Geometry of Surfaces Homework 1

Instructor: **Ling**

Due: None

Exercises for First class 0118

(($\star$) means this exercise may be a little hard, do it carefully!)

1. Is $\gamma(t) = (t^2, t^4)$ a parametrization of the parabola $y = x^2$?

2. Find a parametrization of the following level curve:
$$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$

3. Find the Cartesian equations of the following parametrized curves:
   (i) $\gamma(t) = \big(\cos^2 t, \sin^2 t\big)$;
   (ii) $\gamma(t) = \big(e^t, t^2\big)$.

4. Calculate the tangent vectors of the curves in Exercise 3.

5. ($\star$) The normal line to a curve at a point $\mathbf{p}$ is the straight line passing through $\mathbf{p}$ perpendicular to the tangent line at $\mathbf{p}$. Find the tangent and normal lines to the curve $\gamma(t) = (2\cos t - \cos 2t, 2\sin t - \sin 2t)$ at the point corresponding to $t = \pi/4$.

6. Calculate the arc-length of the catenary $\gamma(t) = (t, \cosh t)$ starting at the point $(0, 1)$.

7. Show that the following curves are unit-speed:
   (i) $\gamma(t) = \Big(\frac{1}{3}(1+t)^{3/2}, \frac{1}{3}(1-t)^{3/2}, \frac{t}{\sqrt{2}}\Big)$.
   (ii) $\gamma(t) = \big(\frac{4}{5}\cos t, 1 - \sin t, -\frac{3}{5}\cos t\big)$.

8. ($\star$) This exercise shows that *a straight line is the shortest curve joining two given points.* Let $\mathbf{p}$ and $\mathbf{q}$ be the two points, and let γ be a curve passing through both, say $\gamma(a) = \mathbf{p}, \gamma(b) = \mathbf{q}$, where $a < b$. Show that, if $\mathbf{u}$ is any unit vector, then there is
$$\dot{\gamma} \cdot \mathbf{u} \leq \|\dot{\gamma}\|,$$
and deduce that
$$(\mathbf{q} - \mathbf{p}) \cdot \mathbf{u} \leq \int_a^b \|\dot{\gamma}\| dt.$$
By taking $\mathbf{u} = (\mathbf{q} - \mathbf{p})/\|\mathbf{q} - \mathbf{p}\|$, show that the length of the part of γ between $\mathbf{p}$ and $\mathbf{q}$ is at least the straight line distance $\|\mathbf{q} - \mathbf{p}\|$.

   Hint: Use Cauchy-Schwarz inequality and the fundamental theorem of calculus. □

9. Which of the following curves are regular?
   (i) $\gamma(t) = (\cos^2 t, \sin^2 t)$ for $t \in \mathbb{R}$.
   (ii) The same curve as in (i), but with $0 < t < \pi/2$.
   (iii) $\gamma(t) = (t, \cosh t)$ for $t \in \mathbb{R}$.
   Find unit-speed reparametrizations of the regular curve(s).

10. The simplest type of singular point of a curve γ is an *ordinary cusp*: a point $\mathbf{p}$ of γ, corresponding to a parameter value t_0, say, is an ordinary cusp if $\dot{\gamma}(t_0) = \mathbf{0}$ and the vectors $\ddot{\gamma}(t_0)$ and $\dddot{\gamma}(t_0)$ are linearly independent (in particular, these vectors must both be nonzero). Show that: The curve $\gamma(t) = (t^m, t^n)$, where m and n are positive integers, has an ordinary cusp at the origin if and only if $(m, n) = (2, 3)$ or $(3, 2)$.

11. ($\star$) Give an example to show that a reparametrization of a closed curve need not be closed.

12. Show that if a curve γ is T_1-periodic and T_2-periodic, then it is $(k_1 T_1 + k_2 T_2)$-periodic for any integers k_1, k_2.

13. Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ be a curve and suppose that T_0 is the smallest positive number such that γ is T_0-periodic. Prove that γ is T-periodic if and only if $T = kT_0$ for some integer k.

14. ($\star$) Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ be a non-constant curve that is T-periodic for some $T > 0$. Show that γ is closed.

