Differentials.

$$f: \mathbb{R}^n \to \mathbb{R}^m.$$

$$f = (f_1, \dots, f_m) \iff f_1, \dots, f_m.$$

Ex.
$$f(x_1, x_2) = \begin{cases} \dfrac{x_1 x_2}{x_1^2 + x_2^2}, & (x_1, x_2) \neq (0,0) \\ 0, & (x_1, x_2) = (0,0) \end{cases}$$

$f(x_1, 0) = 0$, $f(0, x_2) = 0$.

$\lim_{x_1 \to 0} f(x_1, kx_1) = \underline{\dfrac{k}{1+k^2}}$ discontinuous at $(x_1, x_2) = (0,0)$,

Def.

$\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ exists.

- Directional derivatives.

Def. Let $x = (x_1, \dots, x_n) \in U \subset \mathbb{R}^n$, open set. Let $f: U \to \mathbb{R}$ be a function. We define the first order partial derivative of f at x respect to x_i as the limit

$$\frac{\partial f}{\partial x_i}(x) = \lim_{h \to 0} \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{h}.$$

Ex. $f: \mathbb{R}^2 \to \mathbb{R}^3$

$$(x_1, x_2) \mapsto (x_1^2,\ x_1 e^{x_2},\ x_1 x_2).$$

$$\frac{\partial f}{\partial x_1} = (2x_1,\ e^{x_2},\ x_2), \quad \frac{\partial f}{\partial x_2} = (0,\ x_1 e^{x_2},\ x_1).$$

$$\Rightarrow \quad \left(\frac{\partial f}{\partial x_1}\ \frac{\partial f}{\partial x_2} \right) = \begin{pmatrix} 2x_1 & 0 \\ e^{x_2} & x_1 e^{x_2} \\ x_2 & x_1 \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} \end{pmatrix}$$

Jacobi matrix

Def. $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$, assume that the first order partial derivative of f at x exists. Then we define the Jacobi matrix as

$$Jf(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \qquad m \times n$$

Def. Let $x \in U \subset \mathbb{R}^n$, open set. $f: U \to \mathbb{R}^m$ be a function. Let $v \in \mathbb{R}^n$ be a vector of length 1. ($\|v\| = d_2(v, 0) = 1$).

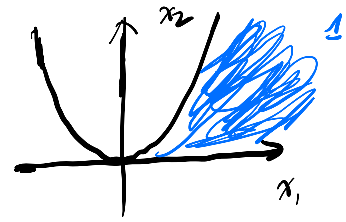
We define the directional derivative of f at x respect to v as

$$D_v f(x) = \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t}.$$

- $v = e_i = (0, \dots, 0, 1, 0, \dots, 0)$. $\quad D_{e_i} f = \frac{\partial f}{\partial x_i}$

Ex. $f(x_1, x_2) = \begin{cases} 1, & x_1, x_2 > 0,\ x_2 < x_1^2 \\ 0, & \text{other}. \end{cases}$

$D_v f(0) = 0.$ discontinuous at $x = 0$.



Differetiability.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \ell.$$

Linear function.

$$\Leftrightarrow \quad f(x+h) - f(x) = \ell h + o(h)$$

Def. Let $U \subset \mathbb{R}^n$ open set. $x_0 \in U$. $f: U \to \mathbb{R}^m$ be a function. We say f is differentiable at x_0 if there exists a linear function $L: \mathbb{R}^n \to \mathbb{R}^m$ such that

$Jf(x_0)h.$

$$\lim_{h \to 0} \frac{\| f(x_0+h) - f(x_0) - L(h) \|}{\|h\|} = 0.$$

L is called the differential of f at x_0, $L = Df_{x_0}$.

Prop. $f: U \to \mathbb{R}^m$. $\|v\| = 1$. $x_0 \in U$.

$$Df_{x_0}(v) = D_v f(x_0).$$

$$\lim_{t \to 0} \frac{\| f(x_0+tv) - f(x) - Df_{x_0}(tv) \|}{t} = 0.$$

$\Updownarrow$

$$\lim_{t\to 0} \frac{\| f(x_0+tv) - f(x_0) - t\, Df_{x_0}(v) \|}{t} = 0. \quad \Rightarrow \quad \lim_{t\to 0} \frac{f(x_0+tv) - f(x_0)}{t} = Df_{x_0}(v)$$

$\Downarrow$

$$\lim_{t\to 0} \frac{f(x_0+tv) - f(x_0)}{t} = D_v f(x_0)$$

$\Leftrightarrow \quad Df_{x_0}(v) = D_v f(x_0).$

Cor.

$$Df_{x_0}(h) = Jf(x_0) \cdot h.$$

Prop. Assume f is differentiable at x_0, then f is continuous at x_0.

$$f(x_0+h) - f(x_0) = Df_{x_0}(h) + o(\|h\|) \to 0, \quad h \to 0.$$

Ex.

