Recall

$K = k_1 k_2$. $\qquad H = \frac{k_1+k_2}{2}$.

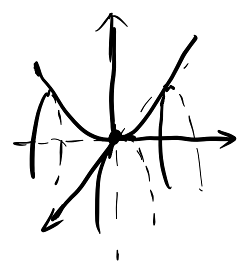
Prop. $K = \frac{LN - M^2}{EG - F^2}$, $H = \frac{LG - 2MF + NE}{2(EG - F^2)}$.

Exam. plane. $K = 0$

unit sphere. $k_1 = k_2 = 1$, $K = 1$, $H = 1$.

cylinder $k_1 = 1$, $k_2 = 0 \Rightarrow K = 0$.

$z = x^2 - y^2$



$k_1 = 2$, $k_2 = -2$.

$\Rightarrow K = -4$ at 0 .

General $K = \frac{-4}{(1 + 4x^2 + 4y^2)^2}$ (x, y)

Exam. $\alpha(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$. $\boxed{f'^2 + g'^2 = 1}$

$E = 1$, $F = 0$, $G = f^2$

$L = f' g'' - f'' g'$, $M = 0$, $N = f g'$

$\Rightarrow \quad K = \dfrac{(\dot f \ddot g - \ddot f \dot g)\,\cancel{f}\,g'}{f^{\cancel{2}}}$

Note that $\boxed{\dot f^2 + \dot g^2 = 1} \xRightarrow{\text{differential}} \dot f \ddot f + \dot g \ddot g = 0.$

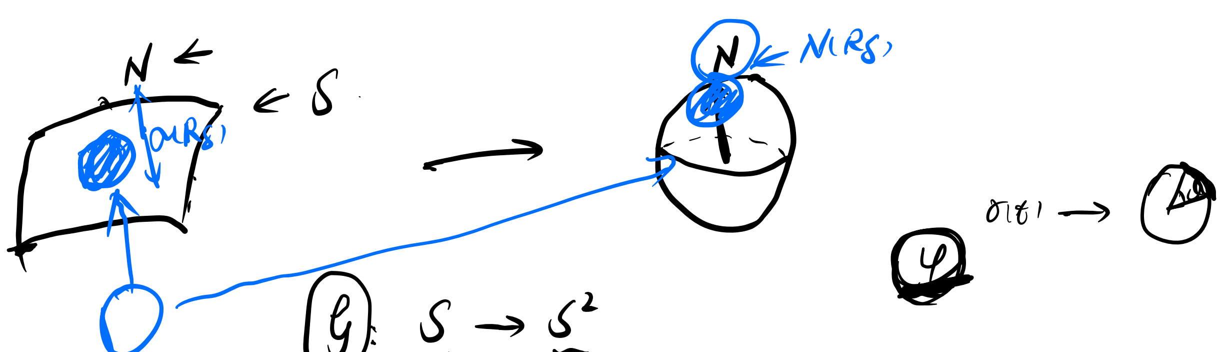
$(\dot f \ddot g - \ddot f \dot g)\dot g = -\dot f^2 \ddot f - \ddot f \dot g^2 = -\ddot f(\boxed{\dot f^2 + \dot g^2}) = -\ddot f$

$\Rightarrow \quad K = -\dfrac{\ddot f}{f}.$

unit sphere $\quad f = \cos u. \quad \Rightarrow \quad K = 1$

cylinder $\quad f = 1. \quad \Rightarrow \quad K = 0$

$\dfrac{d\varphi}{dt} = K.$



$\mathcal{G}: \underset{\sim}{S} \to \underset{\sim}{S^2}$

$\alpha(u,v) \mapsto N(u,v).$

Gauss map, denoted by $\mathcal{G}$

$$D_p\mathcal{G} : T_pS \longrightarrow T_{\mathcal{G}(p)}S^2 \cong T_pS$$ linear map.

$$\Rightarrow \quad D_p\mathcal{G} : T_pS \to T_pS.$$

Def. Weingarten map.

$$\mathcal{W}_{p,S} = -D_p\mathcal{G}$$

Prop. The second fundamental form of S at p

$$\langle\langle v, w\rangle\rangle = \mathrm{II}(v, w) = \langle \mathcal{W}_{p,S}(v), w\rangle, \quad v, w \in T_pS.$$

$$\| $$

$$L\, du(\vec{v})\, du(\vec{w}) + 2M\, du(\vec{v})\, dv(\vec{w}) + N\, dv(\vec{v})\, dv(\vec{w})$$