Geometry of Surfaces Homework 2

Instructor: **Ling**

Due: None

Exercises for Second class 0201

(($\star$) means this exercise may be a little hard, do it carefully!)

1. Compute the curvature of the following curves:
   (i) $\gamma(t)=\left(\frac{1}{3}(1+t)^{3/2},\frac{1}{3}(1-t)^{3/2},\frac{t}{\sqrt{2}}\right)$.
   (ii) $\gamma(t)=\left(\frac{4}{5}\cos t,1-\sin t,-\frac{3}{5}\cos t\right)$.
   (iii) $\gamma(t)=(t,\cosh t)$
   (iv) $\gamma(t)=\left(\cos^3 t,\sin^3 t\right)$

2. ($\star$) Show that, if the curvature $\kappa(t)$ of a regular curve $\gamma(t)$ is >0 everywhere, then $\kappa(t)$ is a smooth function of t. Give an example to show that this may not be the case without the assumption that $\kappa>0$.

3. Show that, if γ is a unit-speed plane curve,
$$\dot{\mathbf{n}}_s=-\kappa_s\mathbf{t}.$$

4. Show that the signed curvature of any regular plane curve $\gamma(t)$ is a smooth function of t. (Compare with Exercise 2.)

5. Let $\gamma(t)$ be a regular plane curve and let λ be a constant. The parallel curve γ^λ of γ is defined by
$$\gamma^\lambda(t)=\gamma(t)+\lambda\mathbf{n}_s(t).$$
   Show that, if $\lambda\kappa_s(t)\neq 1$ for all values of t, then γ^λ is a regular curve and that its signed curvature is $\kappa_s/\left|1-\lambda\kappa_s\right|$.

6. ($\star$) Another approach to the curvature of a unit-speed plane curve γ at a point $\gamma\left(s_0\right)$ is to look for the 'best approximating circle' at this point. We can then define the curvature of γ to be the reciprocal of the radius of this circle.

   Carry out this programme by showing that the centre of the circle which passes through three nearby points $\gamma\left(s_0\right)$ and $\gamma\left(s_0\pm\delta s\right)$ on γ approaches the point
$$\boldsymbol{\epsilon}\left(s_0\right)=\gamma\left(s_0\right)+\frac{1}{\kappa_s\left(s_0\right)}\mathbf{n}_s\left(s_0\right)$$
   as δs tends to zero. The circle $\mathcal{C}$ with centre $\boldsymbol{\epsilon}\left(s_0\right)$ passing through $\gamma\left(s_0\right)$ is called the osculating circle to γ at the point $\gamma\left(s_0\right)$, and $\boldsymbol{\epsilon}\left(s_0\right)$ is called the centre of curvature of $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}\left(s_0\right)$. The radius of $\mathcal{C}$ is $1/\left|\kappa_s\left(s_0\right)\right|=1/\kappa\left(s_0\right)$, where κ is the curvature of γ-this is called the radius of curvature of γ at $\gamma\left(s_0\right)$.

7. ($\star$) A string of length ℓ is attached to the point $\gamma(0)$ of a unit-speed plane curve $\gamma(s)$. Show that when the string is wound onto the curve while being kept taught, its endpoint traces out the curve
$$\iota(s)=\gamma(s)+(\ell-s)\dot{\gamma}(s),$$

where $0 < s < \ell$ and a dot denotes d/ds. The curve ι is called the involute of γ (if γ is any regular plane curve, we define its involute to be that of a unit-speed reparametrization of γ). Suppose that the signed curvature κ_s of γ is never zero, say $\kappa_s(s) > 0$ for all s. Show that the signed curvature of ι is $1/(\ell - s)$.

8. Compute $\kappa, \tau, \mathbf{t}, \mathbf{n}$ and $\mathbf{b}$ for each of the following curves, and verify that the Frenet-Serret equations are satisfied:
(i) $\gamma(t) = \left(\frac{1}{3}(1+t)^{3/2}, \frac{1}{3}(1-t)^{3/2}, \frac{t}{\sqrt{2}}\right)$.
(ii) $\gamma(t) = \left(\frac{4}{5}\cos t, 1 - \sin t, -\frac{3}{5}\cos t\right)$.
Show that the curve in (ii) is a circle, and find its centre, radius and the plane in which it lies.

