参考答案（评分标准由批卷老师协商决定）

1. (10分)求方程$(xy-x^3y^3)\,\mathrm{d}x+(1+x^2)\,\mathrm{d}y=0$满足条件$y(0)=1$的解.

解. $(1+x^2)\dfrac{\mathrm{d}y}{\mathrm{d}x}+xy=x^3y^3$, $\quad y^{-3}y'+\dfrac{x}{1+x^2}y^{-2}=\dfrac{x^3}{1+x^2}$,

$-2y^{-3}y'-2\dfrac{x}{1+x^2}y^{-2}=-2\dfrac{x^3}{1+x^2}$, $\quad (y^{-2})'-\dfrac{2x}{1+x^2}\,y^{-2}=\dfrac{-2x^3}{1+x^2}$,

$$y^{-2}=e^{\int\frac{2x}{1+x^2}\mathrm{d}x}\left[\int\frac{-2x^3}{1+x^2}e^{\int\frac{-2x}{1+x^2}\mathrm{d}x}\mathrm{d}x+c\right]=(1+x^2)\left[\int\frac{-2x^3}{(1+x^2)^2}\,\mathrm{d}x+c\right].$$

$$\int\frac{-2x^3}{(1+x^2)^2}\,\mathrm{d}x=-\int\frac{x^2}{(1+x^2)^2}\,\mathrm{d}(x^2+1)=\int x^2\,\mathrm{d}\frac{1}{1+x^2}$$
$$=\frac{x^2}{1+x^2}-\int\frac{1}{1+x^2}\,\mathrm{d}(x^2+1)=\frac{x^2}{1+x^2}-\ln(1+x^2).$$

所以, $y^{-2}=(1+x^2)\left[\dfrac{x^2}{1+x^2}-\ln(1+x^2)+c\right]=x^2-(1+x^2)\ln(1+x^2)+c(1+x^2)$,

其中c为任意常数. 代入初始条件$y(0)=1$得$c=1$,

从而所求解为$y=1\Big/\sqrt{[2x^2+1-(1+x^2)\ln(1+x^2)]}$.

注意此解写成 $y^{-2}=2x^2+1-(1+x^2)\ln(1+x^2)$ 不准确. □

2. (10分)求方程$x^2y''-3xy'+4y=0\ \ (x>0)$的满足条件$y(1)=1$, $y'(1)=1$的解, 其中$y'=\dfrac{\mathrm{d}y}{\mathrm{d}x}$, $y''=\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

解. 令$x=e^t$, $\ t=\ln x\ \ (x>0)$, 则$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\mathrm{d}y}{\mathrm{d}t}\dfrac{\mathrm{d}t}{\mathrm{d}x}=\dfrac{1}{x}\dfrac{\mathrm{d}y}{\mathrm{d}t}$,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}y'}{\mathrm{d}x}=-\frac{1}{x^2}\frac{\mathrm{d}y}{\mathrm{d}t}+\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)\frac{\mathrm{d}t}{\mathrm{d}x}=-\frac{1}{x^2}\frac{\mathrm{d}y}{\mathrm{d}t}+\frac{1}{x^2}\frac{\mathrm{d}^2y}{\mathrm{d}t^2}.$$

代入原方程得$\dfrac{\mathrm{d}^2y}{\mathrm{d}t^2}-4\dfrac{\mathrm{d}y}{\mathrm{d}t}+4y=0$. 其特征方程为$\lambda^2-4\lambda+4=0$, 特征根为$\lambda_{1,2}=2$.

所以$\dfrac{\mathrm{d}^2y}{\mathrm{d}t^2}-4\dfrac{\mathrm{d}y}{\mathrm{d}t}+4y=0$的通解为$y=(c_1+c_2t)e^{2t}$,

亦即原方程的通解为$y=(c_1+c_2\ln x)x^2$.

$y'=2x(c_1+c_2\ln x)+c_2x=(2c_1+c_2)x+2c_2x\ln x$.

$y(1)=1\ \Rightarrow\ c_1=1;\quad y'(1)=1\ \Rightarrow\ 2+c_2=1,\ c_2=-1$.