Thm. Let $\sigma : U \to \mathbb{R}^3$ be surface. Let $(u_0, v_0) \in U$, let $\delta > 0$ be such the disc

$$R_\delta = \{(u,v) : (u-u_0)^2 + (v-v_0)^2 \le \delta^2\} \subset U.$$

Then

$$\lim_{\delta \to 0} \frac{\mathcal{A}_N(R_\delta)}{\mathcal{A}_\sigma(R_\delta)} = |K|,$$

where

$$\frac{A_N(R_\delta)}{A_\alpha(R_\delta)} = \frac{\text{area of } N(R_\delta)}{\text{area of } \alpha(R_\delta)}$$

Pf.

$$\frac{A_N(R_\delta)}{A_\alpha(R_\delta)} = \frac{\iint_{R_\delta} \| N_u \times N_v \| \, du dv}{\iint_{R_\delta} \| \alpha_u \times \alpha_v \| \, du dv}$$

Lem. $N_u = a\,\alpha_u + b\,\alpha_v$, $\quad N_v = c\,\alpha_u + d\,\alpha_v$.

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = -\mathcal{F}_I^{-1}\mathcal{F}_{II}$$

$$\begin{aligned}
N_u \times N_v &= ad\, \alpha_u \times \alpha_v - bc\, \alpha_u \times \alpha_v \\
&= (ad - bc)\, \alpha_u \times \alpha_v \\
&= \det(-\mathcal{F}_I^{-1}\mathcal{F}_{II})\, \alpha_u \times \alpha_v \\
&= \frac{\det \mathcal{F}_{II}}{\det \mathcal{F}_I}\, \alpha_u \times \alpha_v \\
&= K\, \alpha_u \times \alpha_v .
\end{aligned}$$

$$\lim_{\delta \to 0} \frac{A_N(R_\delta)}{A_\alpha(R_\delta)} = \lim_{\delta \to 0} \frac{\iint_{R_\delta} |K| \, \| \alpha_u \times \alpha_v \| \, du dv}{\iint_{R_\delta} \| \alpha_u \times \alpha_v \| \, du dv} = |K|(u_0, v_0).$$

pf of Lem.

$$N_u \perp N, \quad N_v \perp N. \qquad \|N\| = 1.$$

Then

$$N_u = a\,\sigma_u + b\,\sigma_v, \qquad N_v = c\,\sigma_u + d\,\sigma_v.$$

$$N \cdot \sigma_u = 0$$

$$\Rightarrow \quad N_u \cdot \sigma_u + N \cdot \sigma_{uu} = 0 \quad \Rightarrow \quad \boxed{N_u \cdot \sigma_u = -L}$$

$$\Rightarrow \quad \boxed{N_u \cdot \sigma_v = N_v \cdot \sigma_u = -M}, \quad \boxed{N_v \cdot \sigma_v = -N}$$

$$-L = aE + bF \qquad -M = cE + dF$$

$$-M = aF + bG, \qquad -N = cF + dG$$

$$\Rightarrow \quad -\begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$-\mathcal{F}_{II} = \mathcal{F}_I \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} a & c \\ b & d \end{pmatrix} = -\mathcal{F}_I^{-1} \mathcal{F}_{II}. \qquad \square.$$

Exam. plane $N = \text{const}$. $N(R_\delta) = 0$ $\Rightarrow K = 0$.

$\iint_{R_\delta} \|N_u \times N_v\| \, du dv$

$\alpha(\theta, \varphi) = (\cos\theta \cos\varphi, \cos\theta \sin\varphi, \sin\theta)$

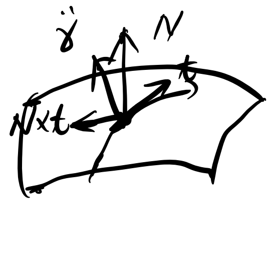
$N = -\alpha$. $N_u \times N_v$

$\frac{A_N(R_\delta)}{A_\alpha(R_\delta)} = 1 \Rightarrow K = 1$.

§7. Geodesic. (测地线)

§7.1 Definition and properties

Def 7.1.1. A unit speed curve γ on a surface α is called a geodesic if $\ddot{\gamma}(t)$ is perpendicular to the surface at $\gamma(t)$. i.e. $\ddot{\gamma} \parallel N$ at $\gamma(t)$.



