mathbb{R}$, and that $\{f_n'(x)\}$ is continuous and converges uniformly to a function $g: [a,b] \to \mathbb{R}$. Then f is differentiable and $g = f'$.

$$\lim_{n\to\infty} f_n'(x) = \left(\lim_{n\to\infty} f_n(x)\right)'.$$

Power series.

$$\sum_{n=0}^{\infty} a_n x^n = \lim_{N\to\infty} \sum_{n=0}^{N} a_n x^n = \lim_{N\to\infty} S_N(x).$$

Def. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. The radius of convergence of the corresponding power series $\sum_{n=0}^{\infty} a_n x^n$ is

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}.$$

Thm. Assume that $R > 0$, then $0 < \delta < R$, $\{S_N\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $(C[-\delta,\delta], d_\infty)$.

Cor. Assume $R > 0$. The power series defines a continuous function

$$S: (-R, R) \to \mathbb{R}.$$
$$x \mapsto \lim_{N\to\infty} S_N(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Cor. $S(x) = \sum_{n=0}^{\infty} a_n x^n$, $x \in (-R, R)$. Then

(1) S is differentiable, and

$$S'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad x \in (-R, R).$$

(2) $F'(x) = S(x)$, then

$$F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} + C.$$

Ex. Compute $\sum_{n=1}^{\infty} n x^{n-1}$.

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x| < 1.$$

$$\sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=1}^{\infty} \frac{d}{dx}(x^n) \overset{?}{=} \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right)$$

$$= \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$$

Compute $\sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1}$.

$$\sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} = \sum_{n=0}^{\infty} \int_0^x t^n \, dt = \int_0^x \sum_{n=0}^{\infty} t^n \, dt$$

$$= \int_0^x \frac{1}{1-t} \, dt$$

$$= -\ln|1-x| + C$$

$$\sum_{n=1}^{\infty} n(n+1) x^{n-1} \qquad \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} x^{n+2}.$$

Differential equations.

$x = 2$. $\quad x^2 = 4$.

Ex. $y'(x) = x$ $\quad$ $y'(x) = y(x)$. $\quad$ $(y'(x))^2 = 1$.

$$\Downarrow \qquad\qquad \Downarrow$$

$$\int y'(x)\,dx = \int x\,dx \qquad y(x) = \int y(x)\,dx$$

$$\Rightarrow \quad y(x) = \frac{1}{2}x^2 + C.$$

$I : C[0,1] \to \mathbb{R}$

$I(y) = y$

Consider

$$\begin{cases} y'(x) = f(x) \\ y(x_0) = y_0. \end{cases}$$

$$\Leftrightarrow \qquad y(x) = y_0 + \int_{x_0}^{x} f(t)\,dt$$

Def. A Cauchy Problem (CP) is the data of a point $(x_0, y_0) \in \mathbb{R}^2$, of a pair $a, b \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$ and of a continuous function

$$f : [x_0 - \alpha, x_0 + \alpha] \times [y_0 - b, y_0 + b] \to \mathbb{R}.$$

A local solution to the (CP) is a differentiable function

$$y : [x_0 - a, x_0 + a] \to [y_0 - b, y_0 + b]$$

satisfying

$$\begin{cases} y'(x) = f(x, y(x)) \quad \text{in } [x_0 - \alpha, x_0 + \alpha] \\ y(x_0) = y_0. \end{cases}$$

Thm (Peano-Picard) Let (x_0, y_0, a, b, f) be the defining data of (CP). If $\frac{\partial f}{\partial y}$ exists and continuous on some open set of $\mathbb{R}^2$ contains (x_0, y_0) then (CP)

$$\begin{cases} y'(x) = f(x, y) \\ y(x_0) = y_0 \end{cases}$$

admits a unique local solution.

Pf. (CP) $\Leftrightarrow$ $y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt$, $x \in [x_0 - \alpha, x_0 + \alpha]$

Let $X = \{ y : [x_0 - \alpha, x_0 + \alpha] \to [y_0 - b, y_0 + b] \text{ is continuous} \}$, d_{L^∞}.

Define $F : (X, d_{L^\infty}) \to (X, d_{L^\infty})$.

$$y \mapsto y_0 + \int_{x_0}^{x} f(t, y(t))\, dt$$

We first show F is well-defined. It suffices to prove

$$\left| \int_{x_0}^{x} f(t, y(t))\, dt \right| \le b.$$

Since $\alpha \le \frac{b}{M}$, where $M = \max_{(x,y) \in \mathbb{R}^2} |f(x, y)|$.

$$\left| \int_{x_0}^{x} f(t, y(t))\, dt \right| \le \int_{x_0}^{x} |f(t, y(t))|\, dt \le M(x - x_0) \le M\alpha \le b.$$

