§2.1 Curvature (曲率)

$\|\vec{a}\|=1$. $\gamma(t)=t\vec{a}+\vec{b}$. $\qquad \dot{\gamma}(t)=\vec{a} \qquad \boxed{\ddot{\gamma}(t)=0}$

$$\gamma(s)=(x_0+R\cos\frac{s}{R},\ y_0+R\sin\frac{s}{R})$$

$$\dot{\gamma}(s)=(-\sin\frac{s}{R},\ \cos\frac{s}{R})$$

$$\ddot{\gamma}(s)=(-\frac{1}{R}\cos\frac{s}{R},\ -\frac{1}{R}\sin\frac{s}{R}) \Rightarrow \|\ddot{\gamma}\|=\frac{1}{R}. \qquad R\to\infty.\ \|\ddot{\gamma}\|\to 0$$

Def 2.1.1 If γ is unit speed curve with parameter s, its curvature $\kappa(s)$ at the $\gamma(s)$ is defined by $\|\ddot{\gamma}(s)\|$.

- $\hat{\gamma}(u)$ $\qquad \boxed{u=\pm s+c}$

$$\boxed{\|\ddot{\hat{\gamma}}(u)\|=\|\ddot{\gamma}(s)\|}$$

Prop 2.1.2 Let $\gamma(t)$ be a regular curve in $\mathbb{R}^n$, then its curvature

$$\kappa=\frac{\|\ddot{\gamma}(\dot{\gamma},\dot{\gamma})-\dot{\gamma}(\dot{\gamma},\ddot{\gamma})\|}{\|\dot{\gamma}\|^4}. \qquad \|\dot{\gamma}\|=1.$$

In particular, in $\mathbb{R}^3$, there is

$$\kappa=\frac{\|\ddot{\gamma}\times\dot{\gamma}\|}{\|\dot{\gamma}\|^3}.$$

Pf. Let $\tilde{\gamma}(s)$ be a unit speed reparameterization of $\gamma(t)$.

$$\dot{\tilde{\gamma}} \frac{ds}{dt} = \gamma'(t) \qquad (\Leftarrow \ \tilde{\gamma}(s(t)) = \gamma(t)).$$

$$\kappa = \|\ddot{\tilde{\gamma}}\| = \left\| \frac{d}{ds}\left(\frac{\gamma'(t)}{ds/dt}\right)\right\|$$

$$= \left\| \frac{d}{dt}\left(\frac{\gamma'(t)}{ds/dt}\right)\frac{dt}{ds}\right\| \qquad \frac{1}{ds/dt}$$

$$= \left\| \frac{\gamma''(t)\frac{ds}{dt} - \gamma'(t)\frac{d^2s}{dt^2}}{(ds/dt)^3}\right\|.$$

Now

$$\left(\frac{ds}{dt}\right)^2 = \|\gamma'(t)\|^2 = (\gamma', \gamma')$$

$$\frac{ds}{dt}\frac{d^2s}{dt^2} = (\gamma'', \gamma').$$

Hence

$$\kappa = \left\| \frac{\gamma''\frac{ds}{dt} - \gamma'\frac{d^2s}{dt^2}}{(ds/dt)^3} \right\|$$

$$= \left\| \frac{\gamma''\|\gamma'\|^2 - \gamma'(\gamma'', \gamma')}{(\|\gamma'\|)^3} \Big/ \|\gamma'\| \right\|$$

$$= \frac{\|\gamma''(\gamma', \gamma') - \gamma'(\gamma'', \gamma')\|}{\|\gamma'\|^4}$$

$$\boxed{\bullet \quad \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}, \vec{c})\vec{b} - (\vec{a}, \vec{b})\vec{c}}$$

$$\gamma' \times (\gamma'' \times \gamma') = (\gamma', \gamma')\gamma'' - (\gamma', \gamma'')\gamma'$$

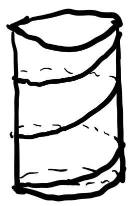
$$\Rightarrow \quad \|\gamma' \times (\gamma'' \times \gamma')\| = \|\gamma'\| \, \|\gamma'' \times \gamma'\|$$