所以, 所求特解为$y=(1-\ln x)x^2$. □

3. (10分)求方程$y''+y'-2y=x+e^x+\sin x$的满足条件$y(0)=-\frac{7}{20}$, $y'(0)=\frac{38}{15}$的解, 其中$y'=\frac{\mathrm{d}y}{\mathrm{d}x}$, $y''=\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

解. $y''+y'-2y=0$的特征方程是$\lambda^2+\lambda-2=0$, 特征根是$\lambda_1=-2$, $\lambda_2=1$.
所以$y''+y'-2y=0$的通解是$y=c_1e^{-2x}+c_2e^x$, 其中c_1,c_2是任意常数.

设$y''+y'-2y=x$的特解是$y_1=ax+b$, 则$y_1'=a, y_1''=0$, 代入方程得
$a-2ax-2b=x$, $a=-\frac{1}{2}$, $b=\frac{a}{2}=-\frac{1}{4}$. 所以, $y_1=-\frac{x}{2}-\frac{1}{4}$.

设$y''+y'-2y=e^x$的特解是$y_2=cxe^x$, 则$y_2'=ce^x+cxe^x, y_2''=2ce^x+cxe^x$,
代入方程得$(2c+cx)+(c+cx)-2cx=1$, $c=\frac{1}{3}$. 所以, $y_2=\frac{x}{3}e^x$.

设$y''+y'-2y=\sin x$的特解是$y_3=A\sin x+B\cos x$,
则$y_3'=A\cos x-B\sin x, y_3''=-A\sin x-B\cos x$, 代入方程得
$(-A\sin x-B\cos x)+(A\cos x-B\sin x)-2(A\sin x+B\cos x)=\sin x$,
$-(3A+B)=1$, $A-3B=0$; $B=-\frac{1}{10}$, $A=-\frac{3}{10}$.
所以, $y_3=-\frac{3}{10}\sin x-\frac{1}{10}\cos x$.

综上, 原方程的通解为
$$y=c_1e^{-2x}+c_2e^x+y_1+y_2+y_3=c_1e^{-2x}+c_2e^x-\frac{x}{2}-\frac{1}{4}+\frac{x}{3}e^x-\frac{3}{10}\sin x-\frac{1}{10}\cos x.$$
$$y'=-2c_1e^{-2x}+c_2e^x-\frac{1}{2}+\frac{1}{3}e^x+\frac{x}{3}e^x-\frac{3}{10}\cos x+\frac{1}{10}\sin x.$$
$$y(0)=-\frac{7}{20}\quad\Rightarrow\quad c_1+c_2-\frac{1}{4}-\frac{1}{10}=-\frac{7}{20}\quad\Rightarrow\quad c_1+c_2=0.$$
$$y'(0)=\frac{38}{15}\quad\Rightarrow\quad -2c_1+c_2-\frac{1}{2}+\frac{1}{3}-\frac{3}{10}=\frac{38}{15}\quad\Rightarrow\quad -2c_1+c_2=3.$$
$$\begin{cases} c_1+c_2=0 \\ -2c_1+c_2=3 \end{cases}\quad\Rightarrow\quad c_1=-1,\ c_2=1.$$
故所求解为 $y=-e^{-2x}+e^x-\frac{x}{2}-\frac{1}{4}+\frac{x}{3}e^x-\frac{3}{10}\sin x-\frac{1}{10}\cos x$. □

4. (10分)设$I(R)=\oint\limits_{x^2+y^2=R^2}\dfrac{x\,\mathrm{d}y-y\,\mathrm{d}x}{(x^2+xy+y^2)^2}$, 证明$\lim\limits_{R\to+\infty}I(R)=0$.

证明. $x^2+y^2=1$的参数式为$\begin{cases}x=R\cos\theta,\\ y=R\sin\theta.\end{cases}\quad 0\leqslant\theta\leqslant 2\pi.$

$$I(R)=\oint\limits_{x^2+y^2=R^2}\frac{x\,\mathrm{d}y-y\,\mathrm{d}x}{(x^2+xy+y^2)^2}=\int_0^{2\pi}\frac{R^2\cos^2\theta+R^2\sin^2\theta}{(R^2+R^2\sin\theta\cos\theta)^2}\,\mathrm{d}\theta=\frac{1}{R^2}\int_0^{2\pi}\frac{\mathrm{d}\theta}{(1+\frac{1}{2}\sin 2\theta)^2}.$$

$$1+\frac{1}{2}\sin 2\theta\geqslant\frac{1}{2}\quad\Rightarrow\quad 0<\frac{1}{(1+\frac{1}{2}\sin 2\theta)^2}<4$$

$$\Rightarrow\quad 0<I(R)\leqslant\frac{8\pi}{R^2}\quad\Rightarrow\quad\lim_{R\to+\infty}I(R)=0.$$ □

