高数 B

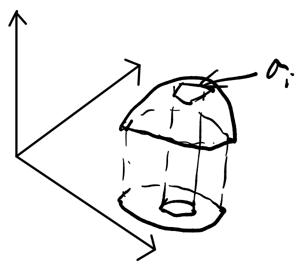
lingwang@stu.pku.edu.cn.

3.26

重积分定义

- 二重积分

$$\iint_D f(x,y)\,d\sigma = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(x_i,y_i)\,\sigma_i$$



$d\sigma = dxdy = dx\wedge dy$.

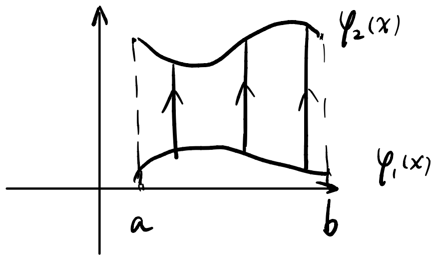
- 三重积分

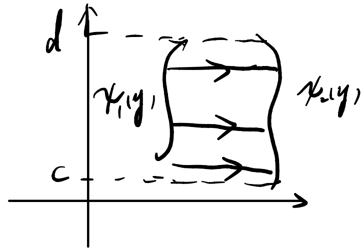
$$\iiint_\Omega f(x,y,z)\,dxdydz = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(x_i,y_i,z_i)\,\Delta V_i$$

$dV = dxdydz = dx\wedge dy\wedge dz$

重积分的计算

- 二重积分



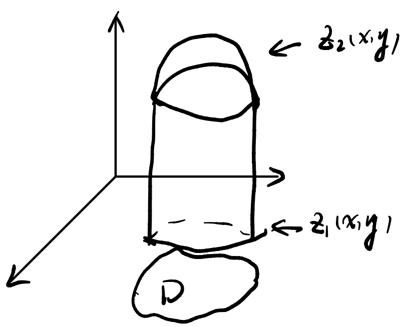


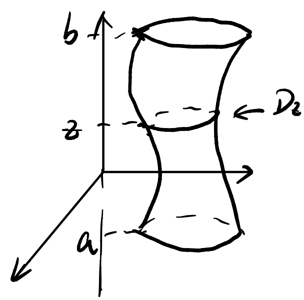
重点是化为累次积分.

$$\iint_D f(x,y)\,dxdy = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy \right) dx$$

$$= \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx \right) dy.$$

- <u>三重积分</u>





$$\iiint_\Omega f(x,y,z)\,dV = \iint_D \left(\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz \right) dxdy$$

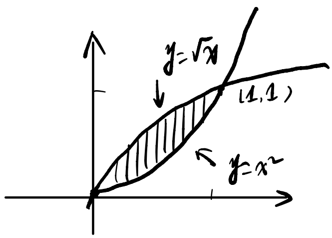
$$= \int_a^b \left(\iint_{D_z} f(x,y,z)\,dxdy \right) dz.$$

- 一般步骤

1. 画出积分区域示意图
2. 选择合适的积分顺序
3. 写出相应的累次积分
4. 计算累次积分

例 1. 求 $I=\iint_D (x^2+2y)\,dx\,dy$，其中 D 是 $y=x^2$ 与 $y=\sqrt{x}$ 所围成的区域.

解. 1. 画出区域示意图.



2, 3. 选择积分顺序写出累次积分.

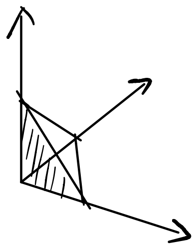
$$I=\int_0^1\left(\int_{x^2}^{\sqrt{x}}(x^2+2y)\,dy\right)dx$$

4. 计算累次积分.

$$I=\int_0^1 (x^2y+y^2)\Big|_{x^2}^{\sqrt{x}}\,dx=\int_0^1 x^{\frac{5}{2}}+x-2x^4\,dx=\frac{27}{70}.$$

例2 求 $I=\iiint_{\Omega}(1-y)e^{-(1-y-z)^2}dV$，其中 Ω 是平面 $x+y+z=1$ 和三个坐标平面相交在第一卦限形成的四面体.

解. 1. 画出区域示意图.



