§1. The inverse function theorem and implicit function theorem.

$f' > 0$

1d. f. $\uparrow$, C^1. f^{-1} exist. $f^{-1}\uparrow$ $f^{-1} \in C^1$.

The inverse function theorem

Thm. Let $U \subset \mathbb{R}^n$ be an open set, and let $f: U \to \mathbb{R}^n$ be a continuously differentiable function. Let $x_0 \in U$ and $Df(x_0): \mathbb{R}^n \to \mathbb{R}^n$ is invertible. Then there exist open set $V \subset U$ with $x_0 \in V$ and $W \subset \mathbb{R}^n$ with $f(x_0) \in W$ such that $f|_V : V \to W$ is invertible and its inverse is continuously differentiable on W.

$$f: V \to W, \quad g: W \to V$$

Remark. $g \circ f|_V = id_V : x \mapsto x$.

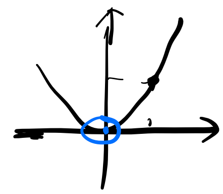
$$Dg(f(x_0)) \cdot Df(x_0) = Id$$

$$\Rightarrow \quad Dg(f(x_0)) = [Df(x_0)]^{-1}$$

Cor. $U \subset \mathbb{R}^n$ open. $f: U \to \mathbb{R}^n$, C^1 at x_0, and $Df(x_0)$ is not invertible, then if a local inverse of f exists, that local inverse is not differentiable at $f(x_0)$.

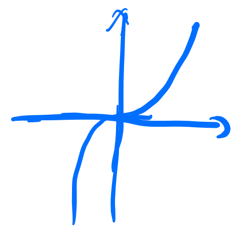
Ex. $f(x) = x^2$. $x_0 = 1$, $x_0 = 0$

- $f'(x) = 2x$ $\quad f'(1) = 2 > 0$. $\quad x = \sqrt{y}$. $\quad x' = \frac{1}{2\sqrt{y}}$ $\quad x'(1) = \frac{1}{2}$



- $f'(x) = 2x$. $\quad f'(0) = 0$.

Ex. $f(x) = x^3$, $x_0 = 0$.



$f'(x) = 3x^2$, $\quad f'(0) = 0$.

$x = \sqrt[3]{y}$ $\quad g(y) = y^{\frac{1}{3}}$. $\quad g'(y) = \frac{1}{3} y^{-\frac{2}{3}}$

Def. Let $f: X \to Y$ be a function. We say that f is local homeomorphism if for $\forall x_0 \in X$, there exist $U \subset X$ with $x_0 \in U$, $V \subset Y$ with $f(x_0) \in V$ such that $f: U \to V$ is homeomorphism.

diffeomorphism.

Cor. Let $f: U \to \mathbb{R}^n$ be continuously differentiable at every point of $U \subset \mathbb{R}^n$ open, and assume that $Df(x_0)$ is invertible for $\forall x_0 \in U$. Then f is a local diffeomorphism.

Exercise. Show that f is diffeomorphism if and only if f is a local diffeomorphism and f is bijective.

Ex. $f: \mathbb{R}^2 \to \mathbb{R}^2$ $\qquad$ $p = (1,1)$.

$(x_1, x_2) \mapsto (x_1^2 - 2x_2, 2x_1^3)$. $\qquad$ $f(1,1) = (-1, 2)$

$$Df(x) = Jf(x) = \begin{pmatrix} 2x_1 & -2 \\ 6x_1^2 & 0 \end{pmatrix}$$

$$Jf(1,1) = \begin{pmatrix} 2 & -2 \\ 6 & 0 \end{pmatrix} \qquad |Jf(1,1)| = \begin{vmatrix} 2 & -2 \\ 6 & 0 \end{vmatrix} = 12 > 0$$

$\Rightarrow$ $Jf(1,1)$ is invertible $\Rightarrow$ $Df(1,1)$ is invertible.

INFT
$\Rightarrow$ f is invertible at $(1,1)$, say $g \circ f = id_U$.

$$Dg(-1,2) = (Df(1,1))^{-1} = \begin{pmatrix} 0 & \frac{1}{6} \\ -\frac{1}{2} & \frac{1}{6} \end{pmatrix}$$

The implicit function theorem.

$$x + y = 0 \iff y = -x$$

Def $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be linear function. Consider

$$L = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m : f(x,y) = 0\}$$

We say that equation $f(x,y) = 0$ implicitly defines y as a function of x if there exists a function $\varphi: \mathbb{R}^n \to \mathbb{R}^m$ such that

$$L = \{(x, \varphi(x)) \in \mathbb{R}^n \times \mathbb{R}^m : x \in \mathbb{R}^n\}.$$

$f(x,y)=0 \iff y=\phi(x)$, s.t. $f(x,\phi(x))=0$.