In $\mathbb{R}^3$, there is

$$\boxed{k = \frac{\|\gamma' \times (\gamma'' \times \gamma')\|}{\|\gamma'\|^4} = \frac{\|\gamma'' \times \gamma'\|}{\|\gamma'\|^3}}.$$

Examp 2.1.3.

$$\gamma(\theta) = \boxed{(a\cos\theta, a\sin\theta, b\theta)} \text{ in } \mathbb{R}^3.$$



$$\gamma'(\theta) = (-a\sin\theta, a\cos\theta, b) \quad \Rightarrow \quad \|\gamma'\| = \sqrt{a^2+b^2}.$$

$$\gamma''(\theta) = (-a\cos\theta, -a\sin\theta, 0)$$

$$\gamma'' \times \gamma' = \begin{vmatrix} i & j & k \\ -a\cos\theta & -a\sin\theta & 0 \\ -a\sin\theta & a\cos\theta & b \end{vmatrix}$$

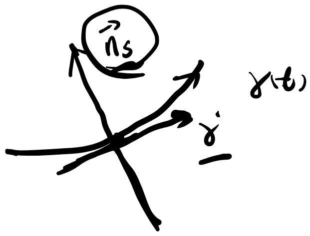
$$= (-ab\sin\theta, ab\cos\theta, -a^2)$$

$$k = \frac{\|(-ab\sin\theta, ab\cos\theta, -a^2)\|}{(\sqrt{a^2+b^2})^3} = \frac{|a|}{a^2+b^2}.$$

If $b=0 \Rightarrow k = \frac{1}{a}$

If $a=0 \Rightarrow k=0$

§2.2 plane curve.



$\underset{\sim}{t} = \dot\gamma$

If $\|\dot\gamma(s)\| = 1$, then $(\ddot\gamma(s), \dot\gamma(s)) = 0$. $\Rightarrow$ $\ddot\gamma \perp \dot\gamma$.

$$\ddot\gamma = k_s \vec{n}_s$$

k_s is called the signed curvature. Since $\|\vec{n}_s\| = 1$,

$$k = \|\ddot\gamma\| = \|k_s \vec{n}_s\| = |k_s|$$

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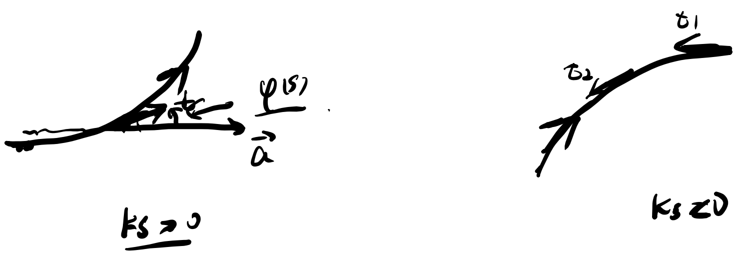


Prop 2.2.1 Let $\gamma(s)$ be unit speed plane curve, and let $\varphi(s)$ be the angle through which a fixed unit vector $\vec{a}$ must be rotated anti-clockwise to bring it into $\dot\gamma(s)$. Then

$$k_s = \frac{d\varphi}{ds}.$$

• $R_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$

$x \mapsto M_\alpha x$, $\quad M_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ $(0 \le \alpha \le 2\pi)$.

$T_a : \mathbb{R}^2 \to \mathbb{R}^2$

$x \mapsto x + a$. $\quad M = T_a \circ R_\alpha$.

Thm 2.2.2. Let $k : (\alpha, \beta) \to \mathbb{R}$ be any smooth function. Then there is a unit speed curve $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ whose signed curvature is k.