2. 3. 选择积分顺序写出累次积分.

$$I=\iint_{D(y,z)}\left(\int_0^{1-y-z}(1-y)e^{-(1-y-z)^2}dx\right)dy\,dz$$

4. 计算累次积分.

$$I=\iint_{D(y,z)}(1-y-z)(1-y)e^{-(1-y-z)^2}dy\,dz$$

$$=\int_0^1\left(\int_0^{1-y}(1-y-z)(1-y)e^{-(1-y-z)^2}dz\right)dy$$

$$=\int_0^1(1-y)\left(\frac{1}{2}e^{-(1-y-z)^2}\right)\Big|_0^{1-y}dy$$

$$=\frac{1}{2}\int_0^1(1-y)\left(1-e^{-(1-y)^2}\right)dy=\frac{1}{4e}.$$

重积分换元

- 二重积分

$$\begin{cases} x = x(\xi,\eta) \\ y = y(\xi,\eta) \end{cases} : D' \to D.$$

$$\iint_D f(x,y)\,dxdy = \iint_{D'} f(x(\xi,\eta), y(\xi,\eta)) \left| \frac{D(x,y)}{D(\xi,\eta)} \right| d\xi d\eta.$$

$$\begin{aligned} dx &= \frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \eta} d\eta \\ dy &= \frac{\partial y}{\partial \xi} d\xi + \frac{\partial y}{\partial \eta} d\eta \end{aligned} \Rightarrow dxdy = \left(\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}\right) d\xi d\eta$$

$$= \left| \frac{D(x,y)}{D(\xi,\eta)} \right| d\xi d\eta.$$

- 三重积分

$$\begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{cases} : \Omega' \to \Omega$$

$$\iiint_\Omega f(x,y,z)\,dV = \iiint_{\Omega'} f(x(u,v,w), y(u,v,w), z(u,v,w)) \,|J|\, dudvdw$$

$$J = \frac{D(x,y,z)}{D(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

- 常见换元方式

极坐标换元

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}, \quad r \geq 0, \quad 0 \leq \theta \leq 2\pi.$$

$$\iint_D f(x,y)\,dx\,dy = \iint_{D'} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta.$$

柱坐标换元

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}, \quad r \geq 0, \quad 0 \leq \theta \leq 2\pi, \quad z \in \mathbb{R}.$$

$$\iiint_\Omega f(x,y,z)\,dV = \iiint_{\Omega'} f(r\cos\theta, r\sin\theta, z)\, r\,dr\,d\theta\,dz.$$

球坐标换元

$$\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases} \quad \rho \geq 0, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \varphi \leq \pi.$$

$$\iiint_\Omega f(x,y,z)\,dV = \iiint_{\Omega'} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,d\rho\,d\theta\,d\varphi.$$

例3. 求 $\iiint_D y^2\,dx\,dy\,dz$，其中 D 是球体 $x^2+y^2+z^2 \le 2z$.

提示.

$$\begin{cases} x = \rho\sin\varphi\cos\theta & 0\le\rho\le 1 \\ y = \rho\sin\varphi\sin\theta & 0\le\theta\le 2\pi \\ z = 1+\cos\varphi & 0\le\varphi\le\pi \end{cases}$$

$$I = \iiint_{\Omega'} \rho^4 \sin^3\varphi \sin^2\theta \, d\rho\, d\varphi\, d\theta = \frac{4}{15}\pi.$$

练习题

1. 求 $I = \iint_D (x+y)\,dx\,dy$，其中 D 是 $y^2=2x$，$x+y=4$ 和 $x+y=12$ 所围成的区域.

2. 求 $I = \iiint_\Omega (y^2+z^2)\,dV$，其中 Ω 代表区域 $0\le z\le x^2+y^2\le 1$.

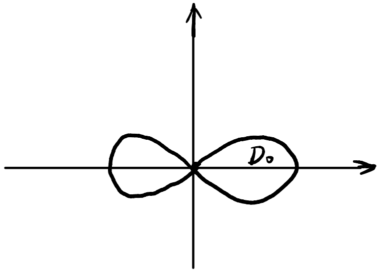
(难) 3. 求 $I = \iiint_\Omega (x+y+z)^2\,dV$，其中 Ω 是 $x^2+y^2\le 2z$ 和 $x^2+y^2+z^2\le 3$ 相交部分.（提示：拆分被积函数并利用对称性）.

重积分在几何中的应用

• 面积和体积的计算

例4. 在平面上计算曲线 $(x^2+y^2)^2 = x^2-y^2$ 所围的面积.

解.



