§4 The First Fundamental Form.

§5.1 Lengths of curves on surfaces (regular).

S . $\sigma(u,v)$. $\leftarrow$ (smooth. injective).

$$\gamma(t) = \sigma(u(t), v(t))$$

$$s = \int_{t_0}^{t} \|\dot{\gamma}(\tau)\| \, d\tau .$$

$$\dot{\gamma}(t) = \sigma_u \dot{u} + \sigma_v \dot{v} .$$

$$s = \int_{t_0}^{t} \| \sigma_u \dot{u} + \sigma_v \dot{v} \| \, dt$$

$$= \int_{t_0}^{t} [(\sigma_u \dot{u} + \sigma_v \dot{v}) \cdot (\sigma_u \dot{u} + \sigma_v \dot{v})]^{\frac{1}{2}} \, dt .$$

$$= \int_{t_0}^{t} [\|\sigma_u\|^2 \dot{u}^2 + 2 \sigma_u \cdot \sigma_v \, \dot{u}\dot{v} + \|\sigma_v\|^2 \dot{v}^2]^{\frac{1}{2}} \, dt .$$

Let

$$E = \|\sigma_u\|^2 \, (= \sigma_u \cdot \sigma_u)$$

$$F = \sigma_u \cdot \sigma_v$$

$$G = \sigma_v \cdot \sigma_v$$

$$\dot{u} = \frac{du}{dt}$$

$$\dot{u}^2 dt^2 = (du)^2 .$$

$$\Rightarrow \quad s = \int_{t_0}^{t} (E \dot{u}^2 + 2F \dot{u}\dot{v} + G \dot{v}^2)^{\frac{1}{2}} \, dt .$$

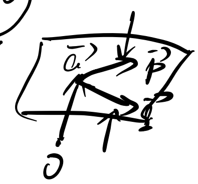
$$= \int_{t_0}^{t} [E \, du^2 + 2F \, du dv + G \, dv^2]^{\frac{1}{2}}$$

$$\Rightarrow \quad \boxed{ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2} \leftarrow \text{First fundamental form}$$

$E = G, \quad F = 0$ conformal (共形)

Example 4.1.1

$$a(u,v) = \vec{a} + u\vec{p} + v\vec{q}, \quad \vec{p}\cdot\vec{q} = 0$$



First fundamental form

$$a_u = \vec{p}, \quad a_v = \vec{q} \qquad \vec{p}\cdot\vec{q} = 0 \;?$$

$$\vec{q} - \vec{p}'$$

$$\vec{p}' = \vec{p} = \frac{\vec{p}}{|\vec{p}|}\cdot \vec{p}\cdot\vec{q}$$

$$|\vec{q}|\cos\theta = |\vec{q}|\cdot\frac{\vec{p}\cdot\vec{q}}{|\vec{p}||\vec{q}|} = \frac{\vec{p}\cdot\vec{q}}{|\vec{p}|}$$

$$\Rightarrow \quad F = a_u\cdot a_v = \vec{p}\cdot\vec{q} = 0$$

$$E = a_u\cdot a_u = |\vec{p}|^2 = 1$$

$$G = a_v\cdot a_v = |\vec{q}|^2 = 1.$$

$$\Rightarrow \quad ds^2 = du^2 + dv^2.$$

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$\mathbb{R}^2$

Example 4.1.2. $\sigma(\theta,\varphi) = (\cos\theta\cos\varphi,\ \cos\theta\sin\varphi,\ \sin\theta)$

$$\sigma_\theta = (-\sin\theta\cos\varphi,\ -\sin\theta\sin\varphi,\ \cos\theta)$$

$$\sigma_\varphi = (-\cos\theta\sin\varphi,\ \cos\theta\cos\varphi,\ 0).$$

$$E = \sigma_\theta\cdot\sigma_\theta = 1, \quad F = \sigma_\theta\cdot\sigma_\varphi = 0.$$

$$G = \sigma_\varphi\cdot\sigma_\varphi = \cos^2\theta.$$

$$\Rightarrow \quad ds^2 = d\theta^2 + \cos^2\theta\, d\varphi^2.$$

Example 4.1.3. cylinder.

