$\gamma(t) = (\cos^3 t, \sin^3 t, 0)$

$\mathbb{R}^3$.

$$K = \frac{\|\gamma'' \times \gamma'\|}{\|\gamma'\|^3}$$

$\gamma'(t) = (-3\cos^2 t \sin t, 3\sin^2 t \cos t, 0)$ $\quad \|\gamma'\| = 3|\sin t \cos t|$

$\gamma''(t) = (6\cos t \sin^2 t - 3\cos^3 t, \ 6\sin t \cos^2 t - 3\sin^3 t, \ 0)$

$$\gamma'' \times \gamma' = \begin{vmatrix} i & j & k \\ 6\cos t \sin^2 t - 3\cos^3 t & 6\sin t \cos^2 t - 3\sin^3 t & 0 \\ -3\cos^2 t \sin t & 3\sin^2 t \cos t & 0 \end{vmatrix}$$

$= (18\cos^2 t \sin^4 t - 9\cos^4 t \sin^2 t)$

$\quad - (-18\cos^4 t \sin^2 t + 9\cos^2 t \sin^4 t)$

$= 18\cos^2 t \sin^2 t - 9\cos^2 t \sin^2 t$

$= 9\cos^2 t \sin^2 t$.

$= (0, 0, 9\cos^2 t \sin^2 t)$.

$$k = \frac{9\cos^2 t\sin^2 t}{[3|\sin t\cos t|]^3} = \frac{1}{3|\sin t\cos t|}.$$

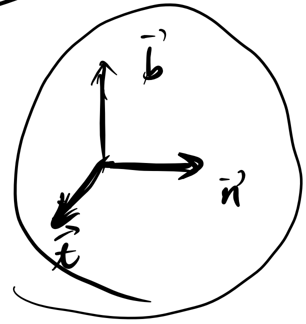
torsion

$k \neq 0$

$$\tau = \frac{(\dot\gamma \times \ddot\gamma)\cdot\dddot\gamma}{\|\gamma''\times\gamma'\|^2} = (\dot\gamma, \ddot\gamma, \dddot\gamma)$$

$\vec t, \vec n, \vec b$

$\dot\gamma(t)$



$$\vec n = \frac{1}{k}\dot t$$

$$\vec b = \vec t \times \vec n$$

$$\dot t = k\vec n$$

$$\dot b = -\tau\vec n$$

torsion

$$\dot n = \lambda t + \mu n + \nu b.$$

$\Rightarrow$ $t\cdot\dot n = \lambda$

$n\cdot\dot n = \mu \Rightarrow \mu = 0$

$b\cdot\dot n = \nu$

$\|t\| = 1$

$\|\dot t\| = k$

$n = \frac{1}{k}\dot t$

$$0 = (n\cdot t)' = \dot n\cdot t + n\cdot\dot t = \underbrace{\dot n\cdot t}_{\lambda} + k$$

$\frac{1}{k}\|\dot t\|^2$

$\Rightarrow \lambda = -k.$

$0 = (b\cdot n)' = \dot{b}\cdot n + b\cdot \dot{n}$ ν.

$= -\tau + \nu$

$\Rightarrow \nu = \tau$.

$$\Rightarrow \quad \begin{aligned} \dot{t} &= \kappa n \\ \dot{n} &= -\kappa t + 0\cdot n + \tau b. \\ \dot{b} &= -\tau n. \end{aligned}$$

$$\Rightarrow \frac{d}{ds}\begin{pmatrix} t \\ n \\ b \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} t \\ n \\ b \end{pmatrix}$$

$\dot{b} = -\tau n$

differential equation.

Frenet-Serret equation

Prop. Let γ be a unit speed curve in $\mathbb{R}^3$, $\kappa = \text{const}$. $\tau \equiv 0$

Then γ is a (part of) circle.

Pf. $\tau \equiv 0 \Rightarrow b = \text{const}$.

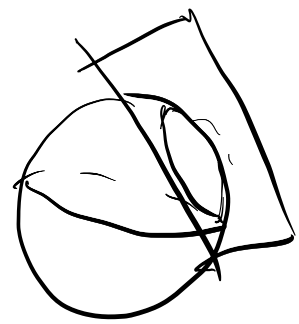
$$\frac{d}{ds}\left(\gamma + \frac{1}{\kappa}n\right) = t + \frac{1}{\kappa}\dot{n} = t + \frac{1}{\kappa}(-\kappa t) = 0$$

$\dot{n} = -\kappa t + \tau b = -\kappa t$

$\gamma + \frac{1}{\kappa} n = a$.

