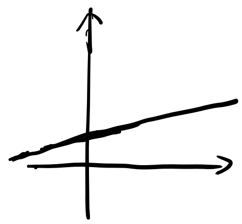
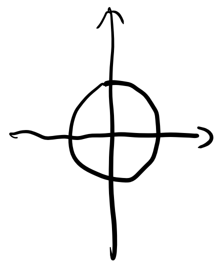
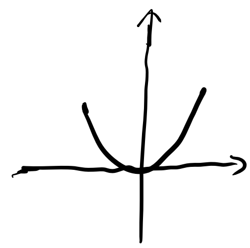
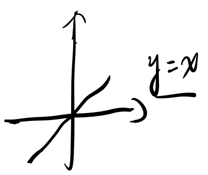






$y = mx + c$ $\quad$ $x^2 + y^2 = 1$ $\quad$ $y = x^2$

<u>Def 1.1.1</u>. A <u>curve</u> in $\mathbb{R}^n$ is a map $\underline{\gamma : (\alpha, \beta) \to \mathbb{R}^n}$, $-\infty \le \alpha < \beta \le +\infty$.

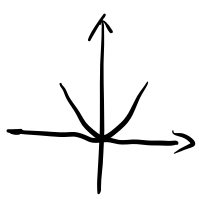


$\gamma(t) : (-\infty, +\infty) \to \mathbb{R}^2$

$\underline{\gamma(t) = (t, t)}$.

$\mathcal{C} = \{ \gamma(t) : t \in (\alpha, \beta) \}$. γ's image is called the <u>parametrization</u> of $\mathcal{C}$.

<u>Exam 1.1.2</u>.



$y = x^2$. <u>parabola</u> (抛物线).

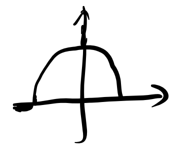
$\gamma(t) = (\gamma_1(t), \gamma_2(t))$, $\gamma_2(t) = \gamma_1(t)^2$.

For example. $\underline{\gamma_1(t) = t, \quad \gamma_2(t) = t^2}$. $\boxed{\tilde{\gamma}_1(t) = t^2, \quad \tilde{\gamma}_2(t) = t^4 \ ?}$

$\tilde{\gamma}_1(t) = t^3$, $\tilde{\gamma}_2(t) = t^6$.

Exam 1.1.3 $\quad x^2+y^2=1$. $\quad t^2+y^2=1 \Rightarrow y=\sqrt{1-t^2}$

$\gamma(t) = (t, \sqrt{1-t^2})$? $\Rightarrow$



Note that $\boxed{\cos^2 t + \sin^2 t = 1}$

$$\boxed{\gamma(t) = (\cos t, \sin t) \qquad t\in(-\infty,+\infty).}$$

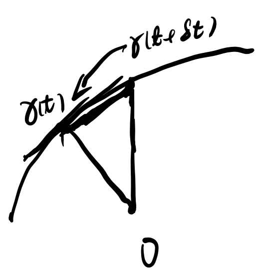
$$\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t)).$$

$$\frac{d\gamma}{dt} = \left(\frac{d\gamma_1}{dt}, \dots, \frac{d\gamma_n}{dt}\right)$$

$$\frac{d^2\gamma}{dt^2} = \left(\frac{d^2\gamma_1}{dt^2}, \dots, \frac{d^2\gamma_n}{dt^2}\right)$$

- If $\frac{d^k\gamma}{dt^k}$ exists, $k=1,2,\dots,n,\dots$, we say γ is a smooth (C^∞) curve.

$$\boxed{\dot\gamma} \qquad \frac{d\gamma}{dt} = \lim_{\delta t\to 0}\frac{\gamma(t+\delta t)-\gamma(t)}{\delta t}$$



- The first derivative $\frac{d\gamma}{dt}$ is called the tangent vector of curve γ at t.

<u>Prop 1.1.6</u>. If the tangent vector of a curve is constant, the image of the curve is a straight line.