9. ($\star$) Let $\gamma(t)$ be a unit-speed curve with $\kappa(t) > 0$ and $\tau(t) \neq 0$ for all t. Show that, if γ is spherical, i.e., if it lies on the surface of a sphere, then

$$\frac{\tau}{\kappa} = \frac{d}{ds}\left(\frac{\dot{\kappa}}{\tau\kappa^2}\right).$$

Conversely, show that if the equation holds, then

$$\rho^2 + (\dot{\rho}\sigma)^2 = r^2$$

for some (positive) constant r, where $\rho = 1/\kappa$ and $\sigma = 1/\tau$, and deduce that γ lies on a sphere of radius r.

10. Let P be an $n \times n$ orthogonal matrix and let $\mathbf{a} \in \mathbb{R}^n$, so that $M(\mathbf{v}) = P\mathbf{v} + \mathbf{a}$ is an isometry of $\mathbb{R}^3$. Show that, if γ is a unit-speed curve in $\mathbb{R}^n$, the curve $\mathbf{\Gamma} = M(\gamma)$ is also unit-speed. Show also that, if $\mathbf{t}, \mathbf{n}, \mathbf{b}$ and $\mathbf{T}, \mathbf{N}, \mathbf{B}$ are the tangent vector, principal normal and binormal of γ and $\mathbf{\Gamma}$, respectively, then $\mathbf{T} = P\mathbf{t}, \mathbf{N} = P\mathbf{n}$ and $\mathbf{B} = P\mathbf{b}$.

11. ($\star$) Let (a_{ij}) be a skew-symmetric 3×3 matrix (i.e., $a_{ij} = -a_{ji}$ for all i, j). Let $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ be smooth functions of a parameter s satisfying the differential equations

$$\dot{\mathbf{v}}_i = \sum_{j=1}^{3} a_{ij}\mathbf{v}_j$$

for $i = 1, 2$ and 3 , and suppose that for some parameter value s_0 the vectors $\mathbf{v}_1(s_0), \mathbf{v}_2(s_0)$ and $\mathbf{v}_3(s_0)$ are orthonormal. Show that the vectors $\mathbf{v}_1(s), \mathbf{v}_2(s)$ and $\mathbf{v}_3(s)$ are orthonormal for all values of s.

12. Show that any open disc in the xy-plane is a surface. Show that every open subset of a surface is a surface.

Geometry of Surfaces Homework 3

Instructor: **Ling**

Due: None

Exercises for Third class 0208

(($\star$) means this exercise may be a little hard, do it carefully!)

1. Show that, if $f(x, y)$ is a smooth function, its graph
$$\{(x, y, z) \in \mathbb{R}^3 \mid z = f(x, y)\}$$
is a smooth surface with atlas consisting of the single regular surface patch
$$\boldsymbol{\sigma}(u, v) = (u, v, f(u, v)).$$

2. Which of the following are regular surface patches (in each case, $u, v \in \mathbb{R}$)
(i) $\boldsymbol{\sigma}(u, v) = (u, v, uv)$.
(ii) $\boldsymbol{\sigma}(u, v) = \left(u, v^2, v^3\right)$.
(iii) $\boldsymbol{\sigma}(u, v) = \left(u + u^2, v, v^2\right)$?

3. Show that the ellipsoid
$$\frac{x^2}{p^2} + \frac{y^2}{q^2} + \frac{z^2}{r^2} = 1,$$
where p, q and r are non-zero constants, is a smooth surface.

4. Find the equation of the tangent plane of each of the following surface patches at the indicated points:
(i) $\boldsymbol{\sigma}(u, v) = \left(u, v, u^2 - v^2\right), (1, 1, 0)$.
(ii) $\boldsymbol{\sigma}(r, \theta) = \left(r \cosh \theta, r \sinh \theta, r^2\right), (1, 0, 1)$.

5. Show that, if $\boldsymbol{\sigma}(u, v)$ is a surface patch, the set of linear combinations of $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ is unchanged when $\boldsymbol{\sigma}$ is reparametrized.

