Chapter 1

Fourier Series and Integrals

1. Fourier coefficients and series

The problem of representing a function f, defined on (an interval of) $\mathbb{R}$, by a trigonometric series of the form

$$f(x) = \sum_{k=0}^{\infty} a_k \cos(kx) + b_k \sin(kx) \tag{1.1}$$

arises naturally when using the method of separation of variables to solve partial differential equations. This is how J. Fourier arrived at the problem, and he devoted the better part of his *Théorie Analytique de la Chaleur* (1822, results first presented to the Institute de France in 1807) to it. Even earlier, in the middle of the 18th century, Daniel Bernoulli had stated it while trying to solve the problem of a vibrating string, and the formula for the coefficients appeared in an article by L. Euler in 1777.

The right-hand side of (1.1) is a periodic function with period 2π, so f must also have this property. Therefore it will suffice to consider f on an interval of length 2π. Using Euler's identity, $e^{ikx} = \cos(kx) + i\sin(kx)$, we can replace the functions $\sin(kx)$ and $\cos(kx)$ in (1.1) by $\{e^{ikx} : k \in \mathbb{Z}\}$; we will do so from now on. Moreover, we will consider functions with period 1 instead of 2π, so we will modify the system of functions to $\{e^{2\pi ikx} : k \in \mathbb{Z}\}$. Our problem is thus transformed into studying the representation of f by

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi ikx}. \tag{1.2}$$

If we assume, for example, that the series converges uniformly, then by multiplying by $e^{-2\pi imx}$ and integrating term-by-term on $(0,1)$ we get

$$c_m = \int_0^1 f(x)e^{-2\pi imx}\,dx$$

because of the orthogonality relationship

$$\int_0^1 e^{2\pi ikx}e^{-2\pi imx}\,dx = \begin{cases} 0 & \text{if } k \neq m \\ 1 & \text{if } k = m. \end{cases} \tag{1.3}$$

Denote the additive group of the reals modulo 1 (that is $\mathbb{R}/\mathbb{Z}$) by $\mathbb{T}$, the one-dimensional torus. This can also be identified with the unit circle, S^1. Saying that a function is defined on $\mathbb{T}$ is equivalent to saying that it is defined on $\mathbb{R}$ and has period 1. To each function $f \in L^1(\mathbb{T})$ we associate the sequence $\{\hat{f}(k)\}$ of Fourier coefficients of f, defined by

$$\hat{f}(k) = \int_0^1 f(x)e^{-2\pi ikx}\,dx. \tag{1.4}$$

The trigonometric series with these coefficients,

$$\sum_{k=-\infty}^{\infty} \hat{f}(k)e^{2\pi ikx}, \tag{1.5}$$

is called the Fourier series of f.

Our problem now consists in determining when and in what sense the series (1.5) represents the function f.

2. Criteria for pointwise convergence

Denote the N-th symmetric partial sum of the series (1.5) by $S_N f(x)$; that is,

$$S_N f(x) = \sum_{k=-N}^{N} \hat{f}(k)e^{2\pi ikx}.$$

Note that this is also the N-th partial sum of the series when it is written in the form of (1.1).

Our first approach to the problem of representing f by its Fourier series is to determine whether $\lim S_N f(x)$ exists for each x, and if so, whether it is equal to $f(x)$. The first positive result is due to P. G. L. Dirichlet (1829), who proved the following convergence criterion: if f is bounded, piecewise continuous, and has a finite number of maxima and minima, then $\lim S_N f(x)$ exists and is equal to $\frac{1}{2}[f(x+) + f(x-)]$. Jordan's criterion, which we prove below, includes this result as a special case.

In order to study $S_N f(x)$ we need a more manageable expression. Dirichlet wrote the partial sums as follows:

$$\begin{aligned} S_N f(x) &= \sum_{k=-N}^{N} \int_0^1 f(t) e^{-2\pi ikt}\, dt \cdot e^{2\pi ikx} \\ &= \int_0^1 f(t) D_N(x-t)\, dt \\ &= \int_0^1 f(x-t) D_N(t)\, dt, \end{aligned}$$

where D_N is the Dirichlet kernel,

$$D_N(t) = \sum_{k=-N}^{N} e^{2\pi ikt}.$$

If we sum this geometric series we get

$$D_N(t) = \frac{\sin(\pi(2N+1)t)}{\sin(\pi t)}. \tag{1.6}$$

This satisfies

$$\int_0^1 D_N(t)\, dt = 1 \qquad \text{and} \qquad |D_N(t)| \le \frac{1}{\sin(\pi\delta)}, \qquad \delta \le |t| \le 1/2.$$

We will prove two criteria for pointwise convergence.

Theorem 1.1 (Dini's Criterion)**.** *If for some x there exists $\delta > 0$ such that*

$$\int_{|t|<\delta} \left| \frac{f(x+t) - f(x)}{t} \right| dt < \infty,$$

then

$$\lim_{N\to\infty} S_N f(x) = f(x).$$

Theorem 1.2 (Jordan's Criterion)**.** *If f is a function of bounded variation in a neighborhood of x, then*

$$\lim_{N\to\infty} S_N f(x) = \frac{1}{2}[f(x+) + f(x-)].$$

At first it may seem surprising that these results are local, since if we modify the function slightly, the Fourier coefficients of f change. Nevertheless, the convergence of a Fourier series is effectively a local property, and if the modifications are made outside of a neighborhood of x, then the behavior of the series at x does not change. This is made precise by the following result.

Theorem 1.3 (Riemann Localization Principle). *If f is zero in a neighborhood of x, then*

$$\lim_{N\to\infty} S_N f(x) = 0.$$

An equivalent formulation of this result is to say that if two functions agree in a neighborhood of x, then their Fourier series behave in the same way at x.

From the definition of Fourier coefficients (1.4) it follows immediately that

$$|\hat{f}(k)| \le \|f\|_1,$$

but a sharper estimate is true which we will use to prove the preceding results.