Furthermore, if $\tilde{\gamma} : (\alpha, \beta) \to \mathbb{R}^2$ - - - , there exists a rigid motion M of $\mathbb{R}^2$ such that

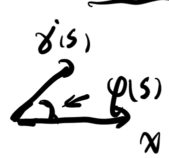
$$\tilde{\gamma}(s) = M(\gamma(s)).$$

Pf. Fixed $s_0 \in (\alpha, \beta)$, define

$$\varphi(s) = \int_{s_0}^{s} k \, du$$

$$\boxed{\gamma(s) = \left(\int_{s_0}^{s} \cos\varphi(t)\, dt, \int_{s_0}^{s} \sin\varphi(t)\, dt \right).}$$

$$\dot{\gamma}(s) = (\cos\varphi(s), \sin\varphi(s)).$$



By Prop 2.2.1, we know

$$k_s = \frac{d\varphi}{ds} = k$$

$\tilde{\gamma}(s)$ unit speed.

$$\tilde{\gamma}'(s) = (\cos\tilde{\varphi}(s), \sin\tilde{\varphi}(s))$$

$$\tilde{\gamma}(s) = \left(\int_{s_0}^{s} \cos\tilde{\varphi}(t)\,dt, \int_{s_0}^{s} \sin\tilde{\varphi}(t)\,dt\right) + \tilde{\gamma}(s_0) \quad a$$

$$\frac{d\tilde{\varphi}}{ds} = k(s) \Rightarrow \tilde{\varphi}(s) = \int_{s_0}^{s} k(u)\,du + \tilde{\varphi}(s_0) = \varphi(s) + \tilde{\varphi}(s_0) \quad \theta$$

$$\cos\tilde{\varphi}(s) = \cos(\varphi(s)+\theta) = \cos\theta\cos\varphi(s) - \sin\theta\sin\varphi(s)$$

$$\sin\tilde{\varphi}(s) = \sin(\varphi(s)+\theta) = \cos\theta\sin\varphi(s) + \sin\theta\cos\varphi(s)$$

$$\Rightarrow \begin{pmatrix} \cos\tilde{\varphi} \\ \sin\tilde{\varphi} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{R_\theta} \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}.$$

$$\tilde{\gamma}(s) = M\gamma(s)$$

$$M = T_a \circ R_\theta$$

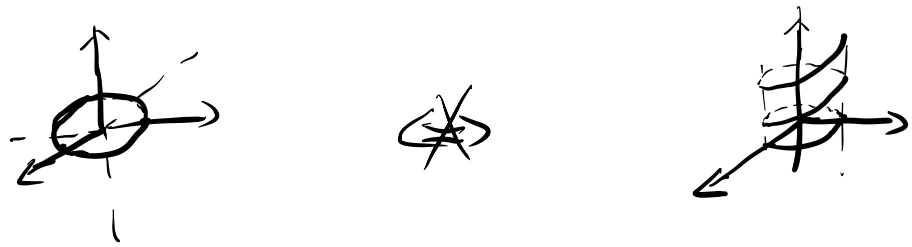
$\boxed{V}$

Exam 2.2.3. Any regular plane curve whose curvature is a positive constant must be a part of a circle.

§2.3 Space curves.

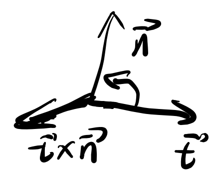
- $\gamma(s) = (\cos s, \ \sin s, \ 0)$ $\qquad$ $\tilde{\gamma}(s) = (\frac{1}{2}\cos s, \ \frac{1}{2}\sin s, \ \frac{1}{2}s)$.

$\Rightarrow \ \kappa(s) = 1$. $\qquad$ $\kappa(s) = \frac{|a|}{a^2+b^2} \underset{a=b=\frac{1}{2}}{=} 1$.



$\gamma(s)$ unit speed.