$\Rightarrow \quad \|\gamma - a\| = \frac{1}{\kappa}$

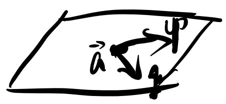
$\Rightarrow \quad |\checkmark|$



Thm 2.3.6. $\kappa > 0$, τ smooth. $\exists!\ \gamma$.

§3. Surfaces in 3-d.

§3.1. What is a surface?



$\subseteq \mathbb{R}^3$

$a(u,v) = \vec{a} + u\vec{p} + v\vec{q}$

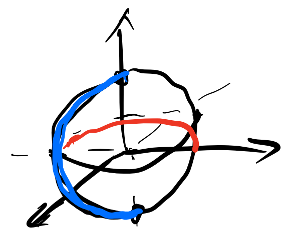
$a: \mathbb{R}^2 \to \mathbb{R}^3$

Def 3.1.1. A surface patch is a smooth injective map $a: U \to \mathbb{R}^3$, where $U \subset \mathbb{R}^2$ is an open set.

open set: $\forall x_0 \in U$, $\exists \delta_{x_0} > 0$, s.t. $\forall x: \|x - x_0\| < \delta \Rightarrow x \in U$.

Exam. $a(\theta, \varphi) = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta)$

$(0 < \varphi < 2\pi, \; -\frac{\pi}{2} < \theta < \frac{\pi}{2})$



$\tilde{a}(\theta, \varphi) = (-\cos\theta\cos\varphi, -\sin\theta, -\cos\theta\sin\varphi)$

Def 4.1.3 A surface patch $\tilde{a}: \tilde{U} \to \mathbb{R}^3$ is a reparametrization of a surface patch $a: U \to \mathbb{R}^3$ if there exists a bijective $\Phi: U \to \tilde{U}$ is smooth, and $\Phi^{-1}: \tilde{U} \to U$ is also smooth.

(differmorphism): $\tilde{a}(\Phi(u,v)) = a(u,v)$, $\forall (u,v) \in U$.

$(\tilde{u}, \tilde{v}) = \Phi(u,v)$, $(u,v) = \Phi^{-1}(\tilde{u}, \tilde{v})$

$$J(\Phi) = \begin{pmatrix} \frac{\partial \tilde{u}}{\partial u} & \frac{\partial \tilde{u}}{\partial v} \\ \frac{\partial \tilde{v}}{\partial u} & \frac{\partial \tilde{v}}{\partial v} \end{pmatrix}. \quad J(\Phi^{-1}) = \begin{pmatrix} \frac{\partial u}{\partial \tilde{u}} & \frac{\partial u}{\partial \tilde{v}} \\ \frac{\partial v}{\partial \tilde{u}} & \frac{\partial v}{\partial \tilde{v}} \end{pmatrix}.$$

$J(\Phi)\cdot J(\Phi^{-1}) = id.$

$\|\dot\gamma\| \neq 0$

Def 6.2.1 Let $a: U \to \mathbb{R}^3$ be smooth surface patch. let $S \subset \mathbb{R}^3$ be its image, let $p \in S$. The Tangent space to S at p is the set of all tangent vectors at p to smooth curves through p.

$\boxed{T_pS}$

$a(u,v)$

a_u, a_v

$a_u = \frac{\partial a}{\partial u}$

Prop 6.2.2 the tangent space to S at p is the subvector space of $\mathbb{R}^3$ spanned by a_u and a_v.