Lemma 1.4 (Riemann-Lebesgue). *If $f \in L^1(\mathbb{T})$ then*

$$\lim_{|k|\to\infty} \hat{f}(k) = 0.$$

Proof. Since $e^{2\pi i x}$ has period 1,

$$\begin{aligned}\hat{f}(k) &= \int_0^1 f(x) e^{-2\pi i k x}\, dx \\ &= -\int_0^1 f(x) e^{-2\pi i k (x+1/2k)}\, dx \\ &= -\int_0^1 f(x-1/2k) e^{-2\pi i k x}\, dx.\end{aligned}$$

Hence,

$$\hat{f}(k) = \frac{1}{2}\int_0^1 [f(x) - f(x-1/2k)] e^{-2\pi i k x}\, dx.$$

If f is continuous, it follows immediately that

$$\lim_{|k|\to\infty} \hat{f}(k) = 0.$$

For arbitrary $f \in L^1(\mathbb{T})$, given $\epsilon > 0$, choose g continuous such that $\|f-g\|_1 < \epsilon/2$ and choose k sufficiently large that $|\hat{g}(k)| < \epsilon/2$. Then

$$|\hat{f}(k)| \le |(f-g)\widehat{\ }(k)| + |\hat{g}(k)| \le \|f-g\|_1 + |\hat{g}(k)| < \epsilon.$$

$\square$

Proof of Theorem 1.3. Suppose that $f(t) = 0$ on $(x - \delta, x + \delta)$. Then

$$\begin{aligned} S_N f(x) &= \int_{\delta \le |t| < 1/2} f(x-t) \frac{\sin(\pi(2N+1)t)}{\sin(\pi t)}\, dt \\ &= (g e^{\pi i \cdot})\widehat{\ }(N) + (g e^{-\pi i \cdot})\widehat{\ }(-N), \end{aligned}$$

where

$$g(t) = \frac{f(x-t)}{2i \sin(\pi t)} \chi_{\{\delta \le |t| < 1/2\}}(t)$$

is integrable. By the Riemann-Lebesgue lemma we conclude that

$$\lim_{N \to \infty} S_N f(x) = 0.$$

□

Proof of Theorem 1.1. Since the integral of D_N equals 1,

$$\begin{aligned} S_N f(x) - f(x) &= \int_{-1/2}^{1/2} [f(x-t) - f(x)] \frac{\sin(\pi(2N+1)t)}{\sin(\pi t)}\, dt \\ &= \int_{|t|<\delta} + \int_{\delta \le |t| < 1/2} . \end{aligned}$$

By the Riemann-Lebesgue lemma both of these integrals tend to 0. The second if we argue as in the previous proof, the first since by hypothesis the function

$$\frac{f(x-t) - f(x)}{\sin(\pi t)} \chi_{\{|t|<\delta\}}(t)$$

is integrable. (Recall that if $|t| < \delta$, $\sin(\pi t)$ and πt are equivalent.) □

Proof of Theorem 1.2. Since every function of bounded variation is the difference of two monotonic functions, we may assume that f is monotonic in a neighborhood of x. Since

$$S_N f(x) = \int_{-1/2}^{1/2} f(x-t) D_N(t)\, dt = \int_0^{1/2} [f(x-t) + f(x+t)] D_N(t)\, dt,$$

it will be enough to show that for g monotonic

$$\lim_{N \to \infty} \int_0^{1/2} g(t) D_N(t)\, dt = \frac{1}{2} g(0+).$$

Further, we may assume that $g(0+) = 0$ and that g is increasing to the right of 0. Given $\epsilon > 0$, choose $\delta > 0$ such that $g(t) < \epsilon$ if $0 < t < \delta$. Then

$$\int_0^{1/2} g(t) D_N(t)\, dt = \int_0^{\delta} + \int_{\delta}^{1/2} .$$

Again by the Riemann-Lebesgue lemma, the second integral tends to 0. We apply the second mean value theorem for integrals[1] to the first integral. Then for some ν, $0 < \nu < \delta$,

$$\int_0^\delta g(t) D_N(t)\,dt = g(\delta-)\int_\nu^\delta D_N(t)\,dt.$$

Furthermore,

$$\begin{aligned}\left|\int_\nu^\delta D_N(t)\,dt\right| &\leq \left|\int_\nu^\delta \sin(\pi(2N+1)t)\left(\frac{1}{\sin(\pi t)} - \frac{1}{\pi t}\right)dt\right| \\ &\quad + \left|\int_\nu^\delta \frac{\sin(\pi(2N+1)t)}{\pi t}\,dt\right| \\ &\leq \int_\nu^\delta \left|\frac{1}{\sin(\pi t)} - \frac{1}{\pi t}\right| dt + 2\sup_{M>0}\left|\int_0^M \frac{\sin(\pi t)}{t}\,dt\right| \\ &\leq C.\end{aligned}$$

Hence,

$$\left|\int_0^\delta g(t) D_N(t)\,dt\right| \leq C\epsilon.$$

□

3. Fourier series of continuous functions

If f satisfies a Lipschitz-type condition in a neighborhood of x, that is, $|f(x+t) - f(x)| \leq C|t|^a$ for some $a > 0$, $|t| < \delta$, then Dini's criterion applies to it. However, continuous functions need not satisfy this condition or any other convergence criterion we have seen. This must be the case because of the following result due to P. du Bois-Reymond (1873).

Theorem 1.5. *There exists a continuous function whose Fourier series diverges at a point.*

Du Bois-Reymond constructed a function with this property, but we will show that one exists by applying the uniform boundedness principle, also known as the Banach-Steinhaus theorem.

Lemma 1.6 (Uniform Boundedness Principle). *Let X be a Banach space, Y a normed vector space, and let $\{T_a\}_{a\in A}$ be a family of bounded linear*

[1]If ϕ is continuous and h monotonic on $[a, b]$, then there exists c, $a < c < b$, such that

$$\int_a^b h\phi = h(b-)\int_c^b \phi + h(a+)\int_a^c \phi.$$

operators from X to Y. Then either

$$\sup_a \|T_a\| < \infty$$

or there exists $x \in X$ such that

$$\sup_a \|T_a x\|_Y = \infty.$$

(Recall that the operator norm of T_a is $\|T_a\| = \sup\{\|T_a x\|_Y : \|x\|_X \leq 1\}$.) A proof of this result can be found, for example, in Rudin [**14**, Chapter 5].

Now let $X = C(\mathbb{T})$ with the norm $\|\cdot\|_\infty$ and let $Y = \mathbb{C}$. Define $T_N : X \to Y$ by

$$T_N f = S_N f(0) = \int_{-1/2}^{1/2} f(t) D_N(t)\, dt.$$

Define the Lebesgue numbers L_N by

$$L_N = \int_{-1/2}^{1/2} |D_N(t)|\, dt;$$

it is immediate that $|T_N f| \leq L_N \|f\|_\infty$. $D_N(t)$ has a finite number of zeros so $\operatorname{sgn} D_N(t)$ has a finite number of jump discontinuities. Therefore, by modifying it on a small neighborhood of each discontinuity, we can form a continuous function f such that $\|f\|_\infty = 1$ and $|T_N f| \geq L_N - \epsilon$. Hence, $\|T_N\| = L_N$. Thus if we can prove that $L_N \to \infty$ as $N \to \infty$, then by the uniform boundedness principle there exists a continuous function f such that

$$\limsup_{N\to\infty} |S_N f(0)| = \infty;$$

that is, the Fourier series of f diverges at 0.