$\vec{t} = \dot{\gamma}$, $\quad \dot{t} = \ddot{\gamma}$



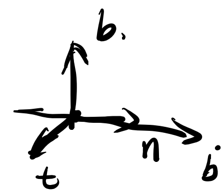
If $\kappa(s) \neq 0$, we define the principal normal (主法向量) of γ at the point $\gamma(s)$ to be the vector

$$\vec{n}(s) = \frac{1}{\kappa(s)}\dot{\vec{t}} \ \left(= \frac{1}{\kappa(s)}\ddot{\gamma}\right).$$

$\Rightarrow \ \|\vec{n}\| = 1$. We define

$$\vec{b} = \vec{t} \times \vec{n} \qquad \|\vec{b}\| = 1$$

be the binormal vector of γ.
(从法向量)



$$b = t \times n, \quad n = b \times t, \quad t = n \times b$$

$$b = t \times n$$

$$\Rightarrow \dot{b} = \dot{t} \times n + t \times \dot{n} = t \times \dot{n}$$

$$\Rightarrow \dot{b} \perp t \qquad \dot{b} \perp b$$

• $\dot{b} = -\tau n$, τ is called torsion of γ (挠率)

If $\tilde{\gamma}(u(s)) = \gamma(s)$. unit speed $u = \pm s + c$

Prop 2.3.1 Let $\gamma(t)$ regular in $\mathbb{R}^3$. $\kappa(t) \neq 0$

$$\tau = \frac{(\dot{\gamma} \times \ddot{\gamma}) \cdot \dddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|^2} \qquad \kappa(t) = \frac{\|\dot{\gamma} \times \ddot{\gamma}\| = 0}{\|\dot{\gamma}\|^3}$$

Exam 2.3.2 $\gamma(\theta) = (a\cos\theta, a\sin\theta, b\theta)$

$$\dot{\gamma}(\theta) = (-a\sin\theta, a\cos\theta, b)$$

$$\ddot{\gamma}(\theta) = (-a\cos\theta, -a\sin\theta, 0)$$

$$\dddot{\gamma}(\theta) = (a\sin\theta, -a\cos\theta, 0)$$

$$\dot{\gamma} \times \ddot{\gamma} = (ab\sin\theta, -ab\cos\theta, a^2)$$

$$(\dot{\gamma}\times\ddot{\gamma})\cdot\dddot{\gamma} = a^2 b\sin^2\theta + a^2 b\cos^2\theta = a^2 b$$

$$\tau = \frac{a^2 b}{a^4 + a^2 b^2} = \frac{b}{a^2+b^2}$$

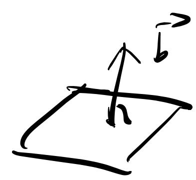
If $b=0$, $\tau = 0$.

Prop 2.2.3 Let $\gamma(t)$ is a regular curve in $\mathbb{R}^3$, $\kappa(t)\neq 0$. (τ can be defined). Then, the image of γ is contained in a plane if and only if $\tau = 0$.

Pf "$\Leftarrow$" If $\tau = 0$. $(\dot{b} = -\tau n)$

$$\Rightarrow \dot{b} = 0 \Rightarrow b \equiv \text{const.}$$

$$\Rightarrow \frac{d}{ds}(\gamma\cdot b) = \dot{\gamma}\cdot b + \gamma\cdot\dot{b} = \dot{\gamma}\cdot b = t\cdot b = 0.$$



$$\Rightarrow \gamma\cdot b = c.$$

$\Rightarrow$ γ is contained in the plane

$$\vec{r}\cdot\vec{b} = c$$

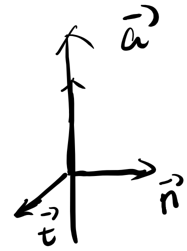
"$\Rightarrow$" If γ is contained in a plane.

$$\gamma\cdot\vec{a} = c \quad (\|\vec{a}\| = 1)$$

$\Rightarrow \quad t\cdot\vec{a}=0 \qquad (\,t=\dot{\gamma}\,)$

$\Rightarrow \quad \dot{t}\cdot\vec{a}=0. \qquad (\,n=\frac{1}{k(s)}\dot{t}\,)$

$\Rightarrow \quad \vec{n}\cdot\vec{a}=0.$



$\Rightarrow \quad \vec{b}=l\vec{a}$

$\Rightarrow \quad \vec{b}=\vec{a}$ or $-\vec{a} \;\Rightarrow\; \vec{b}\equiv\vec{a},\quad \vec{b}\equiv-\vec{a}$

$\Rightarrow \quad \dot{\vec{b}}=0 \;\Rightarrow\; \tau=0$