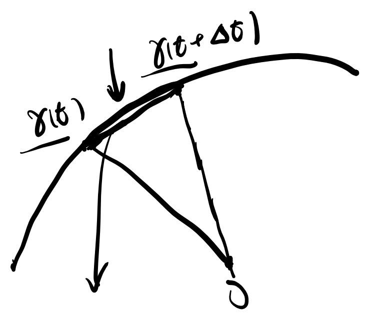
<u>Pf</u>. If $\dot\gamma(t) = a$, $\forall t$. We have

$$\gamma(t) = \int_0^t \dot\gamma(s)\,ds + \gamma(0).$$

$$= \int_0^t a\,ds + \gamma(0)$$

$$= at + \gamma(0).$$

i.e. γ is a straight line. $\boxed{\checkmark}$



$$\underline{\|\gamma(t+\Delta t) - \gamma(t)\|}. \approx \underline{\|\dot\gamma(t)\|\,dt}.$$

$$\Downarrow$$

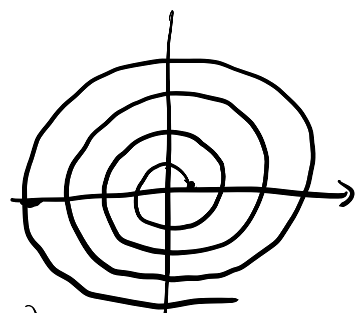
$$\underline{\dot\gamma(t+\theta\Delta t)\,\Delta t}. \approx \dot\gamma(t)\,\Delta t$$

Def 1.2.1 The arc-length of a curve γ starting at point $\gamma(t_0)$ is the $s(t)$ given by

$$\boxed{s(t) = \int_{t_0}^{t} \|\dot\gamma(u)\|\, du} \quad (*)$$

Exam 1.2.2 $\gamma(t) = (e^t \cos t,\ e^t \sin t)$

$e^t \cos t + e^t(-\sin t)$



$\dot\gamma(t) = \left(e^t(\cos t - \sin t),\ e^t(\sin t + \cos t)\right)$

$\|\dot\gamma(t)\| = \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\sin t + \cos t)^2} = e^t \cdot \sqrt{2}$

$s(t) = \int_0^t \|\dot\gamma(u)\|\, du = \int_0^t \sqrt{2}\, e^u\, du = \sqrt{2}(e^t - 1)$ ✓

$s(t) = \int_0^t \|\dot\gamma(u)\|\, du$

$\Rightarrow \quad \dfrac{ds}{dt} = \|\dot\gamma(t)\|$ is called speed of γ.

$= 1$

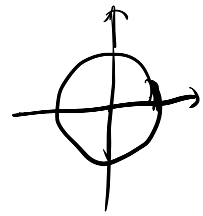
Def 1.3.1 A curve $\tilde\gamma$ is a reparametrization of a curve $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ if there is a smooth function $\phi : (\alpha, \beta) \to \mathbb{R}$ such that

(i) $\dfrac{d\phi}{dt}$ is non-zero on (α, β).

(ii) $\tilde{\gamma}(\phi(t)) = \gamma(t)$ for all $t \in (\alpha, \beta)$.

γ is re... of $\tilde{\gamma}$. $[\phi^{-1}] \leftarrow \exists$ smooth $\leftarrow$ inverse theorem

Exam 1.3.2 $\gamma(t) = (\cos t, \sin t)$.



$\tilde{\gamma}(t) = (\sin t, \cos t)$

$\exists\ \phi$:

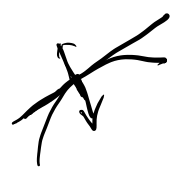
$$\tilde{\gamma}(\phi(t)) = \gamma(t).$$

$$\Leftrightarrow (\sin\phi(t), \cos\phi(t)) = (\cos t, \sin t).$$

$$\phi(t) = \frac{\pi}{2} - t \qquad \frac{d\phi}{dt} = -1 \neq 0.$$

Prop 1.3.3 Let $\gamma(t)$ be a unit speed curve. Then

$$\dot{\gamma}(t) \cdot \ddot{\gamma}(t) = 0$$



Pf.

$$\|\dot{\gamma}(t)\| = 1.$$

$$\|\dot{\gamma}(t)\| = \sqrt{\dot{\gamma}(t) \cdot \dot{\gamma}(t)}$$

or $1 = \|\dot{\gamma}(t)\|^2 = \dot{\gamma}(t) \cdot \dot{\gamma}(t)$

$$\left(\frac{d}{dt}(\vec{a}\cdot\vec{b}) = \frac{d\vec{a}}{dt}\cdot\vec{b} + \vec{a}\cdot\frac{d\vec{b}}{dt} \right)$$

$$0 = \ddot{\gamma}(t)\cdot\dot{\gamma}(t) + \dot{\gamma}(t)\cdot\ddot{\gamma}(t) = 2\,\dot{\gamma}(t)\cdot\ddot{\gamma}(t).$$

$$\Rightarrow \quad \dot{\gamma}(t)\cdot\ddot{\gamma}(t) = 0.$$

Prop 1.3.6. A curve γ has a unit speed reparametrization if and only if $\frac{d\gamma}{dt} \neq 0$.

