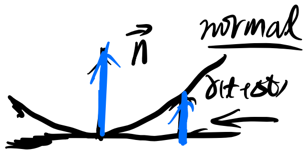
§5 Curvature of surfaces.

§5.1 The second fundamental formula.



$\|\ddot{\gamma}(t)\| = k$

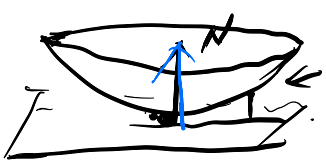
Taylor's formula:

$$\gamma(t+\Delta t) = \gamma(t) + \gamma'(t)\Delta t + \frac{1}{2}\gamma''(t)\Delta t^2 + o(|\Delta t|^2).$$

$$\Rightarrow \quad \gamma(t+\Delta t) - \gamma(t) = \gamma'(t)\Delta t + \frac{1}{2}\gamma''(t)\Delta t^2 + o(|\Delta t|^2) \qquad \|\gamma'(t)\| = 1$$

$$(\gamma(t+\Delta t) - \gamma(t))\cdot\vec{n} = \frac{1}{2}\gamma''(t)\cdot\vec{n}\,\Delta t^2 + o(|\Delta t|^2)$$

$$= \frac{1}{2}k\,\Delta t^2 + o(|\Delta t|^2)$$



Taylor's formula

$$N = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$$

$$\sigma(u+\Delta u, v+\Delta v) = \sigma(u,v) + \sigma_u(u,v)\Delta u + \sigma_v(u,v)\Delta v$$

$$+ \frac{1}{2}(\sigma_{uu}\Delta u^2 + 2\sigma_{uv}\Delta u\Delta v + \sigma_{vv}\Delta v^2) + \text{Higher order.}$$

$$\mathrm{II} = \sigma_{uu}\cdot\vec{N}\,du^2 + 2\sigma_{uv}\cdot\vec{N}\,du\,dv + \sigma_{vv}\cdot\vec{N}\,dv^2$$

$$g_{\mathrm{II}} = L\,du^2 + 2M\,du\,dv + N\,dv^2$$

g_{I}

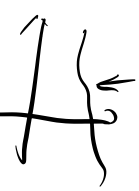
Example plane.

$$\sigma(u,v) = \vec{a} + u\vec{p} + v\vec{q}$$

$\sigma_u = \vec{p}$, $\sigma_v = \vec{q}$, $\sigma_{uu},\ \sigma_{uv},\ \sigma_{vv} = 0$.

$\mathrm{II} = 0$ $\qquad$ $\mathrm{I} = du^2 + dv^2$

$\dot{f}^2 + \dot{g}^2 = 1$



Example $\sigma(u,v) = (f(u)\cos v,\ f(u)\sin v,\ g(u))$.

- $f(u) = \cos u$, $g(u) = \sin u$ $\qquad$ $(\dot{f}\ddot{g} - \ddot{f}\dot{g})\,du^2 + f\dot{g}\,dv^2$

$L = \sigma_{uu}\cdot\vec{N} = 1$, $M = \sigma_{uv}\cdot\vec{N} = 0$, $N = \sigma_{vv}\cdot\vec{N} = \cos^2 u$.

$\mathrm{I} = du^2 + \cos^2 u\,dv^2$.

$\mathrm{II} = du^2 + \cos^2 u\,dv^2$.

- $f(u) = 1$, $g(u) = u$.

$L = M = 0$, $N = 1$

$\mathrm{II} = dv^2$

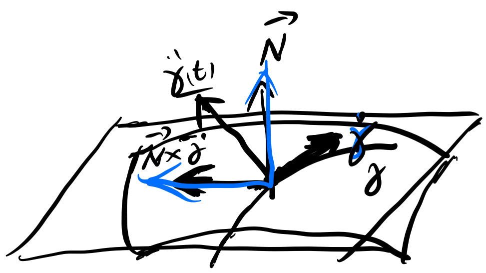
§5.2 The curvature of curves on a surface.

$a(u,v)$. $\quad\quad \gamma(t) = a(u(t), v(t))$

$$\dot\gamma(t) = \dot u\, a_u + \dot v\, a_v.$$

$$\vec N = \frac{a_u \times a_v}{\|a_u \times a_v\|}, \quad \dot\gamma \quad\quad \vec N \cdot \dot\gamma = 0.$$

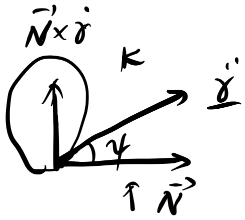
$$(\dot\gamma, \vec N, \vec N \times \dot\gamma) \quad\quad \ddot\gamma(t) \cdot \dot\gamma = 0$$