6. ($\star$) Let $\mathcal{S}$ be a surface, let $\mathbf{p} \in \mathcal{S}$ and let $F : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. Let $\nabla_{\mathcal{S}} F$ be the perpendicular projection of the gradient $\boldsymbol{\nabla} F = (F_x, F_y, F_z)$ of F onto $T_{\mathbf{p}}\mathcal{S}$. Show that, if γ is any curve on $\mathcal{S}$ passing through $\mathbf{p}$ when $t = t_0$, say,
$$(\boldsymbol{\nabla}_{\mathcal{S}} F) \cdot \dot{\gamma}(t_0) = \left.\frac{d}{dt}\right|_{t=t_0} F(\gamma(t)).$$
Deduce that $\nabla_{\mathcal{S}} F = \mathbf{0}$ if the restriction of F to $\mathcal{S}$ has a local maximum or a local minimum at $\mathbf{p}$.

7. Let $f : \mathcal{S}_1 \to \mathcal{S}_2$ be a local diffeomorphism and let γ be a regular curve on $\mathcal{S}_1$. Show that $f \circ \gamma$ is a regular curve on $\mathcal{S}_2$.

8. Calculate the first fundamental forms of the following surfaces:
(i) $\boldsymbol{\sigma}(u, v) = (\sinh u \sinh v, \sinh u \cosh v, \sinh u)$.
(ii) $\boldsymbol{\sigma}(u, v) = \left(u - v, u + v, u^2 + v^2\right)$.
(iii) $\boldsymbol{\sigma}(u, v) = (\cosh u, \sinh u, v)$.
(iv) $\boldsymbol{\sigma}(u, v) = \left(u, v, u^2 + v^2\right)$.
What kinds of surfaces are these?

9. Show that applying an isometry of $\mathbb{R}^3$ to a surface does not change its first fundamental form. What is the effect of a dilation (i.e., a map $\mathbb{R}^3 \to \mathbb{R}^3$ of the form $\mathbf{v} \mapsto a\mathbf{v}$ for some constant $a \neq 0$)?

10. Suppose that a surface patch $\tilde{\boldsymbol{\sigma}}(\tilde{u}, \tilde{v})$ is a reparametrization of a surface patch $\boldsymbol{\sigma}(u, v)$, and let

$$\tilde{E}d\tilde{u}^2 + 2\tilde{F}d\tilde{u}d\tilde{v} + \tilde{G}d\tilde{v}^2 \text{ and } Edu^2 + 2Fdudv + Gdv^2$$

be their first fundamental forms. Show that:

(i) $du = \frac{\partial u}{\partial \tilde{u}} d\tilde{u} + \frac{\partial u}{\partial \tilde{v}} d\tilde{v}, \quad dv = \frac{\partial v}{\partial \tilde{u}} d\tilde{u} + \frac{\partial v}{\partial \tilde{v}} d\tilde{v}.$

(ii) If

$$J = \begin{pmatrix} \frac{\partial u}{\partial \tilde{u}} & \frac{\partial u}{\partial \tilde{v}} \\ \frac{\partial v}{\partial \tilde{u}} & \frac{\partial v}{\partial \tilde{v}} \end{pmatrix}$$

is the Jacobian matrix of the reparametrization map $(\tilde{u}, \tilde{v}) \mapsto (u, v)$, and J^t is the transpose of J, then

$$\begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} = J^t \begin{pmatrix} E & F \\ F & G \end{pmatrix} J.$$

Geometry of Surfaces Homework 4

Instructor: **Ling**

Due: None

Exercises for Fourth class 0215

(($\star$) means this exercise may be a little hard, do it carefully!)

1. Calculate the first fundamental forms of the following surfaces:
   (i) $\boldsymbol{\sigma}(u,v) = (\sinh u \sinh v, \sinh u \cosh v, \sinh u)$.
   (ii) $\boldsymbol{\sigma}(u,v) = \left(u - v, u + v, u^2 + v^2\right)$.
   (iii) $\boldsymbol{\sigma}(u,v) = (\cosh u, \sinh u, v)$.
   (iv) $\boldsymbol{\sigma}(u,v) = \left(u, v, u^2 + v^2\right)$.
   What kinds of surfaces are these?