Lemma 1.7. $L_N = \dfrac{4}{\pi^2} \log N + O(1)$.

Proof.

$$\begin{aligned} L_N &= 2\int_0^{1/2} \left| \frac{\sin(\pi(2N+1)t)}{\pi t} \right| dt + O(1) \\ &= 2\int_0^{N+1/2} \left| \frac{\sin(\pi t)}{\pi t} \right| dt + O(1) \\ &= 2\sum_{k=0}^{N-1} \int_k^{k+1} \left| \frac{\sin(\pi t)}{\pi t} \right| dt + O(1) \\ &= \frac{2}{\pi} \sum_{k=0}^{N-1} \int_0^1 \frac{|\sin(\pi t)|}{t+k}\, dt + O(1) \end{aligned}$$

$$= \frac{2}{\pi}\int_0^1 |\sin(\pi t)| \sum_{k=1}^{N-1} \frac{1}{t+k}\, dt + O(1)$$
$$= \frac{4}{\pi^2}\log N + O(1).$$

□

4. Convergence in norm

The development of measure theory and L^p spaces led to a new approach to the problem of convergence. We can now ask:

(1) Does $\lim_{N\to\infty} \|S_N f - f\|_p = 0$ for $f \in L^p(\mathbb{T})$?

(2) Does $\lim_{N\to\infty} S_N f(x) = f(x)$ almost everywhere if $f \in L^p(\mathbb{T})$?

We can restate the first question by means the following lemma.

Lemma 1.8. *$S_N f$ converges to f in L^p norm, $1 \le p < \infty$, if and only if there exists C_p independent of N such that*

$$\|S_N f\|_p \le C_p \|f\|_p. \tag{1.7}$$

Proof. The necessity of (1.7) follows from the uniform boundedness principle.

To see that it is sufficient, first note that if g is a trigonometric polynomial, then $S_N g = g$ for $N \ge \deg g$. Therefore, since the trigonometric polynomials are dense in L^p (see Corollary 1.11), if $f \in L^p$ we can find a trigonometric polynomial g such that $\|f - g\|_p < \epsilon$, and so for N sufficiently large

$$\|S_N f - f\|_p \le \|S_N(f-g)\|_p + \|S_N g - g\|_p + \|f - g\|_p \le (C_p + 1)\epsilon.$$

□

If $1 < p < \infty$, then inequality (1.7) holds, as we will show in Chapter 3. When $p = 1$, the L^1 operator norm of S_N is again L_N, and so by Lemma 1.7 the answer to the first question is no.

When $p = 2$, the functions $\{e^{2\pi i k x}\}$ form an orthonormal system (by (1.3)) which is complete (i.e. an orthonormal basis) by the density of the trigonometric polynomials in L^2. Therefore, we can apply the theory of Hilbert spaces to get the following.

Theorem 1.9. *The mapping $f \mapsto \{\hat{f}(k)\}$ is an isometry from L^2 to ℓ^2, that is,*

$$\|f\|_2^2 = \sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2.$$

Convergence in norm in L^2 follows from this immediately.

The second question is much more difficult. A. Kolmogorov (1926) gave an example of an integrable function whose Fourier series diverges at every point, so the answer is no if $p = 1$. If $f \in L^p$, $1 < p < \infty$, then the Fourier series of f converges almost everywhere. This was shown by L. Carleson (1965, $p = 2$) and R. Hunt (1967, $p > 1$). Until the result by Carleson, the answer was unknown even for f continuous.

5. Summability methods

In order to recover a function f from its Fourier coefficients it would be convenient to find some other method than taking the limit of the partial sums of its Fourier series since, as we have seen, this approach does not always work well.

One such method, Cesàro summability, consists in taking the limit of the arithmetic means of the partial sums. As is well known, if $\lim a_k$ exists then

$$\lim_{k\to\infty} \frac{a_1 + \cdots + a_k}{k}$$

also exists and has the same value.

Define

$$\begin{aligned}
\sigma_N f(x) &= \frac{1}{N+1} \sum_{k=0}^{N} S_k f(x) \\
&= \int_0^1 f(t) \frac{1}{N+1} \sum_{k=0}^{N} D_k(x-t)\, dt \\
&= \int_0^1 f(t) F_N(x-t)\, dt,
\end{aligned}$$

where F_N is the Fejér kernel,

$$F_N(t) = \frac{1}{N+1} \sum_{k=0}^{N} D_k(t) = \frac{1}{N+1} \left(\frac{\sin(\pi(N+1)t)}{\sin(\pi t)} \right)^2.$$

F_N has the following properties:

$$F_N(t) \geq 0,$$

$$\|F_N\|_1 = \int_0^1 F_N(t)\,dt = 1, \tag{1.8}$$
$$\lim_{N\to\infty} \int_{\delta<|t|<1/2} F_N(t)\,dt = 0 \quad \text{if } \delta > 0.$$

Because F_N is positive, its L^1 norm coincides with its integral and is 1. This is not the case for the Dirichlet kernel: its integral equals 1 because of cancellation between its positive and negative parts while its L^1 norm tends to infinity with N.

Theorem 1.10. *If $f \in L^p$, $1 \le p < \infty$, or if f is continuous and $p = \infty$, then*

$$\lim_{N\to\infty} \|\sigma_N f - f\|_p = 0.$$

Proof. Since $\int F_N = 1$, by Minkowski's inequality we have that

$$\begin{aligned} \|\sigma_N f - f\|_p &= \int_{-1/2}^{1/2} \|f(\cdot - t) - f(\cdot)\|_p F_N(t)\,dt \\ &\le \int_{|t|<\delta} \|f(\cdot - t) - f(\cdot)\|_p F_N(t)\,dt + 2\|f\|_p \int_{\delta<|t|<1/2} F_N(t)\,dt. \end{aligned}$$

Since for $1 \le p < \infty$,

$$\lim_{t\to 0} \|f(\cdot - t) - f(\cdot)\|_p = 0,$$

and the same limit holds if $p = \infty$ and f is continuous, the first term can be made as small as desired by choosing a suitable δ. And for fixed δ, by (1.8) the second term tends to 0. □

Corollary 1.11.