$$\sigma(u,v) = (f(u),\ g(u),\ v).$$

$$f'^2 + g'^2 = 1$$

$$\sigma_u = (f',\ g',\ 0), \quad \sigma_v = (0,\ 0,\ 1).$$

$$E = \sigma_u\cdot\sigma_u = f'^2 + g'^2 = 1.$$

$$F = \sigma_u\cdot\sigma_v = 0.$$

$$G = \sigma_v\cdot\sigma_v = 1.$$

$$\Rightarrow \quad \boxed{ds^2 = du^2 + dv^2}.$$

Prop 4.1.9. A surface $\alpha : U \to \mathbb{R}^3$ is conformal if and only if whenever

$$\pi_1(t) = (u_1(t), v_1(t)) \quad , \quad \pi_2(t) = (u_2(t), v_2(t))$$

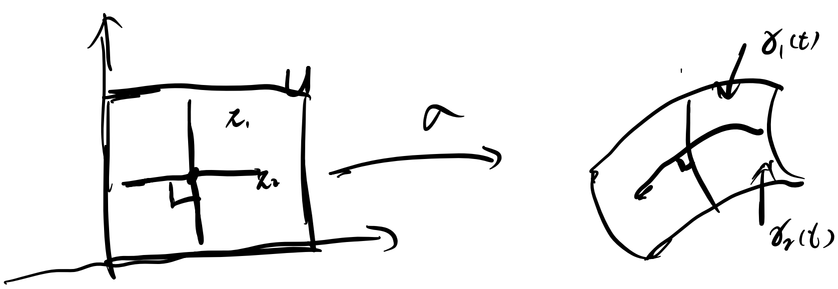
$$\gamma_1(t) = \alpha(\pi_1(t)) \quad , \quad \gamma_2(t) = \alpha(\pi_2(t)).$$

$u_1(t_0) = u_2(t_0) = a$

$v_1(t_0) = v_2(t_0) = b.$

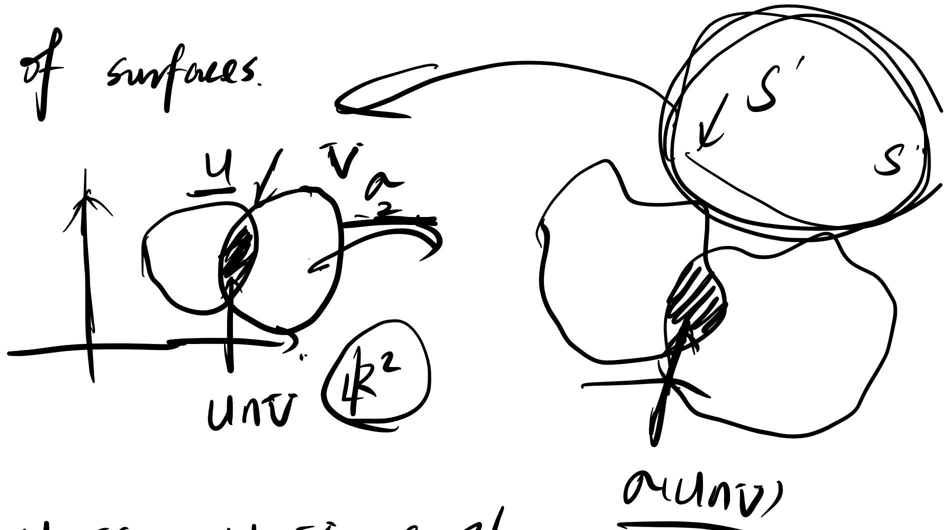
$$\gamma_1(t_0) = \gamma_2(t_0) = \alpha(a,b).$$

$$\boxed{\langle \pi_1(t), \pi_2(t) \rangle = \langle \gamma_1(t), \gamma_2(t) \rangle}$$



§4.2. Isometries of surfaces.

α_1:



$\Phi : U \cap V \to U \cap V$ smooth.

$\Leftrightarrow$ Φ^{-1} smooth.