- $f(x,y)=A_x x+A_y y=0 \iff A_y y=-A_x x$.

A_y is invertible $\iff$ $y=-A_y^{-1}A_x x$.

$f(x,y)=A_x x+A_y y$.

$\boxed{J_y f=A_y}$

<u>Prop</u>. $A=(A_x, A_y)$, A_y is invertible. Then $f(x,y)=0$ implicitly defines y as a function of x.

<u>Def</u>. Let $U\subset\mathbb{R}^n\times\mathbb{R}^m$ open set, and let $(x_0,y_0)\in U$. Let $f:U\to\mathbb{R}^m$ be a function such that $f(x_0,y_0)=0$. Then we say that the equation $f(x,y)=0$ implicitly defines y as a function of x, locally at (x_0,y_0) if there exist $V\subset\mathbb{R}^n$ with $x_0\in V$ and $W\subset\mathbb{R}^m$ with $y_0\in W$ $\phi: V\to W$ such that

$$\{(x,y)\in V\times W:\ f(x,y)=0\}=\{(x,\phi(x))\in V\times W:\ x\in V\}.$$

- $f(x,y)=0 \iff \exists\ \phi: V\to W$, $f(x,\phi(x))=0$.

<u>Thm</u> (IMFT) Let $U\subset\mathbb{R}^n\times\mathbb{R}^m$ be an open set, and let $(x_0,y_0)\in U$. Let $f: U\to\mathbb{R}^m$ be such that $f(x_0,y_0)=0$. Assume that $f\in C^1(U)$.

Writting Jacobi matrix as

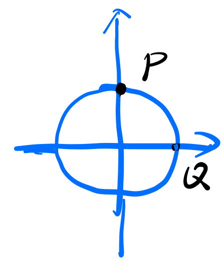
$$Jf(x_0,y_0) = (J_x f(x_0,y_0),\ J_y f(x_0,y_0)).$$

If $J_y f(x_0,y_0)$ is invertible, then equation $f(x,y)=0$ implicitly defines y as a function of x.

<u>Ex</u>. $f: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ $\qquad P=(0,1)$. $\quad Q=(1,0)$

$\qquad (x_1, x_2) \mapsto x_1^2+x_2^2-1$

$$f(x_1,x_2)=0 \iff x_1^2+x_2^2-1=0 \implies$$



- $J_{x_2} f(0,1) = 2x_2 \big|_{(0,1)} = 2 > 0. \implies \exists\ \phi: V \to W$, s.t. $x_2 = \phi(x_1)$

  $x_2 = \sqrt{1-x_1^2}$

- $J_{x_2} f(1,0) = 2x_2 \big|_{(1,0)} = 0.$ $\qquad J_{x_1} f(1,0) = 2x_1 \big|_{(1,0)} = 2 > 0$

- <u>Chain Rule</u>. $\quad f: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$

$$f(x, \phi(x)) = 0.$$

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\phi'(x) = 0 \implies \phi'(x) = -\left(\frac{\partial f}{\partial y}\right)^{-1}\frac{\partial f}{\partial x}.$$

In general, $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$.

$$f(x, \phi(x)) = 0$$

$$\frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \frac{\partial f}{\partial y_j}\frac{\partial \phi_j}{\partial x_i} = 0 \implies \frac{\partial \phi}{\partial x_i} = -(J_y f)^{-1}\frac{\partial f}{\partial x_i}$$

$$\Rightarrow \quad J\phi = -(J_y f)^{-1} J_x f$$

Ex. $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

$(x, y) \mapsto x^3y^2 + xy + 3x - 5$

Find the slope of the tangent line of the curve $\{f(x,y)=0\}$ at $(1,1)$.

$$J_x f(1,1) = 3x^2y + y + 3 \Big|_{(1,1)} = 7 > 0.$$

$$J_y f(1,1) = 2x^3y + x \Big|_{(1,1)} = 3 > 0.$$

$\xrightarrow{\text{IMFT}}$ $\quad y = \phi(x)$. $\qquad \phi'(1) = -3^{-1} \cdot 7 = -\frac{7}{3}$.

$$x^3y^2 + xy + 3x - 5 = 0$$

$$3x^2y^2 + 2x^3y \cdot y' + y + xy' + 3 = 0.$$

- $3 + 2y' + 1 + y' + 3 = 0 \Rightarrow 3y' + 7 = 0 \Rightarrow y' = -\frac{7}{3}$.