(1) *The trigonometric polynomials are dense in L^p, $1 \le p < \infty$.*

(2) *If f is integrable and $\hat{f}(k) = 0$ for all k, then f is identically zero.*

A second summability method is gotten by treating a Fourier series as the formal limit on the unit circle (in the complex plane) of

$$u(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k + \sum_{k=-\infty}^{-1} \hat{f}(k) \bar{z}^{|k|}, \quad z = re^{2\pi i\theta}. \tag{1.9}$$

Since $\{\hat{f}(k)\}$ is a bounded sequence, this function is well defined on $|z| < 1$. It can be rewritten as

$$u(re^{2\pi i\theta}) = \sum_{k=-\infty}^{\infty} \hat{f}(k) r^{|k|} e^{2\pi i k\theta} = \int_{-1/2}^{1/2} f(t) P_r(\theta - t)\,dt,$$

where

$$P_r(t) = \sum_{k=-\infty}^{\infty} r^{|k|} e^{2\pi ikt} = \frac{1-r^2}{1-2r\cos(2\pi t)+r^2}$$

is the Poisson kernel. The Poisson kernel has properties analogous to those of the Fejér kernel:

$$\begin{gathered} P_r(t) \geq 0, \\ \int_0^1 P_r(t)\,dt = 1, \\ \lim_{r\to 1^-} \int_{\delta<|t|<1/2} P_r(t)\,dt = 0 \quad \text{if } \delta > 0. \end{gathered} \tag{1.10}$$

Therefore, we can prove a result analogous to Theorem 1.10.

Theorem 1.12. *If $f \in L^p$, $1 \leq p < \infty$, or if f is continuous and $p = \infty$, then*

$$\lim_{r\to 1^-} \|P_r * f - f\|_p = 0.$$

Since the function u is harmonic on $|z| < 1$, it is the solution to the Dirichlet problem:

$$\begin{aligned} \Delta u &= 0 \quad \text{if } |z| < 1, \\ u &= f \quad \text{if } |z| = 1, \end{aligned}$$

where the boundary condition is interpreted in terms of Theorem 1.12.

In Chapter 2 we will study the almost everywhere convergence of $\sigma_N f(x)$ and $P_r * f(x)$.

6. The Fourier transform of L^1 functions

Given a function $f \in L^1(\mathbb{R}^n)$, define its Fourier transform by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi ix\cdot\xi}\,dx, \tag{1.11}$$

where $x \cdot \xi = x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n$. The following is a list of properties of the Fourier transform:

(1.12) $(\alpha f + \beta g)\widehat{\ } = \alpha\hat{f} + \beta\hat{g}$ (linearity);

(1.13) $\|\hat{f}\|_\infty \leq \|f\|_1$ and $\hat{f}$ is continuous;

(1.14) $\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0$ (Riemann-Lebesgue);

(1.15) $(f * g)\widehat{\ } = \hat{f}\hat{g}$;

(1.16) $(\tau_h f)\hat{}(\xi) = \hat{f}(\xi)e^{2\pi i h\cdot\xi}$, where $\tau_h f(x) = f(x+h)$;

$(fe^{2\pi i h\cdot x})\hat{}(\xi) = \hat{f}(\xi - h)$;

(1.17) if $\rho \in O_n$ (an orthogonal transformation), then

$(f(\rho\cdot))\hat{}(\xi) = \hat{f}(\rho\xi)$;

(1.18) if $g(x) = \lambda^{-n}f(\lambda^{-1}x)$, then $\hat{g}(\xi) = \hat{f}(\lambda\xi)$;

(1.19) $$\left(\frac{\partial f}{\partial x_j}\right)^{\widehat{}}(\xi) = 2\pi i\xi_j\hat{f}(\xi);$$

(1.20) $$(-2\pi i x_j f)\hat{}(\xi) = \frac{\partial \hat{f}}{\partial \xi_j}(\xi).$$

The continuity of $\hat{f}$ follows from the dominated convergence theorem; (1.14) can be proved like Lemma 1.4; the rest follow from a change of variables, Fubini's theorem and integration by parts. In (1.19) we assume that $\partial f/\partial x_j \in L^1$ and in (1.20) that $x_j f \in L^1$.

Unlike on the torus, $L^1(\mathbb{R}^n)$ does not contain $L^p(\mathbb{R}^n)$, $p > 1$, so (1.11) does not define the Fourier transform of functions in those spaces. For the same reason, the formula which should allow us to recover f from $\hat{f}$,

$$\int_{\mathbb{R}^n} \hat{f}(\xi)e^{2\pi i x\cdot\xi}\,d\xi,$$

may not make sense since (1.13) and (1.14) are all that we know about $\hat{f}$, and they do not imply that $\hat{f}$ is integrable. (In fact, $\hat{f}$ is generally not integrable.)

7. The Schwartz class and tempered distributions

A function f is in the Schwartz class, $\mathcal{S}(\mathbb{R}^n)$, if it is infinitely differentiable and if all of its derivatives decrease rapidly at infinity; that is, if for all $\alpha, \beta \in \mathbb{N}^n$,

$$\sup_x |x^\alpha D^\beta f(x)| = p_{\alpha,\beta}(f) < \infty.$$

Functions in C_c^∞ are in $\mathcal{S}$, but so are functions like $e^{-|x|^2}$ which do not have compact support. The collection $\{p_{\alpha,\beta}\}$ is a countable family of seminorms on $\mathcal{S}$, and we can use it to define a topology on $\mathcal{S}$: a sequence $\{\phi_k\}$ converges to 0 if and only if for all $\alpha, \beta \in \mathbb{N}^n$,

$$\lim_{k\to\infty} p_{\alpha,\beta}(\phi_k) = 0.$$

With this topology $\mathcal{S}$ is a Fréchet space (complete and metrizable) and is dense in $L^p(\mathbb{R}^n)$, $1 \le p < \infty$. In particular, $\mathcal{S} \subset L^1$ and (1.11) defines the Fourier transform of a function in $\mathcal{S}$.