$r^2 = \cos 2\theta \geq 0$

$\Rightarrow \theta \in [0, \frac{\pi}{4}] \cup [\frac{3\pi}{4}, \frac{5\pi}{4}] \cup [\frac{7\pi}{4}, 2\pi]$

$$S = \iint_D 1 \, dx\,dy = 4 \iint_{D_0} 1 \, dx\,dy$$

$$= 4 \iint_{D_0'} r \, dr\,d\theta = 4 \int_0^{\frac{\pi}{4}} \left(\int_0^{\sqrt{\cos 2\theta}} r \, dr \right) d\theta$$

$$= 2 \int_0^{\frac{\pi}{4}} \cos 2\theta \, d\theta = 1.$$

- <u>曲面表面积的计算</u> (可与第一型曲面积分联系)

1. $z = f(x, y)$

$$S = \iint_D \sqrt{1 + f_x^2 + f_y^2} \, dx\,dy.$$

2. $\begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases}$

$$S = \iint_{D'} \sqrt{\left|\frac{D(y,z)}{D(u,v)}\right|^2 + \left|\frac{D(z,x)}{D(u,v)}\right|^2 + \left|\frac{D(x,y)}{D(u,v)}\right|^2} \, du\,dv$$

注. $\frac{D(y,z)}{D(u,v)} = \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix}$, $\frac{D(z,x)}{D(u,v)} = \begin{vmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \end{vmatrix}$, $\frac{D(x,y)}{D(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$.

$\boxed{\text{重积分在物理中的应用}}$

- <u>物体的质量</u>

$$M = \iint_D \rho(x,y)\,dxdy$$

核心是微元法.

- <u>物体的质心</u>

$$x_0 = \frac{1}{M(D)} \iint_D x\rho(x,y)\,dxdy.$$

$$y_0 = \frac{1}{M(D)} \iint_D y\rho(x,y)\,dxdy.$$

<u>例5</u>. 第一卦限的立体 Ω 由四个面 $z=0$, $y=1$, $x=y$ 和 $z=xy$ 围成, 密度函数为 $\rho(x,y,z)=1+2z$, 求该立体的质心.

<u>解</u>. 首先计算立体质量:

$$M(\Omega) = \iiint_\Omega (1+2z)\,dxdydz = \iint_{D(x,y)} \int_0^{xy} (1+2z)\,dz\,dxdy$$

$$= \iint_{D(x,y)} (xy+x^2y^2)\,dxdy = \int_0^1 \int_0^y (xy+x^2y^2)\,dx\,dy$$

$$= \int_0^1 \frac{y^3}{2} + \frac{y^5}{3}\,dy = \frac{13}{72}.$$

$$x_0 = \frac{1}{M(\Omega)} \iiint_{\Omega} x(1+2z)\,dxdydz = \frac{1}{M(\Omega)} \iint_{D(x,y)} \int_0^{xy} x(1+2z)\,dz\,dxdy$$

$$= \frac{1}{M(\Omega)} \iint_{D(x,y)} (x^2y+x^3y^2)\,dxdy = \frac{1}{M(\Omega)} \int_0^1 \int_0^y (x^2y+x^3y^2)\,dx\,dy$$

$$= \frac{1}{M(\Omega)} \int_0^1 \frac{y^4}{3}+\frac{y^6}{4}\,dy = \frac{258}{455}.$$

$$y_0 = \frac{1}{M(\Omega)} \iiint_{\Omega} y(1+2z)\,dxdydz = \frac{1}{M(\Omega)} \iint_{D(x,y)} \int_0^{xy} y(1+2z)\,dz\,dxdy$$

$$= \frac{1}{M(\Omega)} \iint_{D(x,y)} (xy^2+x^2y^3)\,dxdy = \frac{1}{M(\Omega)} \int_0^1 \int_0^y (xy^2+x^2y^3)\,dx\,dy$$

$$= \frac{1}{M(\Omega)} \int_0^1 \frac{y^4}{2}+\frac{y^6}{3}\,dy = \frac{372}{455}.$$

$$z_0 = \frac{1}{M(\Omega)} \iiint_{\Omega} z(1+2z)\,dxdydz = \frac{1}{M(\Omega)} \iint_{D(x,y)} \int_0^{xy} z(1+2z)\,dz\,dxdy$$

$$= \frac{1}{M(\Omega)} \iint_{D(x,y)} \left(\frac{x^2y^2}{2}+\frac{2x^3y^3}{3}\right) dxdy = \frac{1}{M(\Omega)} \int_0^1 \int_0^y \frac{x^2y^2}{2}+\frac{2x^2y^3}{3}\,dxdy$$

$$= \frac{1}{M(\Omega)} \int_0^1 \frac{y^5}{6}+\frac{y^7}{6}\,dy = \frac{7}{26}.$$

综上，质心坐标为 $\left(\frac{258}{455}, \frac{372}{455}, \frac{7}{26}\right)$.