$$f(x_1, x_2) = \begin{cases} \frac{x_1 x_2^2}{x_1^2 + x_2^2}, & (x_1, x_2) \neq 0 \\ 0, & (x_1, x_2) = 0. \end{cases} \quad \text{at } (0,0).$$

$$\frac{\partial f}{\partial x_1}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \frac{d}{dx_1}(f(x_1, 0)) = 0.$$

$$\frac{\partial f}{\partial x_1}(0,0) = 0. \qquad Jf(0,0) = (0, 0).$$

$$\lim_{h\to 0} \frac{\| f(h) - f(0,0) - Jf(0,0)\cdot h \|}{\|h\|}$$

$$= \lim_{h\to 0} \frac{\frac{h_1 h_2^2}{h_1^2+h_2^2}}{\sqrt{h_1^2+h_2^2}} = \underline{\lim_{h\to 0} \frac{h_1 h_2^2}{(h_1^2+h_2^2)^{\frac{3}{2}}}}$$ doesn't exist.

$h_1 = kh_2$

$$\lim_{h_2\to 0} \frac{k h_2^3}{(1+k^2) h_2^3} = \frac{k}{1+k^2}$$

Ex.

$$f(x_1,x_2) = \begin{cases} \frac{x_1^2 x_2^2}{x_1^2+x_2^2}, & (x_1,x_2)\neq 0 \\ 0, & (x_1,x_2)=0. \end{cases}$$

$$\frac{h_1^2 h_2^2}{(\sqrt{h_1^2+h_2^2})^3} \qquad 2|h_1 h_2| \leq h_1^2+h_2^2.$$

$$\leq \frac{(h_1^2+h_2^2)^2}{(h_1^2+h_2^2)^{\frac{3}{2}}} \sim (h_1^2+h_2^2)^{1/2} \to 0.$$

Thm. Let $f: U \to \mathbb{R}^m$ be a function defined on an open set. $x_0 \in U$. If f has all partial derivatives on U, and they are all continuous at x_0, then f is differentiable at x_0.

Ex.

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x\neq 0 \\ 0, & x=0. \end{cases}$$ at 0.

Def. We say that f is continuously differentiable at x_0 if its partial derivatives are continuous at x_0. $(f \in C^1)$

Properties of differentials and differentiable functions.

Def. (Operator norm) Let A be matrix in $M_n(\mathbb{R})$. The operator norm of A is defined as

$$\|A\|_{op} = \max_{w \neq 0} \frac{\|Aw\|}{\|w\|} = \max_{\|w\|=1} \|Aw\|.$$

- $$d_{op}(A,B) = \|A-B\|_{op}.$$

Thm. $f \in C^1$ at $x_0 \iff Jf : U \to M_n(\mathbb{R})$ is continuous at x_0.

d_{op}

$(\max |a_{ij}| \le \|A\|_{op} \le \sqrt{mn} \max |a_{ij}|)$

- $$\|Jf(x) - Jf(x_0)\|_{op} \le \sqrt{mn} \max_{i,j} \left| \frac{\partial f_i}{\partial x_j}(x) - \frac{\partial f_i}{\partial x_j}(x_0) \right|$$

- $$\left| \frac{\partial f_i}{\partial x_j}(x) - \frac{\partial f_i}{\partial x_j}(x_0) \right| \le \max_{i,j} \left| \frac{\partial f_i}{\partial x_j}(x) - \frac{\partial f_i}{\partial x_j}(x_0) \right|$$

$$\le \|Jf(x) - Jf(x_0)\|_{op}.$$

• <u>Chain Rule</u>.

Thm. $f: U \to V$, $g: V \to \mathbb{R}^k$, $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$. f is differentiable at x_0, g is differentiable at $f(x_0)$. Then $g \circ f$ is differentiable at x_0, and

$$D(g \circ f)_{x_0} = Dg_{f(x_0)} \circ Df_{x_0}.$$

• $$J(g \circ f)(x_0) = Jg(f(x_0)) \cdot Jf(x_0)$$

$$f(x_0+h) - f(x_0) = Df_{x_0}(h) + o(\|h\|)$$

$$g(f(x_0)+t) - g(f(x_0)) = Dg_{f(x_0)}(t) + o(\|t\|)$$

$$g \circ f(x_0+h) - g \circ f(x_0)$$

$$= g(f(x_0) + Df_{x_0}(h) + o(\|h\|)) - g \circ f(x_0)$$

$$= Dg_{f(x_0)}(Df_{x_0}(h) + o(\|h\|)) + o(\|t\|)$$

$$= Dg_{f(x_0)} \circ Df_{x_0}(h) + o(\|h\|)$$

Thm (Leibniz or product rule). f, g are differentiable at x_0. Then fg is differentiable, and

$$D(fg)_{x_0} = Df_{x_0}\, g + f\, Dg_{x_0}.$$

Ex. $h(x_1, x_2) = (e^{x_1 \sin x_2}, \ln(x_1 x_2))$, $U = \{(x_1, x_2) : x_1 x_2 > 0\}$.