2. Show that applying an isometry of $\mathbb{R}^3$ to a surface does not change its first fundamental form. What is the effect of a dilation (i.e., a map $\mathbb{R}^3 \to \mathbb{R}^3$ of the form $\mathbf{v} \mapsto a\mathbf{v}$ for some constant $a \neq 0$)?

3. Suppose that a surface patch $\tilde{\boldsymbol{\sigma}}(\tilde{u}, \tilde{v})$ is a reparametrization of a surface patch $\boldsymbol{\sigma}(u,v)$, and let

$$\tilde{E} d\tilde{u}^2 + 2\tilde{F} d\tilde{u} d\tilde{v} + \tilde{G} d\tilde{v}^2 \text{ and } E du^2 + 2F du dv + G dv^2$$

   be their first fundamental forms. Show that:
   (i) $du = \frac{\partial u}{\partial \tilde{u}} d\tilde{u} + \frac{\partial u}{\partial \tilde{v}} d\tilde{v}, \quad dv = \frac{\partial v}{\partial \tilde{u}} d\tilde{u} + \frac{\partial v}{\partial \tilde{v}} d\tilde{v}$.
   (ii) If

$$J = \begin{pmatrix} \frac{\partial u}{\partial \tilde{u}} & \frac{\partial u}{\partial \tilde{v}} \\ \frac{\partial v}{\partial \tilde{u}} & \frac{\partial v}{\partial \tilde{v}} \end{pmatrix}$$

   is the Jacobian matrix of the reparametrization map $(\tilde{u}, \tilde{v}) \mapsto (u, v)$, and J^t is the transpose of J, then

$$\begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} = J^t \begin{pmatrix} E & F \\ F & G \end{pmatrix} J.$$

4. Show that every local isometry is conformal. Give an example of a conformal map that is not a local isometry.

5. Show that Enneper's surface

$$\boldsymbol{\sigma}(u,v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2\right)$$

   is conformally parametrized.

6. ($\star$) Let $\Phi : U \to V$ be a diffeomorphism between open subsets of $\mathbb{R}^2$. Write

$$\Phi(u,v) = (f(u,v), g(u,v)),$$

   where f and g are smooth functions on the uv-plane. Show that Φ is conformal if and only if either ($f_u = g_v$ and $f_v = -g_u$) or ($f_u = -g_v$ and $f_v = g_u$). Show that, if $J(\Phi)$ is the Jacobian matrix of Φ, then $\det(J(\Phi)) > 0$ in the first case and $\det(J(\Phi)) < 0$ in the second case.

7. Compute the second fundamental form of the elliptic paraboloid

$$\boldsymbol{\sigma}(u,v) = \left(u, v, u^2 + v^2\right).$$

8. Suppose that the second fundamental form of a surface patch σ is zero everywhere. Prove that $\boldsymbol{\sigma}$ is an open subset of a plane. This is the analogue for surfaces of the theorem that a curve with zero curvature everywhere is part of a straight line.

9. Let a surface patch $\tilde{\boldsymbol{\sigma}}(\tilde{u}, \tilde{v})$ be a reparametrization of a surface patch $\boldsymbol{\sigma}(u, v)$ with reparametrization map$(u, v) = \Phi(\tilde{u}, \tilde{v})$. Prove that

$$\begin{pmatrix} \tilde{L} & \tilde{M} \\ \tilde{M} & \tilde{N} \end{pmatrix} = \pm J^t \begin{pmatrix} L & M \\ M & N \end{pmatrix} J,$$

where J is the Jacobian matrix of Φ and we take the plus sign if $\det(J) > 0$ and the minus sign if $\det(J) < 0$. Deduce from Exercise 6.1.4 that the second fundamental form of a surface patch is unchanged by a reparametrization of the patch which preserves its orientation.

Geometry of Surfaces Homework 5

Instructor: **Ling**

Due: None

Exercises for Fifth class 0222

(($\star$) means this exercise may be a little hard, do it carefully!)