Def 1.3.5. A curve γ whose tangent vector is never zero is said to be regular. (正则曲线)

Pf of Prop 1.3.6. Suppose $\gamma(t)$ has a unit speed re... $\tilde{\gamma}(u)$.

$\exists\ \phi(t): (\alpha,\beta) \to \mathbb{R}$ s.t.

$$\tilde{\gamma}(\phi(t)) = \gamma(t).$$

(chain rule) $$\frac{d\tilde{\gamma}}{du}\frac{d\phi}{dt} = \frac{d\gamma}{dt}$$

$$\left|\frac{d\phi}{dt}\right| = \left\|\frac{d\gamma}{dt}\right\| \Rightarrow \frac{d\gamma}{dt} \neq 0. \quad ✓$$

$\neq 0$

$\Downarrow$

$$\frac{d\phi}{dt} = \pm\left\|\frac{d\gamma}{dt}\right\| \Leftrightarrow \frac{d\phi}{dt} = \pm\frac{ds}{dt}$$

Conversely, suppose that $\frac{d\gamma}{dt} \neq 0$. ($\Rightarrow \|\dot{\gamma}(t)\| \neq 0$).

$$s(t) = \int_{t_0}^{t} \|\dot{\gamma}(u)\| du \Rightarrow \frac{ds}{dt} = \|\dot{\gamma}(t)\| > 0.$$

Choosing $\tilde{\gamma}(s)$ as the reparametrization of $\gamma(t)$, i.e.

$$\tilde{\gamma}(s(t)) = \gamma(t).$$

$$\frac{d\tilde{\gamma}}{ds} \cdot \frac{ds}{dt} = \frac{d\gamma}{dt}$$

$$\Rightarrow \left\|\frac{d\tilde{\gamma}}{ds}\right\| \cdot \frac{ds}{dt} = \left\|\frac{d\gamma}{dt}\right\|$$

$$\Rightarrow \left\|\frac{d\tilde{\gamma}}{ds}\right\| \cancel{\left\|\frac{d\gamma}{dt}\right\|} = \cancel{\left\|\frac{d\gamma}{dt}\right\|} > 0.$$

$$\Rightarrow \left\|\frac{d\tilde{\gamma}}{ds}\right\| = 1.$$

(✓).

$\tilde{\gamma}(s)$.

Cor 1.3.6. $\tilde{\gamma}(\phi(t)) = \gamma(t)$.

$$\Rightarrow \quad \phi = \pm s + c \quad \Leftarrow \quad \frac{d\phi}{dt} = \pm \frac{ds}{dt}$$

const.

Exam 1.3.7 $\gamma(t) = (e^t\cos t,\ e^t\sin t)$

$$s(t) = \int_0^t \|\dot\gamma(u)\|\,du = \sqrt{2}(e^t - 1).$$

$$\tilde\gamma(s(t)) = \gamma(t) \iff \tilde\gamma(s) = \gamma(t(s))$$

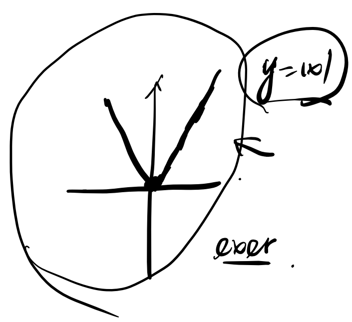
$$t = t(s) = \ln\left(\frac{s}{\sqrt2}+1\right).$$

$$\tilde\gamma(s) = \gamma(t(s)) = \left(\left(\frac{s}{\sqrt2}+1\right)\cos\ln\left(\frac{s}{\sqrt2}+1\right),\ \left(\frac{s}{\sqrt2}+1\right)\sin\ln\left(\frac{s}{\sqrt2}+1\right)\right).$$

Exam 1.3.8 $\gamma(t) = (t, t^2, t^3)$, $-\infty < t < \infty$.

$$\dot\gamma(t) = (1, 2t, 3t^2).$$

$$s(t) = \int_0^t \sqrt{1+4u^2+9u^4}\,du.$$



$t = t(s)$ exists.

$\tilde\gamma(s) = \gamma(t(s))$ ✓.

Exam 1.3.9 $y = x^2$. $\gamma(t) = (t, t^2)$.