The space of bounded linear functionals on $\mathcal{S}$, $\mathcal{S}'$, is called the space of tempered distributions. A linear map T from $\mathcal{S}$ to $\mathbb{C}$ is in $\mathcal{S}'$ if

$$\lim_{k\to\infty} T(\phi_k) = 0 \quad \text{whenever} \quad \lim_{k\to\infty} \phi_k = 0 \quad \text{in } \mathcal{S}.$$

Theorem 1.13. *The Fourier transform is a continuous map from $\mathcal{S}$ to $\mathcal{S}$ such that*

$$\int_{\mathbb{R}^n} f\hat{g} = \int_{\mathbb{R}^n} \hat{f} g \tag{1.21}$$

and

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi. \tag{1.22}$$

Equality (1.22) is referred to as the inversion formula.

To prove Theorem 1.13 we need to compute the Fourier transform of a particular function.

Lemma 1.14. *If $f(x) = e^{-\pi|x|^2}$ then $\hat{f}(\xi) = e^{-\pi|\xi|^2}$.*

Proof. We could prove this result directly by integrating in $\mathbb{C}$, but we will give a different proof here. It is enough to prove this in one dimension, since in $\mathbb{R}^n$ $\hat{f}$ is the product of n identical integrals.

The function $f(x) = e^{-\pi x^2}$ is the solution of the differential equation

$$u' + 2\pi x u = 0,$$
$$u(0) = 1.$$

By (1.19) and (1.20) we see that $\hat{u}$ satisfies the same differential equation with the initial value

$$\hat{u}(0) = \int_{\mathbb{R}} u(x)\, dx = \int_{\mathbb{R}} e^{-\pi x^2}\, dx = 1.$$

Therefore, by uniqueness, $\hat{f} = f$. □

Proof of Theorem 1.13. By (1.19) and (1.20) we have

$$\xi^\alpha D^\beta \hat{f}(\xi) = C(D^\alpha x^\beta f)\widehat{\ }(\xi),$$

so

$$|\xi^\alpha D^\beta \hat{f}(\xi)| \le C\|D^\alpha x^\beta f\|_1.$$

The L^1 norm can be bounded by a finite linear combination of seminorms of f, which implies that the Fourier transform is a continuous map from $\mathcal{S}$ to itself.

Equality (1.21) is an immediate consequence of Fubini's theorem since $f(x)g(y)$ is integrable on $\mathbb{R}^n \times \mathbb{R}^n$.

From (1.18) and (1.21) we get

$$\int f(x)\hat{g}(\lambda x)\,dx = \int \hat{f}(x)\lambda^{-n}g(\lambda^{-1}x)\,dx.$$

If we make the change of variables $\lambda x = y$ in the first integral, this becomes

$$\int f(\lambda^{-1}x)\hat{g}(x)\,dx = \int \hat{f}(x)g(\lambda^{-1}x)\,dx;$$

if we then take the limit as $\lambda \to \infty$, we get

$$f(0)\int \hat{g}(x)\,dx = g(0)\int \hat{f}(x)\,dx.$$

Let $g(x) = e^{-\pi|x|^2}$; then by Lemma 1.14,

$$f(0) = \int \hat{f}(\xi)\,d\xi,$$

which is (1.22) for $x = 0$. If we replace f by $\tau_x f$, then by (1.16),

$$f(x) = (\tau_x f)(0) = \int (\tau_x f)\widehat{\ }(\xi)\,d\xi = \int \hat{f}(\xi)e^{2\pi i x\cdot\xi}\,d\xi.$$

□

If we let $\tilde{f}(x) = f(-x)$, we get the following corollary.

Corollary 1.15. *For $f \in \mathcal{S}$, $(\hat{f})\widehat{\ } = \tilde{f}$, and so the Fourier transform has period 4 (i.e. if we apply it four times, we get the identity operator).*

Definition 1.16. The Fourier transform of $T \in \mathcal{S}'$ is the tempered distribution $\hat{T}$ given by

$$\hat{T}(f) = T(\hat{f}), \qquad f \in \mathcal{S}.$$

By Theorem 1.13, $\hat{T}$ is a tempered distribution, and in particular, if T is an integrable function, then $\hat{T}$ coincides with the Fourier transform defined by equation (1.11). Likewise, if μ is a finite Borel measure (i.e. a bounded linear functional on $C_0(\mathbb{R}^n)$, the space of continuous functions which vanish at infinity), then $\hat{\mu}$ is the bounded continuous function given by

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi}\,d\mu(x).$$

For δ, the Dirac measure at the origin, this gives us $\hat{\delta} = 1$.

Theorem 1.17. *The Fourier transform is a bounded linear bijection from $\mathcal{S}'$ to $\mathcal{S}'$ whose inverse is also bounded.*

Proof. If $T_n \to T$ in $\mathcal{S}'$, then for any $f \in \mathcal{S}$,

$$\hat{T}_n(f) = T_n(\hat{f}) \to T(\hat{f}) = \hat{T}(f).$$

Furthermore, the Fourier transform has period 4, so its inverse is equivalent to applying it 3 times; therefore, its inverse is also continuous. □

If we define $\tilde{T}$ by $\tilde{T}(f) = T(\tilde{f})$, then it follows from Corollary 1.15 that $(\tilde{\hat{T}})\hat{} = T$. And if $\hat{T} \in L^1$ then by the inversion formula we get that

$$T(x) = \int_{\mathbb{R}^n} \hat{T}(\xi) e^{2\pi i x\cdot\xi}\, d\xi;$$

in particular, T is a bounded, continuous function.

8. The Fourier transform on L^p, $1 < p \le 2$

If $f \in L^p$, $1 \le p \le \infty$, then f can be identified with a tempered distribution: for $\phi \in \mathcal{S}$ define

$$T_f(\phi) = \int_{\mathbb{R}^n} f\phi.$$

Clearly this integral is finite. To see that T_f is continuous, suppose that $\phi_k \to 0$ in $\mathcal{S}$ as $k \to \infty$. Then by Hölder's inequality,

$$|T_f(\phi_k)| \le \|f\|_p \|\phi_k\|_{p'}.$$

Then $\|\phi_k\|_{p'}$ is dominated by the L^∞ norm of functions of the form $x^a\phi_k$, and so by a finite linear combination of seminorms of ϕ_k; hence, the left-hand side tends to 0 as $k \to \infty$.

Moreover, when $1 \le p \le 2$ we have that $\hat{f}$ is a function.

Theorem 1.18. *The Fourier transform is an isometry on L^2; that is, $\hat{f} \in L^2$ and $\|\hat{f}\|_2 = \|f\|_2$. Furthermore,*

$$\hat{f}(\xi) = \lim_{R\to\infty} \int_{|x|<R} f(x) e^{-2\pi i x\cdot\xi}\, dx$$

and

$$f(x) = \lim_{R\to\infty} \int_{|\xi|<R} \hat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi,$$

where the limits are in L^2.