$Jh(x_1, x_2)$

$$Jh(x_1, x_2) = \begin{pmatrix} \sin x_2 \, e^{x_1 \sin x_2} & x_1 \cos x_2 \, e^{x_1 \sin x_2} \\ \frac{1}{x_1} & \frac{1}{x_2} \end{pmatrix}$$

$f(x_1, x_2) = (x_1 \sin x_2, x_1 x_2)$, $\quad g(y_1, y_2) = (e^{y_1}, \ln y_2)$

$\Rightarrow \quad h(x_1, x_2) = g \circ f(x_1, x_2)$

$$Jf(x_1, x_2) = \begin{pmatrix} \sin x_2 & x_1 \cos x_2 \\ x_2 & x_1 \end{pmatrix}$$

$$Jg(y_1, y_2) = \begin{pmatrix} e^{y_1} & 0 \\ 0 & \frac{1}{y_2} \end{pmatrix}$$

$Jh(x_1, x_2) = Jg(f(x_1, x_2)) \cdot Jf(x_1, x_2) \cdots$

The Mean Value Theorem.

Def. Let u and v be points in $\mathbb{R}^n$. We define the closed segment from u to v as

$$[u, v] = \{(1-t)u + tv : t \in [0, 1]\}.$$

Open segment.

$$(u, v) = \{(1-t)u + tv : t \in (0, 1)\}.$$

Cor. (MVT) $U \subset \mathbb{R}^n$ open set. $f: U \to \mathbb{R}$ be a differentiable fun.

Let $u, v \in U$ be such that $[u,v] \subset U$. Then there exists a constant $c \in (u,v)$ such that

$$f(u) - f(v) = Jf(c) \cdot (u-v).$$

Pf.

$$\phi : [0,1] \to \mathbb{R}$$
$$t \mapsto f((1-t)u + tv)$$

$$\underline{\phi'(t) = Jf(c) \cdot (v-u)}.$$

$$f(u) - f(v) = \phi(0) - f(1) = \phi'(\xi)(-1)$$
$$= Jf(c)(u-v).$$

Ex.

$$f: [0, 2\pi] \to \mathbb{R}^2$$
$$x \mapsto \begin{pmatrix} \cos x \\ \sin x \end{pmatrix}$$

$\sin^2 x + \cos^2 x = 1$

$$Jf(x) = \begin{pmatrix} -\sin x \\ \cos x \end{pmatrix} \qquad \forall c, \; Jf(c) \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\underline{0 = f(2\pi) - f(0) = \underline{Jf(c)\, 2\pi} \neq 0}$$

$$\underline{0 = \|f(2\pi) - f(0)\| \leq \|Jf(c)\|_{op} \|2\pi - 0\|}.$$

Thm. Let $U \subset \mathbb{R}^n$ open set, $f: U \to \mathbb{R}^m$ be differentiable function. Given $u, v \in U$ such that $[u,v] \subset U$. Then

$$\|f(u)-f(v)\| \le \sup_{c\in[u,v]} \|Jf(c)\|_{op}\, \|u-v\|.$$

Pf.

$$\|f(u)-f(v)\| = \sqrt{(f_1(u)-f_1(v))^2 + \cdots + (f_n(u)-f_n(v))^2}$$

$$\phi(1)-\phi(0) = f(u)-f(v)$$

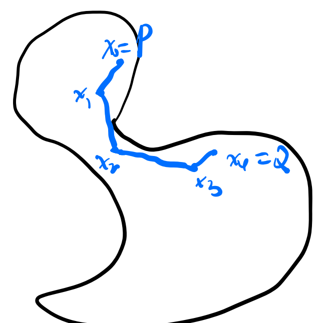
$$\phi(t) = \|f((1-t)u+tv) - f(u)\|.$$

$\boxed{\text{Assume that } u=0,\ f(u)=0.\quad \phi(t) = \|f(tv)\|}$

Def. Let U be a subset of $\mathbb{R}^n$. We say that U is path-connected if for every two points P and Q in U there exist finite sequence

$$P = x_0, x_1, \ldots, x_k = Q,$$

of points of U such that segment $[x_i, x_{i+1}]$ is contained in U.



Cor. $Df_x = 0$, $\forall x \in U$. U path-connected. Then $f = \text{const.}$ in U

$\forall P, Q \in U$, $\quad P = x_0, x_1, \ldots, x_k = Q.$

$$\|f(P)-f(Q)\| \le \|f(P)-f(x_1)\| + \cdots + \|f(x_{k-1})-f(Q)\|.$$

$$\le 0\,\|P-x_1\| + \cdots + 0\cdot\|x_{k-1}-Q\| = 0. \quad \#.$$

<u>Def</u>. (Convex set). We say that a set is convex if for every two points P and Q, the segment $[P, Q]$ is contained in the set.

<u>Cor</u>. $Df_x = 0$ in a convex set $U \Rightarrow f = \text{const.}$ in U.

- <u>domain</u>: <u>connected, open set</u>.