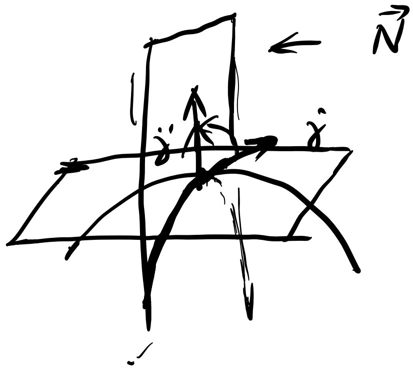
$$\ddot\gamma(t) = k_n \vec N + k_g \vec N \times \dot\gamma$$

k_n normal curvature. $\quad\quad$ k_g geodesic curvature.

$$k^2 = k_n^2 + k_g^2$$



$$k_n = k\cos\psi, \quad k_g = \pm k \sin\psi$$



$\gamma'' \times \vec{N} = 0$

$\underline{k_g = 0}$

$k_n = \pm k$

$\underline{k_n = k\cos\psi} \leftarrow$

§6.3 The normal and principle curvature

Thm. $\gamma(t) = \sigma(u(t), v(t))$ $\Rightarrow$ $\gamma'' = k_n \vec{N} + k_g \vec{N} \times \gamma'$; $k_n = \gamma'' \cdot \vec{N}$

$$k_n = L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2.$$

pf.

$$k_n = \gamma'' \cdot \vec{N} = \vec{N} \cdot \frac{d}{dt}(\gamma')$$

$$= \vec{N} \frac{d}{dt}(\sigma_u \dot{u} + \sigma_v \dot{v})$$

$$= \vec{N}(\sigma_{uu}\dot{u}^2 + 2\sigma_{uv}\dot{u}\dot{v} + \sigma_{vv}\dot{v}^2)$$

$$+ \cancel{\vec{N}(\sigma_u \ddot{u} + \sigma_v \ddot{v})}.$$

$$= L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2$$

$I = E\,du^2 + 2F\,du\,dv + G\,dv^2$, $II = L\,du^2 + 2M\,du\,dv + N\,dv^2$.

$$\mathcal{J}_I = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \qquad \mathcal{J}_{II} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

informly

$$I = (du, dv)\, \mathcal{J}_I \begin{pmatrix} du \\ dv \end{pmatrix}$$

$$t_1 = \xi_1 a_u + \eta_1 a_v \qquad t_2 = \xi_2 a_u + \eta_2 a_v$$

(ξ_1, η_1) $\qquad$ (ξ_2, η_2)

$$t_1 \cdot t_2 = (\xi_1 a_u + \eta_1 a_v) \cdot (\xi_2 a_u + \eta_2 a_v)$$

$$= (\xi_1, \eta_1)\, \mathcal{J}_I \begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix}$$

$$T_1 = \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} \qquad T_2 = \begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix}$$

$$t_1 \cdot t_2 = T_1^t \mathcal{J}_I T_2$$

$$\dot{\gamma}(t) = \xi a_u + \eta a_v \qquad T = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

$$\boxed{k_n = T^t \mathcal{J}_{II} T}$$

Def. The principal curvature of $\boxed{\text{a surface}}$ are the roots of

variable

$$\det(\boxed{\mathcal{J}_{II}} - x\mathcal{J}_I) = 0 \iff \det(\mathcal{J}_I^{-1}\mathcal{J}_{II} - xI) = 0$$

k_1, k_2 or $k_1 = k_2$ $\qquad k_i \qquad \mathcal{J}_I^{-1}\mathcal{J}_{II}$ eigenvalue

$\exists$ non-zero column vectore $T_i = \binom{\xi}{\eta}$ s.t.

$$(\mathcal{J}_{II} - k_i \mathcal{J}_I) T_i = 0 \qquad i = 1, 2$$

Def. $\vec{t} = \xi \sigma_u + \eta \sigma_v$ principal vector corresponding to k.

Prop. Let k_1, k_2 be the principle curvature at p

(a) If $k_1 \neq k_2$, $t_1 \cdot t_2 = 0$.

(b) If $k_1 = k_2$, every tangent vector at p is a principle vector.