$\Phi : \alpha_2^{-1} \circ \alpha_1$

Def Let $\sigma_1 : U_1 \to \mathbb{R}^3$ and $\sigma_2 : U_2 \to \mathbb{R}^3$. $S_1 = \sigma_1(U_1)$, $S_2 = \sigma_2(U_2)$

$f : S_1 \to S_2$ is said to be smooth if $\exists\, F : U_1 \to U_2$ smooth s.t.

$$F = \sigma_2^{-1} \circ f \circ \sigma_1$$

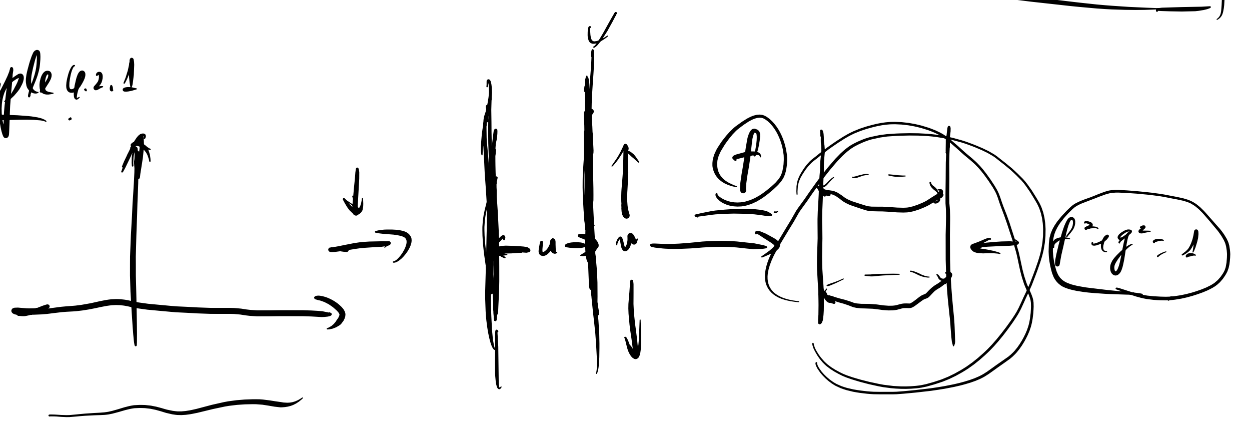
$$\boxed{f(\sigma_1(u,v)) = \sigma_2(F(u,v))}.$$

$$\begin{tikzcd} S_1 \arrow[r, "f"] & S_2 \\ U_1 \arrow[u, "\sigma_1"] \arrow[r, "F"] & U_2 \arrow[u, "\sigma_2"'] \end{tikzcd}$$

$$\boxed{f \circ \sigma_1 = \sigma_2 \circ F}$$

The $f : S_1 \to S_2$ is called a **diffeomorphism** if f is bijective and smooth, and f^{-1} is smooth.

Example 4.2.1



$$f(0,u,v) = (\cos u, \sin u, v).$$

$$\sigma_1(u,v) = (0,u,v) \qquad u \in (0, 2\pi),\ v \in \mathbb{R}$$

$$\sigma_2 = (\cos u, \sin u, v) \qquad F(u,v) = (u,v).$$

$$\Rightarrow \quad \boxed{f \circ \sigma_1} \ f(0,u,v) = (\cos u, \sin u, v)$$

$$= \sigma_2(u,v) = \boxed{\sigma_2 \circ F}$$

f diffeomorphism

Def 4.2.2. Let $\sigma_1 : U_1 \to \mathbb{R}^3$, $\sigma_2 : U_2 \to \mathbb{R}^3$, $S_1 = \sigma_1(U_1)$. $S_2 = \sigma_2(U_2)$. A diffeomorphism $f : S_1 \to S_2$ is an isometry if it takes curves in S_1 to curves in S_2 with the same length.

$$\boxed{\begin{array}{l} \gamma_1(t) \subset S_1 \xrightarrow{f} \gamma_2(t) = f(\gamma_1(t)) \subset S_2. \\ \text{length}(\gamma_1) = \text{length}(\gamma_2) \end{array}}$$

Thm 4.2.3. Two surfaces are isometric if and only if they have $\sigma_1 : U \to \mathbb{R}^3$, and $\sigma_2 : U \to \mathbb{R}^3$ such that their first fundamental form are the same.