The identity $\|\hat{f}\|_2 = \|f\|_2$ is referred to as the Plancherel theorem.

Proof. Given $f, h \in \mathcal{S}$, let $g = \bar{\hat{h}}$, so that $\hat{g} = \bar{h}$. Then by (1.21) we have that

$$\int_{\mathbb{R}^n} f\bar{h} = \int_{\mathbb{R}^n} \hat{f}\bar{\hat{h}}. \tag{1.23}$$

If we let $h = f$ then we get $\|f\|_2 = \|\hat{f}\|_2$ for $f \in \mathcal{S}$. Since $\mathcal{S}$ is dense in L^2, the Fourier transform extends to all f in L^2 with equality of norms.

Finally, the continuity of the Fourier transform implies the given formulas for f and $\hat{f}$ as limits in L^2, since $f\chi_{B(0,R)}$ and $\hat{f}\chi_{B(0,R)}$ converge to f and $\hat{f}$ in L^2. □

If $f \in L^p$, $1 < p < 2$, then it can be decomposed as $f = f_1 + f_2$, where $f_1 \in L^1$ and $f_2 \in L^2$. (For example, let $f_1 = f\chi_{\{x:|f(x)|>1\}}$ and $f_2 = f - f_1$.) Therefore, $\hat{f} = \hat{f}_1 + \hat{f}_2 \in L^\infty + L^2$. However, by applying an interpolation theorem we can get a sharper result.

Theorem 1.19 (Riesz-Thorin Interpolation). *Let* $1 \le p_0, p_1, q_0, q_1 \le \infty$, *and for* $0 < \theta < 1$ *define* p *and* q *by*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

If T *is a linear operator from* $L^{p_0} + L^{p_1}$ *to* $L^{q_0} + L^{q_1}$ *such that*

$$\|Tf\|_{q_0} \le M_0\|f\|_{p_0} \quad \text{for } f \in L^{p_0}$$

and

$$\|Tf\|_{q_1} \le M_1\|f\|_{p_1} \quad \text{for } f \in L^{p_1},$$

then

$$\|Tf\|_q \le M_0^{1-\theta} M_1^{\theta}\|f\|_p \quad \text{for } f \in L^p.$$

The proof of this result uses the so-called "three-lines" theorem for analytic functions; it can be found, for example, in Stein and Weiss [**18**, Chapter 5] or Katznelson [**10**, Chapter 4].

Corollary 1.20 (Hausdorff-Young Inequality). *If* $f \in L^p$, $1 \le p \le 2$, *then* $\hat{f} \in L^{p'}$ *and*

$$\|\hat{f}\|_{p'} \le \|f\|_p.$$

Proof. Apply Theorem 1.19 using inequality (1.13), $\|\hat{f}\|_\infty \le \|f\|_1$, and the Plancherel theorem, $\|\hat{f}\|_2 = \|f\|_2$. □

We digress to give another corollary of Riesz-Thorin interpolation which is not directly related to the Fourier transform but which will be useful in later chapters.

Corollary 1.21 (Young's Inequality). *If $f \in L^p$ and $g \in L^q$, then $f * g \in L^r$, where $1/r + 1 = 1/p + 1/q$, and*

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

Proof. If we fix $f \in L^p$ we immediately get the inequalities

$$\|f * g\|_p \leq \|f\|_p \|g\|_1$$

and

$$\|f * g\|_\infty \leq \|f\|_p \|g\|_{p'}.$$

The desired result follows by Riesz-Thorin interpolation. □

9. The convergence and summability of Fourier integrals

The problem of recovering a function from its Fourier transform is similar to the same problem for Fourier series. We need to determine if and when

$$\lim_{R \to \infty} \int_{B_R} \hat{f}(\xi) e^{2\pi i x \cdot \xi} \, d\xi = f(x),$$

where $B_R = \{Rx : x \in B\}$, B is an open convex neighborhood of the origin, and the limit is understood either as in L^p or as pointwise almost everywhere. If we define the partial sum operator S_R by

$$(S_R f)\widehat{\ } = \chi_{B_R} \hat{f},$$

then this problem is equivalent to determining if

$$\lim_{R \to \infty} S_R f = f.$$

Analogous to Lemma 1.8, a necessary and sufficient condition for convergence in norm is that

$$\|S_R f\|_p \leq C_p \|f\|_p,$$

where C_p is independent of R. When $n = 1$ this is the case; we will prove this in Chapter 3. We will also prove several partial results when $n > 1$, but in general there is no convergence in norm when $p \neq 2$. We will discuss this in Chapter 8.

In the case $n = 1$, if $B = (-1, 1)$ then

$$S_R f(x) = D_R * f(x),$$

where D_R is the Dirichlet kernel,

$$D_R(x) = \int_{-R}^{R} e^{2\pi i x\xi}\, d\xi = \frac{\sin(2\pi R x)}{\pi x}.$$

This is clearly not integrable, but it is in $L^q(\mathbb{R})$ for any $q > 1$, so $D_R * f$ is well defined if $f \in L^p$, $1 < p < \infty$.

Almost everywhere convergence depends on the bound

$$\| \sup_R |S_R f| \|_p \le C_p \|f\|_p.$$

This holds if $1 < p < \infty$ (the Carleson-Hunt theorem) but we cannot prove it here.

For the Fourier transform, the method of Cesàro summability consists in taking integral averages of the partial sum operators,

$$\sigma_R f(x) = \frac{1}{R}\int_0^R S_t f(x)\, dt,$$

and determining if $\lim \sigma_R f(x) = f(x)$. When $n = 1$ and $B = (-1, 1)$,

$$\sigma_R f(x) = F_R * f(x),$$

where F_R is the Fejér kernel,

$$F_R(x) = \frac{1}{R}\int_0^R D_t(x)\, dt = \frac{\sin^2(\pi R x)}{R(\pi x)^2}. \tag{1.24}$$

Unlike the Dirichlet kernel, the Fejér kernel is integrable. Since it has properties analogous to (1.8), one can prove that in L^p, $1 \le p < \infty$,

$$\lim_{R\to\infty} \sigma_R f = f.$$

The proof is similar to that of Theorem 1.10. In Chapter 2 we will prove two general results from which we can deduce convergence in L^p and pointwise almost everywhere for this and the following summability methods.