Pf. (a) $t_i = \xi_i \sigma_u + \eta_i \sigma_v$. $T_i = \binom{\xi_i}{\eta_i}$ $i = 1, 2$.

$$t_1 \cdot t_2 = T_1^t \mathcal{J}_I T_2$$

$$\mathcal{J}_{II} T_1 = k_1 \mathcal{J}_I T_1, \quad \mathcal{J}_{II} T_2 = k_2 \mathcal{J}_I T_2.$$

$$T_2^t \mathcal{J}_{II} T_1 = k_1 T_2^t \mathcal{J}_I T_1, \quad T_1^t \mathcal{J}_{II} T_2 = k_2 T_1^t \mathcal{J}_I T_2.$$

$$\Rightarrow (T_2^t \mathcal{J}_{II} T_1)^t = T_1^t \mathcal{J}_{II} T_2 = k_1 T_1^t \mathcal{J}_I T_2$$

$$\Rightarrow k_1 T_1^t \mathcal{J}_I T_2 = k_2 T_1^t \mathcal{J}_I T_2$$

$\overset{k_1 \neq k_2}{\Longrightarrow} \quad T_1^t \mathcal{J}_I T_2 = 0$

$\Rightarrow \quad t_1 \cdot t_2 = 0$

(b) $t_1 \cdot t_2 = 0 \qquad T_i = \begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix} \qquad \|t_i\| = 1$

$$A = \begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix}$$

$$A^t \mathcal{J}_I A = \begin{pmatrix} \overset{t_1 \cdot t_1}{T_1^t \mathcal{J}_I T_1} & \overset{t_1 \cdot t_2}{T_1^t \mathcal{J}_I T_2} \\ T_2^t \mathcal{J}_I T_1 & T_2^t \mathcal{J}_I T_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $\quad \mathcal{G}_{II} = A^t \mathcal{J}_I A \qquad$ symmetric.

$$(\mathcal{G}_{II})^t = (A^t \mathcal{J}_I A)^t = A^t \mathcal{J}_I A = \mathcal{G}_{II}.$$

$\exists$ orthogonal matrix B, s.t.

$$B^t \mathcal{G}_{II} B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Let $\quad C = AB.$

$$\Rightarrow \quad C^t \mathcal{J}_I C = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = k \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$C^t g_1 C = (AB)^t g_1 AB$$
$$= B^t \underbrace{A^t g_1 A} B$$
$$= B^t B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\det C \neq 0$

$$\det(g_2 - k g_1) = 0 \iff \det\left(C^t (g_2 - k g_1) C\right) = 0.$$
$$\iff \det\left(\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} - k \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = 0$$

$$\iff (\lambda_1 - k)(\lambda_2 - k) = 0$$

$$\Rightarrow \lambda_1 = \lambda_2 = k.$$

$$C^t g_2 C = k \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = k\, C^t g_1 C$$

$$\iff g_2 - k g_1 = 0.$$

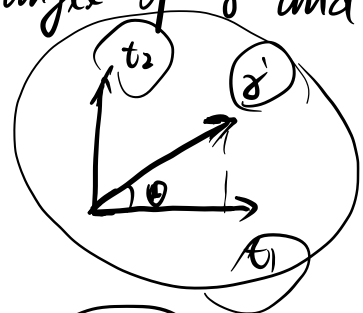
$$\iff \forall T, \quad (g_2 - k g_1) T = 0.$$

$t =$ [illegible] principle vector

Euler's Thm. k_1, k_2, t_1, t_2

$$k_n = k_1 \cos^2\theta + k_2 \sin^2\theta$$

where θ is the angle of γ' and t_1



$t_1 \cdot t_2 = 0$

Pf. Let $t_i = \xi_i a_u + \eta_i a_v$. $T_i = \begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix}$

$$\gamma' = \xi a_u + \eta a_v \qquad T = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

$$\gamma' = t_1 \cos\theta + t_2 \sin\theta \;-$$

$$= (\xi_1 \cos\theta + \xi_2 \sin\theta)\, a_u + (\eta_1 \sin\theta + \eta_2 \cos\theta)\, a_v .$$

$$\Rightarrow \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \cos\theta \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} + \sin\theta \begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix} \Leftrightarrow T = \cos\theta\, T_1 + \sin\theta\, T_2$$

$$k_n = T^t \mathcal{J}_{II} T$$

$$= (\cos\theta\, T_1 + \sin\theta\, T_2)^t \mathcal{J}_{II} (\cos\theta\, T_1 + T_2)$$

$$= \cos^2\theta\, T_1^t \mathcal{J}_{II} T_1 + \cos\theta \sin\theta \left(\cancel{T_1^t \mathcal{J}_{II} T_2 +} T_2^t \mathcal{J}_{II} T_1 \right)$$

$k_1 (T_1^t \mathcal{J}_{II} T_1)$ ...