- **转动惯量**

$$J_x = \iiint_\Omega (y^2+z^2)\rho(x,y,z)\,dx\,dy\,dz$$

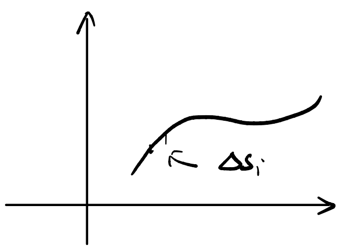
$$J_{xy} = \iiint_\Omega z^2 \rho(x,y,z)\,dx\,dy\,dz$$

其余类似（关于谁转动就把它去掉）

曲线积分

- **定义和计算**

第一型曲线积分（数值函数）



$$\int_L f(x,y)\,ds = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(x_i,y_i)\Delta s_i$$

其中 $\lambda = \max\limits_{1\le i\le n} \Delta s_i$ 　　 $ds = \sqrt{dx^2+dy^2}$

第二型曲线积分（向量值函数）

$$\int_L \vec{F}(x,y)\cdot d\vec{s} = \int_L P\,dx + Q\,dy = \lim_{\lambda\to 0}\sum_{i=1}^{n} F(x,y)\cdot(\Delta x_i, \Delta y_i)$$

其中 $\lambda = \max\limits_{1\le i\le n} \Delta s_i$. $d\vec{s} = (dx, dy)$

例1. 考虑摆线 L: $x = t - \sin t$, $y = 1 - \cos t$, $t \in [0, 2\pi]$. 计算

$$I = \int_L x\,ds.$$

解.

$$I = \int_L x\,ds = \int_0^{2\pi} (t - \sin t)\sqrt{(1-\cos t)^2 + \sin^2 t}\;dt$$

$$= \int_0^{2\pi} (t - \sin t)\sqrt{2 - 2\cos t}\;dt$$

$$= 2\int_0^{2\pi} (t - \sin t)\sin\frac{t}{2}\;dt = 8\pi.$$

例2. 计算第二型曲线积分 $\int_{\widehat{AB}} (-y)\,dx + x\,dy$, 其中 $\widehat{AB}$ 为单位圆周 $x^2+y^2=1$ 的上半部分, 方向从 $A(1,0)$ 到 $B(-1,0)$

解. 令

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases} \quad t \in [0, \pi].$$

$$I = \int_{\widehat{AB}} (-y)\,dx + x\,dy = \int_0^{\pi} \big((-\sin t)(-\sin t) + \cos t\cdot\cos t\big)\,dt$$

$$= \int_0^{\pi} 1\;dt = \pi.$$

- <u>两类曲线积分的关系</u>

注意到单位切向量

$$\vec{t} = \frac{1}{\Delta s_i}(\Delta x_i, \Delta y_i)$$

于是

$$\boxed{\int_L F(x,y)\cdot d\vec{s}} = \lim_{\lambda\to 0}\sum_{i=1}^{n} F(x_i,y_i)\cdot(\Delta x_i, \Delta y_i) = \lim_{\lambda\to 0}\sum_{i=1}^{n} F(x_i,y_i)\cdot\vec{t}\,\Delta s_i$$

$$= \boxed{\int_L F\cdot\vec{t}\,ds.}$$

<u>例13</u>. 计算第二型曲面积分 $\int_\Gamma x\,dx + y\,dy + z\,dz$，其中 Γ 为球面 $x^2+y^2+z^2=1$ 与平面 $x+y+z=0$ 的交线，从 z 轴看去取逆时针方向.

解. 法一：可取 Γ 的参数方程

$$\begin{cases} x = \frac{1}{\sqrt{2}}\cos t + \frac{1}{\sqrt{6}}\sin t \\ y = -\frac{1}{\sqrt{2}}\cos t + \frac{1}{\sqrt{6}}\sin t \\ z = -\frac{2}{\sqrt{6}}\sin t \end{cases}$$

计算得 $I=0$.　（细节比较繁琐，可尝试）.

法二、注意到 $F(x,y,z)=(x,y,z)$ 为球面 $x^2+y^2+z^2=1$ 在每个点 (x,y,z) 的单位外法向 $\vec{n}$、于是

$$I=\int_\Gamma x\,dx+y\,dy+z\,dz=\int_\Gamma \vec{n}\cdot\vec{t}\,ds=0.$$

$\boxed{\text{练习题}}$

1. 计算曲线积分 $I=\oint_L |y|\,ds$，其中 L：$(x^2+y^2)^2=x^2-y^2$.

(提示：可用极坐标下 $ds=\sqrt{r^2+(r')^2}\,d\theta$).

2. 计算第二型曲线积分 $I=\oint_L \dfrac{x\,dy-y\,dx}{x^2+y^2}$，其中 L 为 $x^2+y^2=1$.

(注：此题不适用 Green 公式，故据定义求之).