5. (10分)设L为空间曲线$\begin{cases}x^2+y^2=1\\ x+z=1\end{cases}$, 其正向为自$z$轴正向看下来的逆时针方向. 计算积分$I=\int_L(y-z+\sin^2x)\,\mathrm{d}x+(z-x+\sin^2y)\,\mathrm{d}y+(x-y+\sin^2z)\,\mathrm{d}z$.

解.【法一】记L为边界线的椭圆盘(平面)上侧为S, 则S的单位法向量为$\boldsymbol{n}=\dfrac{(1,0,1)}{\sqrt{2}}$.

$$I\xlongequal{Stokes}\iint\limits_S\begin{vmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ (y-z+\sin^2x) & (z-x+\sin^2y) & (x-y+\sin^2z)\end{vmatrix}\mathrm{d}S$$

$$=\iint\limits_S\frac{-4}{\sqrt{2}}\,\mathrm{d}S=\frac{-4}{\sqrt{2}}\times\pi\sqrt{2}=-4\pi.$$

【法二】记$P=y-z+\sin^2x$, $Q=z-x+\sin^2y$, $R=x-y+\sin^2z$, 则

$$\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}=-1-1=-2,\quad\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}=1-(-1)=2,\quad\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=-1-1=-2.$$

$$I=\iint\limits_S(-2)\,\mathrm{d}y\,\mathrm{d}z+2\,\mathrm{d}z\,\mathrm{d}x+(-2)\,\mathrm{d}x\,\mathrm{d}y=(-2)\times\pi\cdot1\cdot1+0+(-2)\times\pi 1^2=-4\pi.$$ □

6. (10分)计算积分$I=\iint\limits_D (x+y+xy)^2\,\mathrm{d}\sigma$, 其中$D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\leqslant 1\}$.

解. 根据区域对称性, $\iint\limits_D xy\,\mathrm{d}\sigma=0,\quad \iint\limits_D x^2y\,\mathrm{d}\sigma=0,\quad \iint\limits_D xy^2\,\mathrm{d}\sigma=0.$

所以, $I=\iint\limits_D (x^2+y^2+x^2y^2)\,\mathrm{d}\sigma \xlongequal{\text{极坐标}} \int_0^{2\pi}\mathrm{d}\theta\int_0^1 r^2\,r\,\mathrm{d}r+\int_0^{2\pi}\mathrm{d}\theta\int_0^1 r^4\sin^2\theta\cos^2\theta r\,\mathrm{d}r$

$=2\pi\times\frac{1}{4}+\frac{1}{6}\int_0^{2\pi}\sin^2\theta\cos^2\theta\,\mathrm{d}\theta=\frac{\pi}{2}+\frac{1}{24}\int_0^{2\pi}\sin^2 2\theta\,\mathrm{d}\theta$

$=\frac{\pi}{2}+\frac{1}{24}\int_0^{2\pi}\frac{1-\cos 4\theta}{2}\,\mathrm{d}\theta=\frac{\pi}{2}+\frac{\pi}{24}=\frac{13\pi}{24}.$ □

【法二】

$I=4\int_0^1\mathrm{d}x\int_0^{\sqrt{1-x^2}}(x^2+y^2+x^2y^2)\,\mathrm{d}y=4\int_0^1\left[x^2\sqrt{1-x^2}+(1+x^2)\,\frac{1}{3}\,\sqrt{1-x^2}^3\right]\mathrm{d}x$

$\xlongequal{x=\sin\theta} 4\int_0^{\frac{\pi}{2}}\left[\sin^2\theta\cos\theta+(1+\sin^2\theta)\frac{\cos^3\theta}{3}\right]\cos\theta\,\mathrm{d}\theta$

$=4\int_0^{\frac{\pi}{2}}\left[\sin^2\theta\cos^2\theta+\frac{1}{3}\cos^4\theta+\frac{1}{3}\sin^2\theta\cos^4\theta\right]\mathrm{d}\theta.$