The method of Abel-Poisson summability consists in introducing the factor $e^{-2\pi t|\xi|}$ into the inversion formula. Then for any $t > 0$ the integral converges, and we take the limit as t tends to 0. If we instead introduce the factor $e^{-\pi t^2|\xi|^2}$, we get the method of Gauss-Weierstrass summability. More precisely, we define the functions

$$u(x,t) = \int_{\mathbb{R}^n} e^{-2\pi t|\xi|} \hat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi, \tag{1.25}$$

$$w(x,t) = \int_{\mathbb{R}^n} e^{-\pi t^2|\xi|^2} \hat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi, \tag{1.26}$$

and then try to determine if

$$\lim_{t\to 0^+} u(x,t) = f(x), \tag{1.27}$$

$$\lim_{t\to 0^+} w(x,t) = f(x) \tag{1.28}$$

in L^p or pointwise almost everywhere.

One can show that $u(x,t)$ is harmonic in the half-space $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,\infty)$. When $n = 1$ we have an equivalent formula analogous to (1.9):

$$u(z) = \int_0^\infty \hat{f}(\xi)e^{2\pi i z\xi}\,d\xi + \int_{-\infty}^0 \hat{f}(\xi)e^{2\pi i\bar{z}\xi}\,d\xi, \quad z = x+it, \tag{1.29}$$

which immediately implies that u is harmonic. The limit (1.27) can be interpreted as the boundary condition of the Dirichlet problem,

$$\begin{aligned} \Delta u &= 0 \quad \text{on } \mathbb{R}^{n+1}_+, \\ u(x,0) &= f(x), \quad x \in \mathbb{R}^n. \end{aligned}$$

It follows from (1.25) that

$$u(x,t) = P_t * f(x),$$

where $\hat{P}_t(\xi) = e^{-2\pi t|\xi|}$. One can prove by a simple calculation if $n = 1$, and a more difficult one when $n > 1$ (see Stein and Weiss [**18**, p. 6]), that

$$P_t(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}\frac{t}{(t^2+|x|^2)^{\frac{n+1}{2}}}. \tag{1.30}$$

This is called the Poisson kernel.

In the case of Gauss-Weierstrass summability, one can show that the function $\tilde{w}(x,t) = w(x,\sqrt{4\pi t})$ is the solution of the heat equation

$$\begin{aligned} \frac{\partial \tilde{w}}{\partial t} - \Delta\tilde{w} &= 0 \quad \text{on } \mathbb{R}^{n+1}_+, \\ \tilde{w}(x,0) &= f(x) \quad x \in \mathbb{R}^n, \end{aligned}$$

and (1.28) can be interpreted as the initial condition for the problem. We also have the formula

$$w(x,t) = W_t * f(x),$$

where W_t is the Gauss-Weierstrass kernel,

$$W_t(x) = t^{-n}e^{-\pi|x|^2/t^2}. \tag{1.31}$$

This formula can be proved using Lemma 1.14 and (1.18).

10. Notes and further results

10.1. References.

The classic reference on trigonometric series is the book by Zygmund [**21**], which will also be a useful reference for results in the next few chapters. However, this work can be difficult to consult at times. Another comprehensive reference on trigonometric series is the book by Bary [**1**].

There are excellent discussions of Fourier series and integrals in Katznelson [**10**] and Dym and McKean [**4**]. The book by R. E. Edwards [**5**] is an exhaustive study of Fourier series from a more modern perspective. The article by Weiss [**20**] and the book by Körner [**12**] are also recommended. An excellent historical account by J. P. Kahane on Fourier series and their influence on the development of mathematical concepts is found in the first half of [**9**]. The book *Fourier Analysis and Boundary Value Problems*, by E. González-Velasco (Academic Press, New York, 1995), contains many applications of Fourier's method of separation of variables to partial differential equations and also contains historical information. (Also see by the same author, *Connections in mathematical analysis: the case of Fourier series*, Amer. Math. Monthly **99** (1992), 427–441.) The book by O. G. Jørsboe and L. Melbro (*The Carleson-Hunt Theorem on Fourier Series*, Lecture Notes in Math. **911**, Springer-Verlag, Berlin, 1982) is devoted to the proof of this theorem. The original references for this are the articles by L. Carleson (*On convergence and growth of partial sums of Fourier series*, Acta Math. **116** (1966), 135–157) and R. Hunt (*On the convergence of Fourier series*, Orthogonal Expansions and their Continuous Analogues (Proc. Conf., Edwardsville, Ill., 1967), pp. 235–255, Southern Illinois Univ. Press, Carbondale, 1968). Kolmogorov's example of an L^1 function whose Fourier series diverges everywhere appeared in *Une série de Fourier-Lebesgue divergente partout* (C. R. Acad. Sci. Paris **183** (1926), 1327–1328).

10.2. Multiple Fourier series.

Let $\mathbb{T}^n$ be the n-dimensional torus (which we can identify with the quotient group $\mathbb{R}^n/\mathbb{Z}^n$). A function defined on $\mathbb{T}^n$ is equivalent to a function on $\mathbb{R}^n$ which has period 1 in each variable. If $f \in L^1(\mathbb{R}^n)$ then we can define its Fourier coefficients by

$$\hat{f}(\nu) = \int f(x) e^{-2\pi i x\cdot\nu}\, dx, \quad \nu \in \mathbb{Z}^n,$$

and construct the Fourier series of f with these coefficients,

$$\sum_{\nu\in\mathbb{Z}^n} \hat{f}(\nu) e^{2\pi i x\cdot\nu}.$$

One can prove several results similar to those for Fourier series in one variable, but one needs increasingly restrictive regularity conditions as n increases. See Stein and Weiss [**18**, Chapter 7].

10.3. The Poisson summation formula.

Let f be a function such that for some $\delta > 0$,

$$|f(x)| \leq A(1+|x|)^{-n-\delta} \quad \text{and} \quad |\hat{f}(\xi)| \leq A(1+|\xi|)^{-n-\delta}.$$

(In particular, f and $\hat{f}$ are both continuous.) Then

$$\sum_{\nu \in \mathbb{Z}^n} f(x+\nu) = \sum_{\nu \in \mathbb{Z}^n} \hat{f}(\nu) e^{2\pi i x \cdot \nu}.$$

This equality (or more precisely, the case when $x = 0$) is known as the Poisson summation formula and is nothing more than the inversion formula. The left-hand side defines a function on $\mathbb{T}^n$ whose Fourier coefficients are precisely $\hat{f}(\nu)$.