1. Compute the second fundamental form of the elliptic paraboloid

$$\boldsymbol{\sigma}(u,v) = \left(u, v, u^2+v^2\right).$$

2. Suppose that the second fundamental form of a surface patch σ is zero everywhere. Prove that $\boldsymbol{\sigma}$ is an open subset of a plane. This is the analogue for surfaces of the theorem that a curve with zero curvature everywhere is part of a straight line.

3. Calculate the Gauss map $\mathcal{G}$ of the paraboloid $\mathcal{S}$ with equation $z = x^2+y^2$. What is the image of $\mathcal{G}$?

4. ($\star$)Let γ be a regular, but not necessarily unit-speed, curve on a surface. Prove that (with the usual notation) the normal and geodesic curvatures of γ are

$$\kappa_n = \frac{\langle\langle\dot{\gamma},\dot{\gamma}\rangle\rangle}{\langle\dot{\gamma},\dot{\gamma}\rangle} \text{ and } \kappa_g = \frac{\ddot{\gamma}\cdot(\mathbf{N}\times\dot{\gamma})}{\langle\dot{\gamma},\dot{\gamma}\rangle^{3/2}}.$$

5. Show that the normal curvature of any curve on a sphere of radius r is $\pm 1/r$.

6. Calculate the principal curvatures of the helicoid

$$\boldsymbol{\sigma}(u,v) = (v\cos u, v\sin u, \lambda u).$$

7. ($\star$)Show that, if $\gamma(t) = \boldsymbol{\sigma}(u(t), v(t))$ is a unit-speed curve on a surface patch σ with first fundamental form $Edu^2 + 2Fdudv + Gdv^2$, the geodesic curvature of γ is

$$\kappa_g = (\ddot{v}\dot{u} - \dot{v}\ddot{u})\sqrt{EG-F^2} + A\dot{u}^3 + B\dot{u}^2\dot{v} + C\dot{u}\dot{v}^2 + D\dot{v}^3,$$

where A, B, C and D can be expressed in terms of E, F, G and their derivatives. Find A, B, C, D explicitly when $F = 0$.

8. A curve γ on a surface $\mathcal{S}$ is called asymptotic if its normal curvature is everywhere zero. Show that any straight line on a surface is an asymptotic curve. Show also that a curve γ with positive curvature is asymptotic if and only if its binormal $\mathbf{b}$ is parallel to the unit normal of $\mathcal{S}$ at all points of γ.

9. Prove that the asymptotic curves on the surface

$$\boldsymbol{\sigma}(u,v) = (u\cos v, u\sin v, \ln u)$$

are given by

$$\ln u = \pm(v+c),$$

where c is an arbitrary constant.

Geometry of Surfaces   Homework 6

Instructor: **Ling**

Due: None

Exercises for Sixth class 0301

(($\star$) means this exercise may be a little hard, do it carefully!)

1. Show that the Gaussian and mean curvatures of the surface $z=f(x,y)$, where f is a smooth function, are
$$K=\frac{f_{xx}f_{yy}-f_{xy}^2}{\left(1+f_x^2+f_y^2\right)^2}, \quad H=\frac{\left(1+f_y^2\right)f_{xx}-2f_xf_yf_{xy}+\left(1+f_x^2\right)f_{yy}}{2\left(1+f_x^2+f_y^2\right)^{3/2}}.$$

2. Show that the Gaussian and mean curvatures of a surface $\mathcal{S}$ are smooth functions on $\mathcal{S}$.

3. Show the Gaussian curvature of a ruled surface
$$\boldsymbol{\sigma}(u,v)=\gamma(u)+v\boldsymbol{\delta}(u)$$
is non-positive.

4. In the notation of Exercise 3, show that if $\boldsymbol{\delta}$ is the principal normal $\mathbf{n}$ of γ or its binormal $\mathbf{b}$, then $K=0$ if and only if γ is planar.

5. ($\star$) Let $\boldsymbol{\sigma}: U\to\mathbb{R}^3$ be a patch of a surface $\mathcal{S}$. Show that the image under the Gauss map of the part $\boldsymbol{\sigma}(R)$ of $\mathcal{S}$ corresponding to a region $R\subseteq U$ has area
$$\int_R |K| d\mathcal{A}_{\boldsymbol{\sigma}}$$
where K is the Gaussian curvature of $\mathcal{S}$.