$4\int_0^{\frac{\pi}{2}}\sin^2\theta\cos^2\theta\,\mathrm{d}\theta=\int_0^{\frac{\pi}{2}}\sin^2 2\theta\,\mathrm{d}\theta=\frac{1}{2}\int_0^{\frac{\pi}{2}}(1-\cos 4\theta)\,\mathrm{d}\theta=\frac{\pi}{4};$

$\frac{4}{3}\int_0^{\frac{\pi}{2}}\cos^4\theta\,\mathrm{d}\theta=\frac{1}{3}\int_0^{\frac{\pi}{2}}(1+\cos 2\theta)^2\,\mathrm{d}\theta=\frac{1}{3}\int_0^{\frac{\pi}{2}}\left(1+2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right)\mathrm{d}\theta=\frac{\pi}{4};$

$\frac{4}{3}\int_0^{\frac{\pi}{2}}\sin^2\theta\cos^4\theta\,\mathrm{d}\theta=\frac{4}{3}\int_0^{\frac{\pi}{2}}\frac{1-\cos 2\theta}{2}\left(\frac{1+\cos 2\theta}{2}\right)^2\mathrm{d}\theta$

$=\frac{1}{6}\int_0^{\frac{\pi}{2}}\left[1+\cos 2\theta-\cos^2 2\theta-\cos^3 2\theta\right]\mathrm{d}\theta$

$=\frac{1}{6}\int_0^{\frac{\pi}{2}}\left[1-\frac{1+\cos 4\theta}{2}-(1-\sin^2 2\theta)\cos\theta\right]\mathrm{d}\theta=\frac{1}{6}\times\frac{1}{2}\times\frac{\pi}{2}=\frac{\pi}{24}.$

所以, $I=\frac{\pi}{4}+\frac{\pi}{4}+\frac{\pi}{24}=\frac{13\pi}{24}.$ □

7. (10分)计算积分$I=\iint\limits_D\left(\frac{3x^2\sin y}{y}+2e^{x^2}\right)\mathrm{d}\sigma$, 其中$D$由$y=x, y=x^3$围成.

解. $I_0=\int_0^1 2e^{x^2}\,\mathrm{d}x\int_{x^3}^{x}\mathrm{d}y+\int_0^1\frac{\sin y}{y}\,\mathrm{d}y\int_y^{\sqrt[3]{y}}3x^2\,\mathrm{d}x$

$=\int_0^1 2(x-x^3)e^{x^2}\,\mathrm{d}x+\int_0^1\frac{\sin y}{y}(y-y^3)\,\mathrm{d}y=\int_0^1(1-x^2)e^{x^2}\,\mathrm{d}(x^2)+\int_0^1\sin y(1-y^2)\,\mathrm{d}y$

$=\int_0^1(1-t)e^t\,\mathrm{d}t-\int_0^1(1-y^2)\,\mathrm{d}\cos y$

$=(1-t)e^t\Big|_0^1+\int_0^1 e^t\,\mathrm{d}t-(1-y^2)\cos y\Big|_0^1-2\int_0^1 y\cos y\,\mathrm{d}y\ =-1+e^t\Big|_0^1+1-2\int_0^1 y\,\mathrm{d}\sin y$

$=(e-1)-2y\sin y\Big|_0^1+2\int_0^1\sin y\,\mathrm{d}y=e-1-2\sin 1-2\cos y\Big|_0^1=e-1-2\sin 1-2\cos 1+2$

$=e+1-2(\sin 1+\cos 1).$

$I=2I_0=2(e-1)-4(\sin 1+\cos 1).$ □

8. (10分)计算积分$I=\iiint\limits_{\Omega}\frac{(x+y+z)^2\sqrt{1+x^2+y^2}}{(x^2+y^2+z^2)(1+x^2+y^2+z^2)}\,\mathrm{d}v$, 其中$\mathrm{d}v$即$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$，$\Omega$是由曲面 $z=\sqrt{1+x^2+y^2}$, $z=\sqrt{3(1+x^2+y^2)}$, $x^2+y^2=1$ 所围成的区域.