10.4. Gibbs phenomenon.

Let $f(x) = \operatorname{sgn}(x)$ on $(-1/2, 1/2)$. By Dirichlet's criterion, for example, we know that $S_N f(x)$ converges to $f(x)$ for all x. To the right of 0 the partial sums oscillate around 1 but, contrary to what one might expect, the amount by which they overstep 1 does not tend to 0 as N increases. One can show that

$$\lim_{N\to\infty} \sup_x S_N f(x) = \frac{2}{\pi} \int_0^\pi \frac{\sin(y)}{y}\, dy \approx 1.17898\ldots\ .$$

This phenomenon occurs whenever a function has a jump discontinuity. It is named after J. Gibbs, who announced it in Nature **59** (1899), although it had already been discovered by H. Wilbraham in 1848. See Dym and McKean [**4**, Chapter 1] and the paper by E. Hewitt and R. E. Hewitt (*The Gibbs-Wilbraham phenomenon: an episode in Fourier analysis*, Arch. Hist. Exact Sci. **21** (1979/80), 129–160).

Gibbs phenomenon is eliminated by replacing pointwise convergence by Cesàro summability. For if $m \le f(x) \le M$, then by the first two properties of Féjer kernels in (1.8), $m \le \sigma_N f(x) \le M$. In fact, it can be shown that if $m \le f(x) \le M$ on an interval (a, b), then for any $\epsilon > 0$, $m - \epsilon \le \sigma_N f(x) \le M + \epsilon$ on $(a+\epsilon, b-\epsilon)$ for N sufficiently large.

10.5. The Hausdorff-Young inequality.

Corollary 1.20 was gotten by an immediate application of Riesz-Thorin interpolation. But in fact a stronger inequality is true: if $1 \le p \le 2$ then

$$\|\hat{f}\|_{p'} \le \left(\frac{p^{1/p}}{(p')^{1/p'}} \right)^{n/2} \|f\|_p.$$

This inequality is sharp since equality holds for $f(x) = e^{-\pi|x|^2}$. This result was proved by W. Beckner (*Inequalities in Fourier analysis*, Ann. of Math. **102** (1975), 159–182); the special case when p is even was proved earlier by K. I. Babenko (*An inequality in the theory of Fourier integrals* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **25** (1961), 531–542).

In the same article, Beckner also proved a sharp version of Young's inequality (Corollary 1.21).

10.6. Eigenfunctions for the Fourier transform in $L^2(\mathbb{R})$.

Since the Fourier transform has period 4, if f is a function such that $\hat{f} = \lambda f$, we must have that $\lambda^4 = 1$. Hence, $\lambda = \pm 1, \pm i$ are the only possible eigenvalues of the Fourier transform. Lemma 1.14 shows that $\exp(-\pi x^2)$ is an eigenfunction associated with the eigenvalue 1. The Hermite functions give the remaining eigenfunctions: for $n \geq 0$,

$$h_n(x) = \frac{(-1)^n}{n!} \exp(\pi x^2) \frac{d^n}{dx^n} \exp(-\pi x^2)$$

satisfies $\hat{h}_n = (-i)^n h_n$. If we normalize these functions,

$$e_n = \frac{h_n}{\|h_n\|_2} = [(4\pi)^{-n}\sqrt{2}n!]^{1/2} h_n,$$

we get an orthonormal basis of $L^2(\mathbb{R})$ such that

$$\hat{f} = \sum_{n=0}^{\infty} (-i)^n \langle f, e_n \rangle e_n.$$

Thus $L^2(\mathbb{R})$ decomposes into the direct sum $H_0 \oplus H_1 \oplus H_2 \oplus H_3$, where on the subspace H_j, $0 \leq j \leq 3$, the Fourier transform acts by multiplying functions by i^j.

This approach to defining the Fourier transform in $L^2(\mathbb{R})$ is due to N. Wiener and can be found in his book (*The Fourier Integral and Certain of its Applications*, original edition, 1933; Cambridge Univ. Press, Cambridge, 1988). Also see Dym and McKean [**4**, Chapter 2].

In higher dimensions, the eigenfunctions of the Fourier transform are products of Hermite functions, one in each coordinate variable. Also see Chapter 4, Section 7.2.

10.7. Interpolation of analytic families of operators.

The Riesz-Thorin interpolation theorem has a powerful generalization due to E. M. Stein. (See Stein and Weiss [**18**, Chapter 5].) Let $S = \{z \in \mathbb{C} : 0 \leq \operatorname{Re} z \leq 1\}$ and let $\{T_z\}_{z \in S}$ be a family of operators. This family is said to be admissible if given two functions $f, g \in L^1(\mathbb{R}^n)$, the mapping

$$z \mapsto \int_{\mathbb{R}^n} T_z(f) g \, dx$$

is analytic on the interior of S and continuous on the boundary, and if there exists a constant $a < \pi$ such that

$$e^{-a|\operatorname{Im} z|} \log \left| \int_{\mathbb{R}^n} T_z(f) g \, dx \right|$$

is uniformly bounded for all $z \in S$.

Theorem 1.22. *Let $\{T_z\}$ be an admissible family of operators, and suppose that for $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and $y \in \mathbb{R}$,*

$$\|T_{iy}f\|_{q_0} \leq M_0(y)\|f\|_{p_0} \quad \text{and} \quad \|T_{1+iy}f\|_{q_1} \leq M_1(y)\|f\|_{p_1},$$

where for some $b < \pi$

$$\sup_{y\in\mathbb{R}} e^{-b|y|} \log M_j(y) < \infty, \quad j = 1, 2.$$

Then for $0 < \theta < 1$, $\operatorname{Re} z = \theta$ and p, q defined as in Theorem 1.19, there exists a constant M_θ such that

$$\|T_z f\|_q \leq M_\theta \|f\|_p.$$

10.8. Fourier transforms of finite measures.

As we noted above, if μ is a finite Borel measure then $\hat{\mu}$ is a bounded, continuous function. The collection of all such functions obtained in this way is characterized by the following result.

Theorem 1.23. *If h is a bounded, continuous function, then the following are equivalent:*

(1) *$h = \hat{\mu}$ for some positive, finite Borel measure μ;*
(2) *h is positive definite: given any $f \in L^1(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} h(x-y)f(x)\bar{f}(y)\,dxdy \geq 0.$$

This theorem is due to S. Bochner (*Lectures on Fourier Integrals*, Princeton Univ. Press, Princeton, 1959; translated from *Vorlesungen über Fouriersche Integrale*, Akad. Verlag, Leipzig, 1932). Also see Katznelson [**10**, Chapter 6].