Pf. "$\Leftarrow$" $\quad S_1 = \sigma_1(U)$, $\quad S_2 = \sigma_2(U)$. $\quad ds^2 = E du^2$

Let $f: S_1 \to S_2$ be the map such that

$$f(\sigma_1(u,v)) = \sigma_2(u,v).$$

$$\boxed{f = \sigma_2 \circ \sigma_1^{-1}} \Longrightarrow \boxed{\begin{array}{l} f \text{ smooth.} \\ \cancel{f^{-1} \text{ smooth}} \end{array}}$$

$\Downarrow$

f diffeomorphism

$t \mapsto (u(t), v(t))$.

$$\boxed{\begin{array}{l} \gamma_1(t) = \sigma_1(u(t), v(t)) \subset S_1. \\ \gamma_2(t) = \sigma_2(u(t), v(t)) \subset S_2 \end{array}}$$

$$f(\gamma_1(t)) = f(\sigma_1(u(t), v(t))) = \sigma_2(u(t), v(t)) = \gamma_2(t).$$

$$\boxed{\boxed{l(\gamma_1)} = \int_{t_0}^{t} ds = \int_{t_0}^{t} [E du^2 + 2F du\, dv + G dv^2]^{\frac{1}{2}} = \boxed{l(\gamma_2)}}$$

$\Leftrightarrow \quad l(\gamma_1) = l(f(\gamma_1))$

"$\Rightarrow$" $\quad \tilde{\sigma}_1 : U_1 \to \mathbb{R}^3$, $\quad \tilde{\sigma}_2 : U_2 \to \mathbb{R}^3$. $\quad S_1 = \tilde{\sigma}_1(U_1)$,

$S_2 = \tilde{\sigma}_2(U_2)$.

$f: S_1 \to S_2 \qquad \exists\, F: U_1 \to U_2$.

s.t.

$$\boxed{f \circ \tilde{\sigma}_1 = \tilde{\sigma}_2 \circ F}.$$

Let $\alpha_1 = \hat{\alpha}_1$, $\alpha_2 = \hat{\alpha}_2 \circ F$.

$$f(\gamma_1(t)) = f(\alpha_1(u(t), v(t))) = \hat{\alpha}_2 \circ f(u(t), v(t))$$

$$= \alpha_2(u(t), v(t)) = \gamma_2(t)$$

$l(\gamma_1(t)) = l(\gamma_2(t))$.

$$\int_{t_0}^{t} (E_1 \dot{u}^2 + 2F_1 \dot{u}\dot{v} + G_1 \dot{v}^2)^{\frac{1}{2}} dt = \int_{t_0}^{t} (E_2 \dot{u}^2 + 2F_2 \dot{u}\dot{v} + G_2 \dot{v}^2)^{\frac{1}{2}} dt.$$

$$\Leftrightarrow \quad E_1 \dot{u}^2 + 2F_1 \dot{u}\dot{v} + G_1 \dot{v}^2 = E_2 \dot{u}^2 + 2F_2 \dot{u}\dot{v} + G_2 \dot{v}^2$$

(i) $u = u_0 + t - t_0$, $v = v_0$. $\dot{u} = 1$, $\dot{v} = 0$.

$\Rightarrow E_1 = E_2$

(ii) $u = u_0$, $v = v_0 + t - t_0$ —

$\Rightarrow G_1 = G_2$.

(iii) $u = u_0 + t - t_0$, $v = v_0 + t - t_0 \Rightarrow \dot{u} = 1$, $\dot{v} = 1$

$\Rightarrow \cancel{E_1} + 2F_1 + G_1 = \cancel{E_2} + 2F_2 + G_2$.

$\Rightarrow F_1 = F_2$.

plan $\quad ds^2 = du^2 + dv^2$ $\quad$ cylinder: $\quad ds^2 = du^2 + dv^2$

$\Rightarrow$ plane is isometric to cylinder

$\boxed{\text{tangent developables}}$ $\quad \sigma(u,v) = \boxed{\gamma(u) + v\dot\gamma(u)}$

$(\kappa > 0)$

$\Rightarrow \quad \sigma_u \times \sigma_v = -\kappa v \mathbf{b} \neq 0 \iff \kappa > 0$.

Prop 6.2.4 Any tangent developable is isometric to plane.

Pf. $\quad \sigma(u,v) = \gamma(u) + v\dot\gamma(u)$.

$\sigma_u = \dot\gamma(u) + v\ddot\gamma(u)$, $\quad \sigma_v = \dot\gamma(u)$.