解. 首先，由区域对称性，$\iiint\limits_{\omega}\frac{xy\,\sqrt{1+x^2+y^2}}{(x^2+y^2+z^2)(1+x^2+y^2+z^2)}\,\mathrm{d}v=0$,

$$\iiint\limits_{\omega}\frac{xz\,\sqrt{1+x^2+y^2}}{(x^2+y^2+z^2)(1+x^2+y^2+z^2)}\,\mathrm{d}v=0,$$

$$\iiint\limits_{\omega}\frac{yz\,\sqrt{1+x^2+y^2}}{(x^2+y^2+z^2)(1+x^2+y^2+z^2)}\,\mathrm{d}v=0.$$

所以，$I=\iiint\limits_{\Omega}\frac{\sqrt{1+x^2+y^2}}{1+x^2+y^2+z^2}\,\mathrm{d}v$.

区域Ω的柱面坐标表示为$\begin{cases}0\leqslant\theta\leqslant 2\pi\\ 0\leqslant r\leqslant 1\\ \sqrt{1+r^2}\leqslant|z|\leqslant\sqrt{3(1+r^2)}\end{cases}$

所以，$I=\int_0^{2\pi}\mathrm{d}\theta\int_0^1 r\,\mathrm{d}r\int_{\sqrt{1+r^2}}^{\sqrt{3(1+r^2)}}\frac{\sqrt{1+r^2}}{1+r^2+z^2}\,\mathrm{d}z$

$$=2\pi\int_0^1 r\,\mathrm{d}r\ \arctan\frac{z}{\sqrt{1+r^2}}\Big|_{z=\sqrt{1+r^2}}^{z=\sqrt{3(1+r^2)}}=2\pi\times\frac{1}{2}\times\frac{\pi}{12}=\frac{\pi^2}{12}.\qquad\square$$

9. 计算积分$I=\oint\limits_{\Gamma}\left(\frac{y^2+y+4x^2}{4x^2+y^2}+\sin x^2\right)\mathrm{d}x+\left(\frac{4x^2-x+y^2}{4x^2+y^2}+\sin y^2\right)\mathrm{d}y$, 其中$\Gamma$是$x^2+y^2=9\ (y\geqslant 0)$, $\frac{x^2}{9}+\frac{y^2}{16}=1\ (y\leqslant 0)$所组成的闭曲线的逆时针方向.

解. $I=\oint\limits_{\Gamma}\frac{y}{4x^2+y^2}\,\mathrm{d}x+\frac{-x}{4x^2+y^2}\,\mathrm{d}y+\oint\limits_{\Gamma}(1+\sin x^2)\,\mathrm{d}x+(1+\sin y^2)\,\mathrm{d}y$.

据*Green*公式或直接积分，$\oint\limits_{\Gamma}(1+\sin x^2)\,\mathrm{d}x+(1+\sin y^2)\,\mathrm{d}y=0$.

另一方面，$\frac{\partial}{\partial y}\left(\frac{y}{4x^2+y^2}\right)=\frac{1}{4x^2+y^2}-y\frac{2y}{(4x^2+y^2)^2}=\frac{4x^2-y^2}{(4x^2+y^2)^2}$, $(x,y)\neq(0,0)$.

$\frac{\partial}{\partial x}\left(\frac{-x}{4x^2+y^2}\right)=\frac{-1}{4x^2+y^2}+x\frac{8x}{(4x^2+y^2)^2}=\frac{4x^2-y^2}{(4x^2+y^2)^2}$, $(x,y)\neq(0,0)$.

令Γ_1 为 $x^2+\frac{y^2}{4}=1$的逆时针方向，其所围成的区域记为D，则由*Green*公式，

$$I=\oint\limits_{\Gamma}\frac{y}{4x^2+y^2}\,\mathrm{d}x+\frac{-x}{4x^2+y^2}\,\mathrm{d}y=\oint\limits_{\Gamma_1}\frac{y}{4x^2+y^2}\,\mathrm{d}x+\frac{-x}{4x^2+y^2}\,\mathrm{d}y$$

$$=\frac{1}{4}\oint\limits_{\Gamma_1}y\,\mathrm{d}x-x\,\mathrm{d}y\ \overset{Green}{=\!=\!=}\ \frac{1}{4}\iint\limits_{D}(-2)\,\mathrm{d}\sigma\ =-\frac{1}{2}\times 2\pi=-\pi.$$

10. (10分)设曲面S是柱体$\Omega = \left\{(x,y,z) \,\middle|\, x^2+y^2 \leqslant 1,\ 0 \leqslant z \leqslant 1\right\}$的表面的外侧. 计算下列积分:

(1) $I_1 = \iint\limits_S (y-z)|x|\,\mathrm{d}y\,\mathrm{d}z + (z-x)|y|\,\mathrm{d}z\,\mathrm{d}x + (x-y)z\,\mathrm{d}x\,\mathrm{d}y$;

(2) $I_2 = \iint\limits_S (y-z)x^2\,\mathrm{d}y\,\mathrm{d}z + (z-x)y^2\,\mathrm{d}z\,\mathrm{d}x + (x-y)z^2\,\mathrm{d}x\,\mathrm{d}y$;

(3) $I_3 = \iint\limits_S (y-z)x^3\,\mathrm{d}y\,\mathrm{d}z + (z-x)y^3\,\mathrm{d}z\,\mathrm{d}x + (x-y)z^3\,\mathrm{d}x\,\mathrm{d}y$.

解. (1)由几何对称性和面积微元的对称性, $I_1 = 0$.

具体地说, 记圆柱体的侧面外侧为S_1, 圆柱体的上表面（圆盘）上侧为S_2, 下表面（圆盘）下侧为S_3.

则$\iint\limits_{S_1}(y-z)|x|\,\mathrm{d}y\,\mathrm{d}z = \iint\limits_{S_1} y|x|\,\mathrm{d}y\,\mathrm{d}z - \iint\limits_{S_1} z|x|\,\mathrm{d}y\,\mathrm{d}z = 0-0=0$（前后抵消）;

$\iint\limits_{S_1}(z-x)|y|\,\mathrm{d}z\,\mathrm{d}x = \iint\limits_{S_1} z|y|\,\mathrm{d}z\,\mathrm{d}x - \iint\limits_{S_1} x|y|\,\mathrm{d}z\,\mathrm{d}x = 0-0=0$（左右抵消）;

在S_1上$\mathrm{d}x\,\mathrm{d}y = 0 \quad \Rightarrow \quad \iint\limits_{S_1}(x-y)z\,\mathrm{d}x\,\mathrm{d}y = 0.$

在S_2, S_3上, $\mathrm{d}y\,\mathrm{d}z = 0$, $\mathrm{d}z\,\mathrm{d}x = 0 \quad \Rightarrow \quad \iint\limits_{S_1 \bigcup S_2} (y-z)|x|\,\mathrm{d}y\,\mathrm{d}z + (z-x)|y|\,\mathrm{d}z\,\mathrm{d}x = 0.$

在S_2上$z=0 \quad \Rightarrow \quad \iint\limits_{S_2}(x-y)z\,\mathrm{d}x\,\mathrm{d}y = 0,.$

在S_1上$z=1$, $\mathrm{d}x\,\mathrm{d}y = \mathrm{d}\sigma \quad \Rightarrow \quad \iint\limits_{S_1}(x-y)z\,\mathrm{d}x\,\mathrm{d}y = \iint\limits_{S_1}(x-y)\,\mathrm{d}\sigma = 0$（对称性）.

(2)【法一】用(1)的同样的方法可知$I_2 = 0$.

【法二】$I_2 = \iint\limits_S (y-z)x^2\,\mathrm{d}y\,\mathrm{d}z + (z-x)y^2\,\mathrm{d}z\,\mathrm{d}x + (x-y)z^2\,\mathrm{d}x\,\mathrm{d}y$

$$\overset{Gauss}{=\!=\!=} \iiint\limits_\Omega 2\left[(yx-zx)+(zy-xy)+(xz-yz)\right]\mathrm{d}v = 0.$$

(3) $I_3 = \iint\limits_S (y-z)x^3\,\mathrm{d}y\,\mathrm{d}z + (z-x)y^3\,\mathrm{d}z\,\mathrm{d}x + (x-y)z^3\,\mathrm{d}x\,\mathrm{d}y$

$$\overset{Gauss}{=\!=\!=} \iiint\limits_\Omega 3\left[(y-z)x^2+(z-x)y^2+(x-y)z^2\right]\mathrm{d}v$$

$$\overset{对称性}{=\!=\!=\!=} \iiint\limits_\Omega 3\left[zy^2 - zx^2\right]\mathrm{d}v = 0,$$

其中, $\iiint\limits_\Omega \left[-x^2z - xy^2 + (x-y)z^2\right]\mathrm{d}v = 0.$ □