To show F is contraction. $\forall\, y_1, y_2 \in X$, we have

$$d_{L^\infty}(F(y_1), F(y_2)) = \max_{x \in [x_0 - \alpha, x_0 + \alpha]} \left| \int_{x_0}^{x} f(t, y_1(t)) - f(t, y_2(t))\, dt \right|$$

$$\le \max_{x\in[x_0-\alpha,\, x_0+\alpha]} \int_{x_0}^{x} |f(t, y_1(t)) - f(t, y_2(t))|\, dt$$

$$= \max_{x\in[x_0-\alpha,\, x_0+\alpha]} \int_{x_0}^{x} \left|\frac{\partial f}{\partial y}(t,\xi)\,(y_1(t) - y_2(t))\right| dt$$

$$\le L \max_{x\in[x_0-\alpha,\, x_0+\alpha]} \int_{x_0}^{x} |y_1(t) - y_2(t)|\, dt.$$

$$\le \alpha L\ d_{L^\infty}(y_1, y_2).$$

where $L = \max_{(x,y)\in\mathbb{R}^2} \left|\frac{\partial f}{\partial y}(x,y)\right|$. It suffices to chose $\alpha < \frac{1}{L}$,

then

$$d_{L^\infty}(F(y_1), F(y_2)) \le \alpha L\ d_{L^\infty}(y_1, y_2), \quad \text{where } \alpha L < 1,$$

i.e. F is contraction.

$\boxed{\alpha < \min\{a, \frac{b}{M}, \frac{1}{L}\}}$. By CMT, there exists unique $y \in X$, s.t. $F(y) = y$, i.e.

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt. \qquad \#$$

Ex.

$$\begin{cases} y'(x) = 3y^{\frac{2}{3}}(x) \\ y(x_0) = y_0 \end{cases} \quad (CP)$$

Solu. $f(x,y) = 3y^{\frac{2}{3}}$. $\mathbb{R}\times[0,+\infty)$, $\frac{\partial f}{\partial y} = 2y^{-\frac{1}{3}}$. $\mathbb{R}\times(0,+\infty)$

① when $y_0 \neq 0$, by Peano-Picard theorem, (CP) admits a unique solution.

$$\int_{x_0}^{x} \frac{y'(t)}{3y^{\frac{2}{3}}(t)}\,dt = \int_{x_0}^{x} 1\,dt$$

$$\Leftrightarrow \quad y^{\frac{1}{3}}(x) - y^{\frac{1}{3}}(x_0) = x - x_0$$

$$\Rightarrow \quad y^{\frac{1}{3}}(x) = y^{\frac{1}{3}}(x_0) + x - x_0. \quad \Leftrightarrow \quad y(x) = (x - x_0 + y_0^{\frac{1}{3}})^3.$$

② when $y_0 = 0$. $\quad y(x) = (x-x_0)^3$. or $y(x) = 0$, $\forall x$.

<u>Exercise 5.3.13</u>.

$$X = \{ y \in C[-\tfrac{1}{12}, \tfrac{1}{12}] ; \ y(x) \in [0,2] \} \subset C[-\tfrac{1}{12}, \tfrac{1}{12}]$$

1. (X, d_{L^∞}) is complete.
2. $F : C[-\frac{1}{12}, \frac{1}{12}] \to C[-\frac{1}{12}, \frac{1}{12}]$

$$y \mapsto 1 + \int_0^x (y^2(t) + y(t) + t)\,dt$$

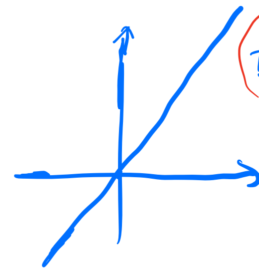
3. $F(y) \in [0,2]$, $y \in X$ $\quad (F : X \to X)$.
4. F is contraction.
5. exists a unique solution. (CMT).

Introduction to multi-variable calculus.

linear algebra.

Def. Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a function. We say f is linear if

1. $f(v+w) = f(v) + f(w)$
2. $f(\lambda v) = \lambda f(v)$. $\Rightarrow$ $(f(0)=0)$



$f(x)=2x$

Remark. $f \Leftrightarrow$ matrix.

$$f(x) = Ax$$

Ex. $f: \mathbb{R}^3 \to \mathbb{R}^2$, $f(x_1, x_2, x_3) = (2x_1 - x_2, x_3)$.

Def. $\det: M_n(\mathbb{R}) \to \mathbb{R}$.

$A \mapsto \det(A)$.

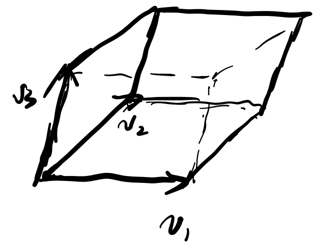
- $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 3 & 2 \end{pmatrix}$ $\det A = \dots$

  $Id = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $\det Id = 1$

- $\det(A+B) \neq \det(A) + \det(B)$.

  $\det(AB) = \det A \det B$.

$V = \det A$ $\quad$ $A = (v_1, v_2, v_3)$



Lem. Let $L: \mathbb{R}^n \to \mathbb{R}^m$ be a linear function. Then L is continuous.

Pf. Let A be the matrix corresponding to L, the i-th component of $L(x_1, \dots, x_n)$ is $\sum_{j=1}^{n} a_{ij} x_j$. Define

$$M = \max_{1 \le i \le m} \left| \sum_{j=1}^{n} a_{ij} \right|$$

$\forall \varepsilon > 0$, choosing $\delta = \frac{\varepsilon}{M}$, $\max_{1 \le i \le n} |y_i - x_i| < \delta$, there is

$$\max_{1 \le i \le m} \left| \sum_{j=1}^{n} a_{ij} (x_j - y_j) \right| \le M \max_{1 \le i \le m} |x_i - y_i|$$

$$< M\delta$$

$$= \varepsilon .$$