$+ \sin^2\theta \, T_2^t \, \mathcal{J}_{\mathrm{II}} \, T_2$

$= k_1 \cos^2\theta + k_2 \sin^2\theta.$

$\theta = 0$

Cor. $k_n = k_1 \cos^2\theta + k_2 \sin^2\theta.$

$k_n = k_1.$

$\theta = \frac{\pi}{2}$

$k_n = k_2$

Exam. $z = k' x^2 + k'' y^2.$

$\alpha(u,v) = (u, v, k'u^2 + k''v^2)$

$\mathrm{I} = du^2 + dv^2.$

$\mathrm{II} = 2k' du^2 + 2k'' dv^2.$

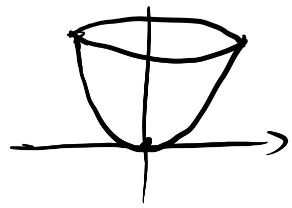
$\mathcal{J}_{\mathrm{I}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{J}_{\mathrm{II}} = \begin{pmatrix} 2k' & 0 \\ 0 & 2k'' \end{pmatrix}$

$\det(\mathcal{J}_{\mathrm{II}} - k \mathcal{J}_{\mathrm{I}}) = \begin{vmatrix} 2k' - k & 0 \\ 0 & 2k'' - k \end{vmatrix} = 0$

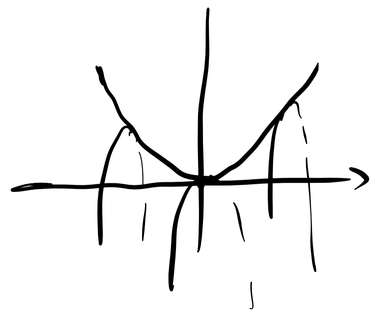
$\Rightarrow \quad k_1 = 2k' \quad k_2 = 2k''$

$\Rightarrow \quad z = \frac{1}{2}(k_1 x^2 + k_2 y^2)$

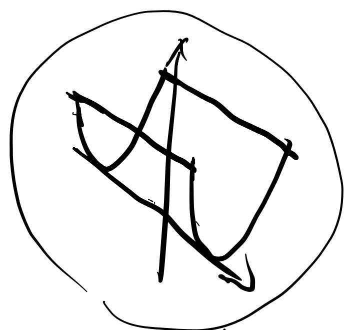
$k_1 > 0,\ k_2 > 0$



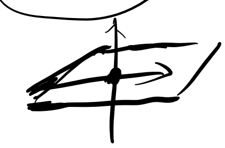
$k_1 < 0,\ k_2 > 0$



$k_1 = 0,\ k_2 \neq 0$



$k_1 = 0,\ k_2 = 0$



Example $\quad x(\theta, \varphi) = (\cos\theta\cos\varphi,\ \cos\theta\sin\varphi,\ \sin\theta)$

$$I = d\varphi^2 + \cos^2\theta\, d\theta^2$$

$$II = d\varphi^2 + \cos^2\theta\, d\theta^2$$

$$\begin{vmatrix} 1-k & 0 \\ 0 & \cos^2\theta - k\cos^2\theta \end{vmatrix} = 0$$

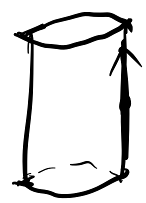
$\Rightarrow \quad k_1 = k_2 = 1$

<u>Example</u>. $\quad \alpha(u,v) = (\cos u, \sin u, v)$

$$\mathrm{I} = du^2 + dv^2$$

$$\mathrm{II} = dv^2$$

$$\begin{vmatrix} 0-k & 0 \\ 0 & 1-k \end{vmatrix} = 0$$



$\Rightarrow \quad k_1 = 0, \quad k_2 = 1.$

§6. Guassian Curvature and the Guass Map.

<u>Def</u>. Let k_1 and k_2 be the principle curvatures of the surface. Then, the Guassian curvature of the surface is

$$K = k_1 k_2.$$

$\det(\mathrm{I}^{-1}\mathrm{II})$

and its mean curvature is

$$H = \frac{1}{2}(k_1 + k_2).$$

trace

$$\det(\mathrm{II} - k\,\mathrm{I}) = 0 \Leftrightarrow \det(\mathrm{I}^{-1}\mathrm{II} - k\,\mathrm{1}) = 0.$$

Prop (i) $K = \dfrac{LN - M^2}{EG - F^2} \left(= \det \mathcal{F}_2^{-1} \mathcal{F}_1 = \dfrac{\det \mathcal{F}_2}{\det \mathcal{F}_1} \right)$

(ii) $H = \dfrac{LG - 2MF + NE}{2(EG - F^2)}$.

(iii) The principle curvatures are $H \pm \sqrt{H^2 - K}$.

(iii) $K = k_1 k_2 \qquad H = \frac{1}{2}(k_1 + k_2)$

$$k^2 + ak + b = 0$$

$$\Rightarrow \quad b = k_1 k_2, \quad a = -(k_1 + k_2).$$

$$b = K, \quad a = -2H$$

$$\Rightarrow \quad k^2 - 2Hk + K = 0$$

$$\Rightarrow \quad k = \frac{2H \pm \sqrt{(2H)^2 - 4K}}{2} = H \pm \sqrt{H^2 - K}$$