$\ddot{\gamma} = k_n N + k_g N \times \vec{t}$.

γ is a geodesic $\Leftrightarrow$ $k_g = 0$.

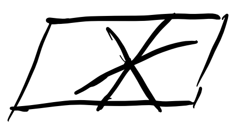
Exercise. $\|\dot{\gamma}\| = \text{const}$.

Prop 7.1.2. Any (part of a) straight line on a surface is a geodesic.

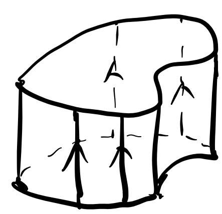
pf. $\gamma(t) = a + bt$.

$\underline{\ddot{\gamma}(t) = 0 \parallel N}$. $\boxed{\checkmark}$.

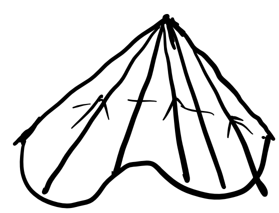
Exam 7.1.3 ①



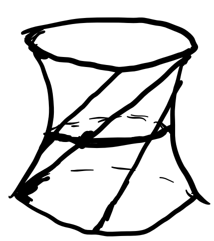
②



③



④

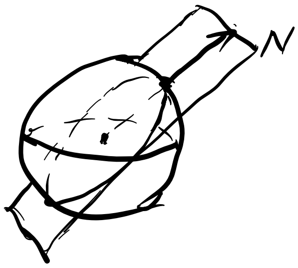


$a(u,v) = \gamma(u) + v\,\delta(u)$

Prop 7.1.4 Any normal section of surface is a geodesic.

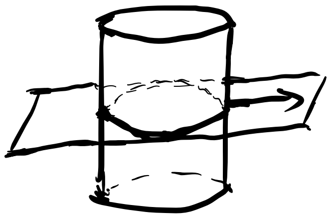
$\underline{\ddot{\gamma} \parallel N} \Rightarrow$ geodesic ✓.

Exam 7.1.5



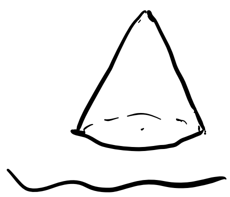
great circles on sphere are geodesic.

Exam 7.1.6



$\sigma(u,v) = (f(u), g(u), v)$

$\Rightarrow N(u,v) = \dfrac{(\dot f, -\dot g, 0)}{\sqrt{\dot f^2 + \dot g^2}}$



Thm 7.1.7. A unit speed curve $\gamma(t) = \sigma(u(t), v(t))$ is geodesic if and only if

$$\begin{cases} \dfrac{d}{dt}(E\dot u + F\dot v) = \dfrac{1}{2}(E_u \dot u^2 + 2F_u \dot u\dot v + G_u \dot v^2). \\ \dfrac{d}{dt}(F\dot u + G\dot v) = \dfrac{1}{2}(E_v \dot u^2 + 2F_v \dot u\dot v + G_v \dot v^2). \end{cases}$$

Pf. γ is geodesic if and only if $\ddot\gamma \perp \sigma_u$, $\ddot\gamma \perp \sigma_v$.

$\dot\gamma = \dot u \sigma_u + \dot v \sigma_v$

$$\Longleftrightarrow \begin{cases} \frac{d}{dt}(\dot u \sigma_u + \dot v \sigma_v)\cdot \sigma_u = 0 \\ \frac{d}{dt}(\dot u \sigma_u + \dot v \sigma_v)\cdot \sigma_v = 0 \end{cases}$$

Exercise

we compute

$$\frac{d}{dt}\left((\dot u \sigma_u + \dot v \sigma_v)\cdot \sigma_u\right) = \frac{d}{dt}(E\dot u + F\dot v) - (\dot u \sigma_u + \dot v \sigma_v)\frac{d\sigma_u}{dt}$$

$$= \frac{d}{dt}(E\dot u + F\dot v) - (\dot u \sigma_u + \dot v \sigma_v)(\dot u \sigma_{uu} + \dot v \sigma_{uv})$$

$$= \frac{d}{dt}(E\dot u + F\dot v) - (\dot u^2 \sigma_{uu}\cdot\sigma_u + \dot u \dot v(\sigma_u \sigma_{uv} + \sigma_v \sigma_{uu}) + \dot v^2 \sigma_{uv}\cdot\sigma_v)$$