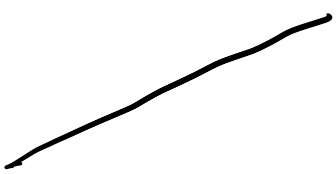
$$\dot\gamma(t) = (1, 2t) \neq 0.$$

$\|\dot{\gamma}(t)\| = \sqrt{1+4t^2} > 0$.

$\boxed{\text{Exer} \quad \text{unit speed reparametrization of } \gamma.}$

$\bar{\gamma}(t) = (t^3, t^6)$.

$\frac{d\bar{\gamma}}{dt} = (3t^2, 6t^5)$ $\quad t=0 \quad \frac{d\bar{\gamma}}{dt} = (0,0)$



$t \in [0, 2\pi]$

$(\cos t, \sin t)$

$t \in (-\infty, +\infty)$

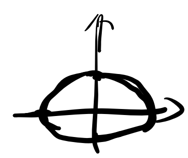
Def 1.4.1. Let $\gamma: \mathbb{R} \to \mathbb{R}^n$ be a smooth curve and let $T \in \mathbb{R}$. We say that γ is T-periodic if

$$\gamma(t+T) = \gamma(t).$$

If γ is not constant and $T \neq 0$, then γ is said to be closed.

Def 1.4.2. The period of a closed curve γ is the smallest positive number T such that γ is T-periodic.

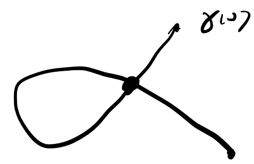
Exam 1.4.3. $\gamma(t) = (p\cos t, q\sin t)$ $\quad p > q$



- $$l(\gamma) = \int_0^T \|\dot{\gamma}(t)\|\,dt.$$

$\gamma(t+\tilde{T}) = \gamma(t),\ \forall t \implies \tilde{T} = kT$

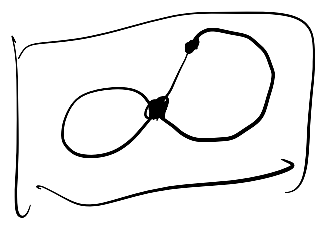
$\implies$ $\tilde{\gamma}(s)$ $\quad l(\gamma) = T$



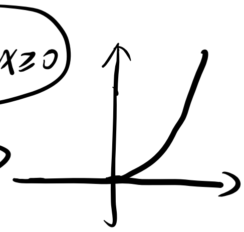
Def 1.4.6. A curve γ is said to have a self-intersection at p of the curve if ~~there~~ are

(i) $\gamma(a) = \gamma(b),\ a \neq b$

(ii) if γ is closed with period T, then $a-b \neq kT$



- (t^2, t^4) $\uparrow$ $x = t^2 \geq 0$ $\rightarrow$ $y = x^2,\ x \geq 0$



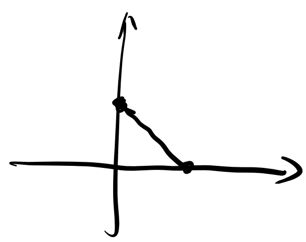
$\gamma : (\alpha, \beta) \to \mathbb{R}^n$

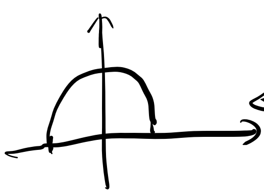
$\forall p \in \ell, \ \exists t_0, \ \gamma(t_0) = p$

$\gamma(t) = (\cos^2 t, \sin^2 t)$ $\qquad y = \sqrt{1-x^2} \Rightarrow (\cos t, \sin t)$

$\Downarrow$

$\gamma \begin{cases} x = \cos^2 t \\ y = \sin^2 t \end{cases} \Rightarrow x + y = 1. \Rightarrow y = 1 - x, \ 0 \le x \le 1, 0 \le y \le 1$





$\leftarrow (x, \sqrt{1-x^2})$

$(\cos t, \sin t), \ t \in [0, \pi]$

$\gamma(t) = (t, \cosh t)$

$\uparrow \quad \cosh t = \frac{e^t + e^{-t}}{2}$ $\qquad (\cos t)' = -\sin t$

$\dot{\gamma}(t) = (1, \sinh t)$ $\qquad \cosh^2 t - \sinh^2 t = 1$

$\|\dot{\gamma}(t)\| = \sqrt{1 + \sinh^2 t} = \sqrt{\cosh^2 t} = \cosh t.$

$$s(t) = \int_0^t \|\gamma'(u)\|\, dt = \int_0^t \cosh u \, du = \sinh u \Big|_0^t$$

$$= \sinh t$$