Pf. $\gamma(t) = a(u(t), v(t))$ chain rule.

$$\dot\gamma(t) = a_u \cdot \frac{du}{dt} + a_v \frac{dv}{dt}$$
$$= a\, a_u + b\, a_v \in \langle a_u, a_v \rangle$$

$a_u, a_v \in T_pS. \iff T_pS = \langle a_u, a_v \rangle$

a_u , a_v linearly independent (in $\mathbb{R}^3$) $\Longleftrightarrow$ $a_u \times a_v \neq 0$

$\gamma \neq 0$

$a_u \neq 0, a_v \neq 0$

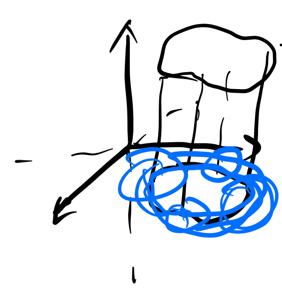
Def 4.2.3. A surface $a: U \to \mathbb{R}^3$ is regular if $a_u \times a_v \neq 0$ for $(u,v) \in U$.

Def. $\vec{N} = \dfrac{a_u \times a_v}{\|a_u \times a_v\|}$ normal vector of S.

$t = a\,a_u + b\,a_v$

$t \cdot \vec{N} = a\,a_u \cdot N + b\,a_v \cdot N = 0, \quad \forall t \in T_pS.$

Exam. (generalized) Cylinder



$a(u,v) = (f(u), g(u), v)$

$a_u = (f', g', 0) \qquad a_v = (0, 0, 1)$

$$\sigma_u \times \sigma_v = \begin{vmatrix} i & j & k \\ f' & g' & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

manifold 流形

$$= (g', -f', 0) \neq 0$$

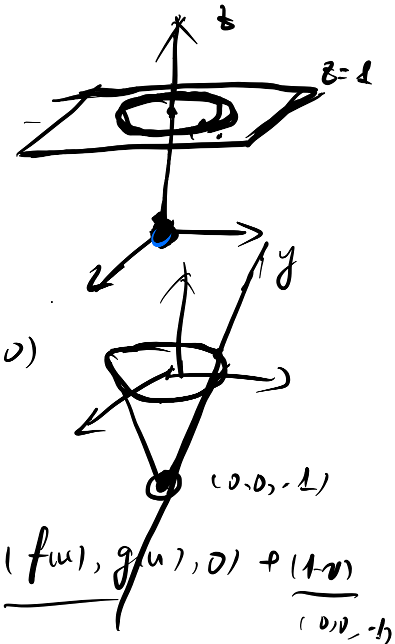
$\Leftrightarrow \quad (g')^2 + [f']^2 \neq 0.$ $\qquad \gamma(t) = (f(t), g(t))$

$f = \cos u, \quad g = \sin u.$

$\Leftrightarrow \quad \dot\gamma(t) \neq 0.$

If $\gamma(t)$ is regular, then $\sigma(u,v)$ is also regular.

Exam (generalized) cone



$$\sigma(u,v) = (f(u), g(u), 1),$$

$(f(u), g(u), 0)$

$\sigma_u = v(f', g', 0)$

$\sigma_v = (f, g, 1)$

$v(f(u), g(u), 0) + (1-v)(0,0,-1)$

$$\sigma_u \times \sigma_v = \begin{vmatrix} i & j & k \\ vf' & vg' & 0 \\ f & g & 1 \end{vmatrix}$$

$$= v(g', -f', f'g - fg')$$

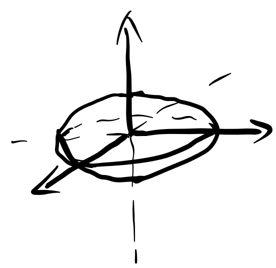
$$\alpha_u \times \alpha_v \neq 0 \Leftrightarrow |v| \sqrt{(g')^2 + (f')^2 + (f'g - fg')^2} \neq 0.$$

$$\Leftrightarrow \underline{v \neq 0 \text{ or } (g')^2 + (f')^2 \neq 0}.$$

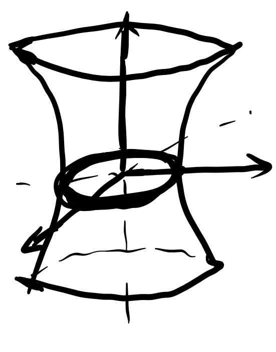
$\dot{\gamma}(t) \neq 0$

Exam.