Proof of IMFT and INFT.

IMFT $\Rightarrow$ INFT.

$F : U \to \mathbb{R}^n$ $\qquad$ $JF(x)$ is invertible.

Construct

$f: U \times \mathbb{R}^n \to \mathbb{R}^n$

$(x, y) \mapsto F(x) - y$. $\qquad f(x,y) = F(x) - y$.

$x = G(y)$. $\quad 0 = f(G(y), y) = F(G(y)) - y \Leftrightarrow F(G(y)) = y$.

$J_x f(x_0, y_0) = JF(x_0)$ is invertible. #.

INFT $\Rightarrow$ IMFT.

$f: U \to \mathbb{R}^m \qquad U \subset \mathbb{R}^n \times \mathbb{R}^m$ open.

$f(x_0, y_0) = 0$. $\quad \phi: V \to W \quad f(x, \phi(x)) = 0$. $\quad J_y f(x_0, y_0)$ is invertible

Construct

$F: U \to \mathbb{R}^n \times \mathbb{R}^m$

$(x, y) \mapsto (x, f(x,y))$

$F(x_0, y_0) = (x_0, f(x_0, y_0)) = (x_0, 0)$

$$JF(x_0, y_0) = \left. \begin{pmatrix} Id & 0 \\ J_x f & J_y f \end{pmatrix} \right|_{(x_0, y_0)} \text{ is invertible.}$$

$\exists G$, $\quad F \circ G = id \qquad G = (x, G_2(x,y))$

$F \circ G = (G_1, f(x, G_2(x,y))) = (x, y)$

Let $\phi(x) = G_2(x, 0)$.

$\Leftrightarrow \quad y = f(x, G_2(x,y))$

$\overset{y=0}{\Rightarrow} \quad 0 = f(x, \phi(x))$. #

Proof of IMFT

$\mathrm{cont}(V_\alpha, W_\beta) = \{ g : V_\alpha \to W_\beta \text{ is continuous} \}$.

$$\Omega(\psi) : V_\alpha \to W_\beta$$
$$x \mapsto \psi(x) - J_y f(x_0,y_0)^{-1} f(x, \psi(x))$$

- $\| \Omega(\psi)(x) - y_0 \| \le \beta$.

$\Leftrightarrow \quad \| \psi(x) - J_y f(x_0,y_0)^{-1} f(x, \psi(x)) - y_0 \| \le \beta$.

Note that

$G(y) = y - J \cdots f(x,y)$

$$\| (\psi(x) - J_y f(x_0,y_0)^{-1} f(x, \psi(x))) - (y_0 - J_y f(x_0,y_0)^{-1} f(x,y_0)) + J_y f(x_0,y_0)^{-1} f(x,y_0) \|$$

$$\le \underbrace{\| Id - J_y f(x_0,y_0)^{-1} J_y f(x,y) \|_{op}}_{1/2\sqrt{m}} \underbrace{\| \psi(x) - y_0 \|}_{\beta} + \underbrace{\| J_y f(x_0,y_0)^{-1} \| \, \| f(x,y_0) \|}_{\le \frac{\beta}{2}}$$

$\le \beta$.

$\| \Omega(\psi_1)(x) - \Omega(\psi_2)(x) \|$

$G(y) := y - \cdots f(x,y)$

$= \| (\psi_1(x) - J_y f(x_0,y_0)^{-1} f(x, \psi_1(x))) - (\psi_2(x) - J_y f(x_0,y_0)^{-1} f(x, \psi_2(x)) \|$

$\leq \| Id - J_y f(x_0,y_0)^{-1} J_y f(x,y) \|_{op} \| \psi_1(x) - \psi_2(x) \|$.

$\leq \frac{1}{2\sqrt{m}} \| \psi_1(x) - \psi_2(x) \|$

$\exists \phi \in \mathrm{cont}(U_\alpha, W_\beta)$, $\Omega(\phi) = \phi$.

$\Leftrightarrow \quad \cancel{\phi}(x) = \cancel{\phi}(x) - J_y f(x_0,y_0)^{-1} f(x, \phi(x))$

$\Leftrightarrow \quad f(x, \phi(x)) = 0$.

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<u>Lagrange multipliers</u>

$f(x,y) : U \to \mathbb{R}$ $\qquad x+y=0$

$F(x,y,\lambda) = f(x,y) - \lambda(x+y)$