$\Rightarrow \quad E = 1 + v^2\kappa^2$, $\quad F = 1$, $\quad G = 1$

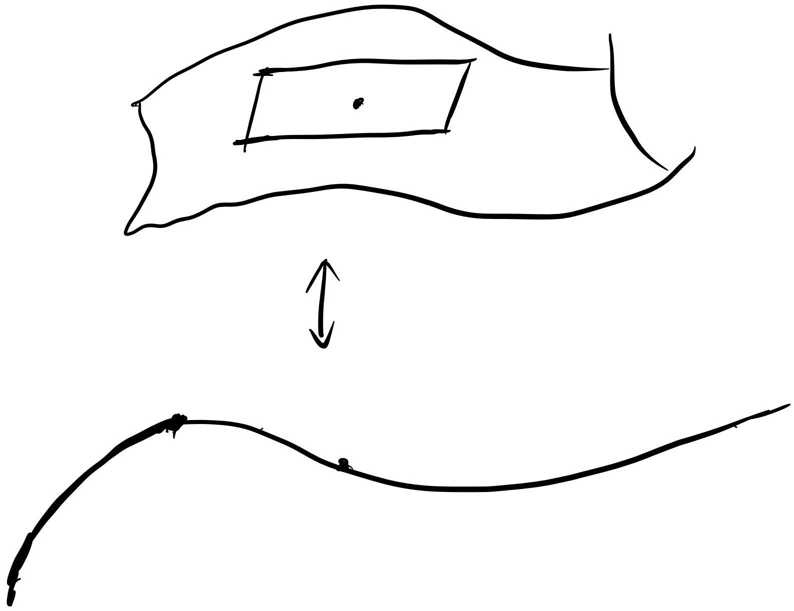
$\Rightarrow \quad ds^2 = (1 + v^2\kappa^2)\, du^2 + 2du\, dv + dv^2$.

$\kappa > 0$ $\quad \exists \quad \tau(t)$.

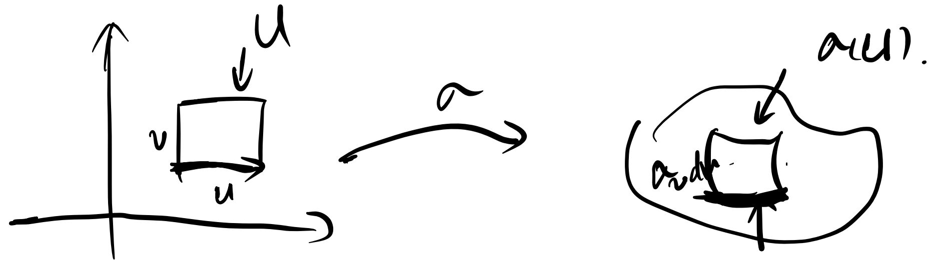
$P: \tau(t) + v\dot{\tau}(t)$

$\Rightarrow \quad ds^2 = (1+v^2\kappa^2)\,du^2 + 2\,du\,dv + dv^2$

Remark. Any surfaces isometric to plane is plane, a (generalised) cylinder, a (generalised) cone, a tangent developable.



§6.3 Surface area.



$|\vec{u}\times\vec{v}|$.

$\sigma(U) \approx \sigma_u du$.

$$\|\sigma_u du \times \sigma_v dv\| = \|\sigma_u\times\sigma_v\|\, du\, dv$$

Def 10.3.1 Let $\sigma: U \to \mathbb{R}^3$. $A_\sigma(R)$ is the area of the $\sigma(R)$

$$A_\sigma(R) = \iint_R \|\sigma_u\times\sigma_v\|\, du\, dv.$$

Prop 10.3.2 $\|\sigma_u\times\sigma_v\| = (EG-F^2)^{\frac{1}{2}}$

Pf.

$$\|\sigma_u\times\sigma_v\|^2 = \|\sigma_u\|^2\|\sigma_v\|^2 \sin^2\theta \quad = (1-\cos^2\theta)$$

$(\|\sigma_u\|\|\sigma_v\|\cos\theta)^2$

$$= \|\sigma_u\|^2\|\sigma_v\|^2 - \|\sigma_u\|^2\|\sigma_v\|^2\cos^2\theta. \quad \sigma_u\cdot\sigma_v$$

$$= EG - F^2$$

$$\Leftrightarrow \quad A_\sigma(R) = \iint_R (EG-F^2)^{\frac{1}{2}}\, du\, dv$$