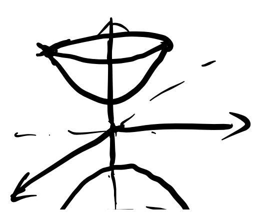
(i) ellipsoid : $\frac{x^2}{p^2} + \frac{y^2}{q^2} + \frac{z^2}{r^2} = 1$



(ii) hyperboloid of one sheet $\underline{\frac{x^2}{p^2} + \frac{y^2}{q^2} - \frac{z^2}{r^2}} = 1$

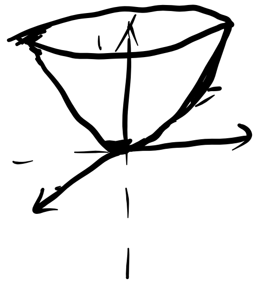


(iii) hyperboloid of two sheets. $\underline{\frac{x^2}{p^2} - \frac{y^2}{q^2} - \frac{z^2}{r^2}} = 1$

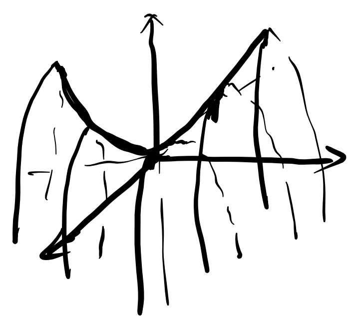


(iv) elliptic paraboloid $\frac{x^2}{p^2}+\frac{y^2}{q^2}=z$

$\sigma(x,y) = (x, y, x^2+y^2)$

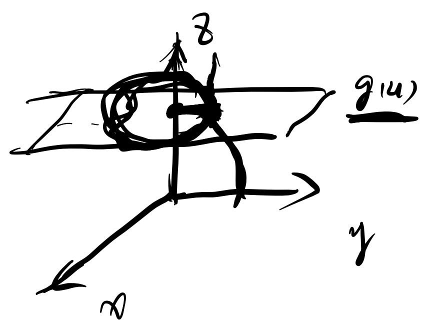


(v) hyperbolic parabolic $\frac{x^2}{p^2}-\frac{y^2}{q^2}=z$



P61 - P63

Exam.

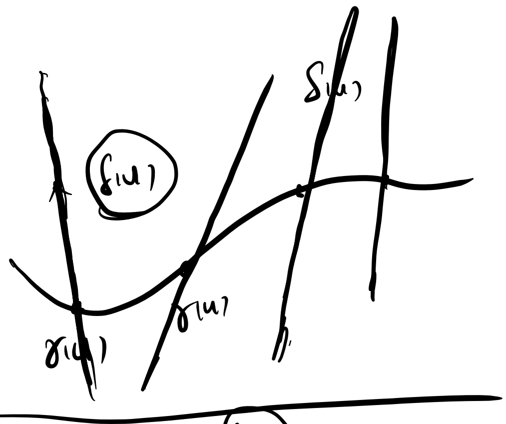


Exercise.

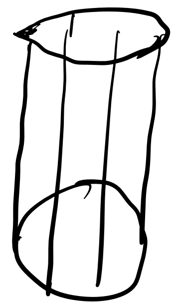
$\gamma(u) = (0, f(u), g(u))$

$$\boxed{\sigma(u,v) = (f(u)\cos v, f(u)\sin v, g(u))}$$

Exam

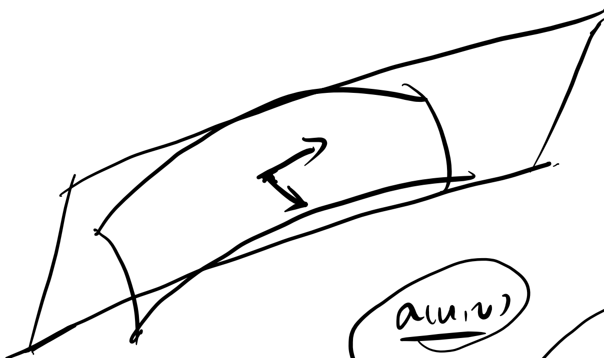


$\gamma(u) + v\,\delta(u)$



$$\sigma(u,v) = \gamma(u) + v\,\delta(u)$$

$\mathbb{R}^2$



$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

positive

$\sigma(u,v)$

$$g = \begin{pmatrix} \|\sigma_u\|^2 & \sigma_u \cdot \sigma_v \\ \sigma_u \cdot \sigma_v & \|\sigma_v\|^2 \end{pmatrix}$$