6. Prove that any geodesic has constant speed.

7. ($\star$) Let $\gamma(t)$ be a geodesic on an ellipsoid $\mathcal{S}$. Let $2R(t)$ be the length of the diameter of $\mathcal{S}$ parallel to $\dot\gamma(t)$, and let $S(t)$ be the distance from the centre of $\mathcal{S}$ to the tangent plane $T_{\boldsymbol{\gamma}(t)}\mathcal{S}$. Show that the curvature of γ is $S(t)/R(t)^2$, and that the product $R(t)S(t)$ is independent of t.

8. ($\star$) Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two surfaces that intersect in a curve $\mathcal{C}$, and let γ be a unit-speed parametrization of $\mathcal{C}$.

   (i) Show that if γ is a geodesic on both $\mathcal{S}_1$ and $\mathcal{S}_2$ and if the curvature of γ is nowhere zero, then $\mathcal{S}_1$ ad $\mathcal{S}_2$ touch along γ (i.e., they have the same tangent plane at each point of $\mathcal{C}$). Give an example of this situation.

   (ii) Show that if $\mathcal{S}_1$ and $\mathcal{S}_2$ intersect orthogonally at each point of $\mathcal{C}$, then γ is a geodesic on $\mathcal{S}_1$ if and only if $\dot{\mathbf{N}}_2$ is parallel to $\mathbf{N}_1$ at each point of $\mathcal{C}$ (where $\mathbf{N}_1$ and $\mathbf{N}_2$ are unit normals of $\mathcal{S}_1$ and $\mathcal{S}_2$). Show also that, in this case, γ is a geodesic on both $\mathcal{S}_1$ and $\mathcal{S}_2$ if and only if $\mathcal{C}$ is part of a straight line.

9. Construct a smooth function with the properties in the class we want in the following steps:

   (i) Show that, for all integers n (positive and negative), $t^n e^{-1/t^2}$ tends to 0 as t tends to 0 .

   (ii) Deduce from (i) that the function
$$\theta(t)=\begin{cases} e^{-1/t^2} & \text{if } t>0 \\ 0 & \text{if } t\leq 0\end{cases}$$

is smooth everywhere.

(iii) Show that the function

$$\psi(t) = \theta(1+t)\theta(1-t)$$

is smooth everywhere, that $\psi(t) > 0$ if $-1 < t < 1$, and that $\psi(t) = 0$ otherwise.

(iv) Show that the function

$$\phi(t) = \psi\left(\frac{t - t_0}{\eta}\right)$$

has the properties we want.

Geometry of Surfaces Homework 7

Instructor: **Ling**

Due: None

Exercises for Seventh class 0308

(($\star$) means this exercise may be a little hard, do it carefully!)

1. A surface patch has first and second fundamental forms

$$\cos^2 v du^2 + dv^2 \text{ and } -\cos^2 v du^2 - dv^2,$$

respectively. Show that the surface is an open subset of a sphere of radius one. Write down a parametrization of S^2 with these first and second fundamental forms.

2. Show that there is no surface patch whose first and second fundamental forms are

$$du^2 + \cos^2 u dv^2 \text{ and } \cos^2 u du^2 + dv^2,$$

respectively.

3. Show that if a surface patch has first fundamental form $e^\lambda \left(du^2 + dv^2\right)$, where λ is a smooth function of u and v, its Gaussian curvature K satisfies

$$\Delta\lambda + 2Ke^\lambda = 0$$

where Δ denotes the Laplacian $\partial^2/\partial u^2 + \partial^2/\partial v^2$.

4. Show that there is no isometry between any region of a sphere and any region of a (generalised) cylinder or a (generalised) cone.

5. The first fundamental form of Poincaré disc is

$$\frac{4\left(dv^2 + dw^2\right)}{\left(1 - v^2 - w^2\right)^2}.$$

Calculate its Gaussian curvature. (In particular, Poincaré disc is a conformal model of hyperbolic geometry.)