Note that

$$E_u = \frac{\partial}{\partial u}(\sigma_u\cdot\sigma_u) = 2\sigma_{uu}\cdot\sigma_u$$

$$F_u = \frac{\partial}{\partial u}(\sigma_u\cdot\sigma_v) = \sigma_{uu}\cdot\sigma_v + \sigma_u\cdot\sigma_{uv}$$

$$G_u = \frac{\partial}{\partial u}(\sigma_v\cdot\sigma_v) = 2\sigma_{uv}\cdot\sigma_v$$

$$\Rightarrow 0 = \frac{d}{dt}(E\dot u + F\dot v) - \frac{1}{2}(E_u \dot u^2 + 2F_u \dot u\dot v + 2G_u \dot v^2).$$

$\Rightarrow$ First equation $\boxed{\checkmark}$

$$\begin{cases} \ddot{u} = f(u, v, \dot{u}, \dot{v}) \\ \ddot{v} = g(u, v, \dot{u}, \dot{v}) \end{cases} \qquad f, g \in C^\infty.$$

$u(t_0) = a$, $v(t_0) = b$, $\dot{u}(t_0) = c$, $\dot{v}(t_0) = d$

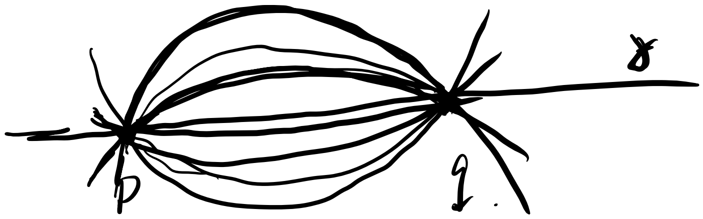
$\Rightarrow \exists\, u, v \quad \exists!$ geodesic $\quad t \in (t_0 - \varepsilon, t_0 + \varepsilon) \quad \gamma(t)$

Prop 7.1.8. Let P be a point in the surface, let $\vec{t}$ be a unit tangent vector to σ at P. Then, there exists a unique unit speed geodesic γ on σ which passes through P and has tangent vector $\vec{t}$ there.

Cor 7.1.9. An isometry between two surfaces takes the geodesics of one surface to the geodesics of the other.

Exam 7.1.10.

§7.2 Geodesics as shortest paths.



$\gamma_\tau(t) : (-\varepsilon, \varepsilon) \to S$. $\tau \in (-\delta, \delta)$.

$-\varepsilon < a < b < \varepsilon$

(1) $\gamma_\tau(a) = p$, $\gamma_\tau(b) = q$, $\forall \tau \in (-\delta, \delta)$

(2) $\gamma : (-\delta, \delta) \times (-\varepsilon, \varepsilon) \to S$.

$(\tau, t) \mapsto \gamma_\tau(t)$, (smooth).

$$L(\gamma_\tau) = \int_a^b \|\dot\gamma_\tau\| \, dt .$$ $\tau = 0$, $\gamma_0 = \gamma(t)$.

Thm 7.2.1. The unit speed curve γ is a geodesic if and only if

$$\frac{d}{d\tau} L(\gamma_\tau) \Big|_{\tau=0} = 0.$$

Pf.
$$\frac{d}{d\tau} L(\gamma_\tau) = \frac{d}{d\tau} \int_a^b \|\dot\gamma_\tau\| \, dt$$
$$= \int_a^b \frac{\partial}{\partial \tau} \Big(\underbrace{E\dot u^2 + 2F \dot u \dot v + G \dot v^2}_{g(\tau, t)} \Big)^{\frac{1}{2}} dt$$

$$\underline{E(u(\tau,t),v(\tau,t))} = \frac{1}{2}\int_a^b g^{-\frac{1}{2}} \frac{\partial g}{\partial \tau}\, dt.$$

$$\frac{\partial g}{\partial \tau} = \left(\frac{\partial E}{\partial \tau}\right)\dot u^2 + 2\frac{\partial F}{\partial \tau}\dot u \dot v + \frac{\partial G}{\partial \tau}\dot v^2 + 2E\dot u\frac{\partial \dot u}{\partial \tau} + 2F\left(\frac{\partial \dot u}{\partial \tau}\dot v + \dot u\frac{\partial \dot v}{\partial \tau}\right) + 2G\dot v\frac{\partial \dot v}{\partial \tau})$$

$$= \left(E_u\frac{\partial u}{\partial \tau} + E_v\frac{\partial v}{\partial \tau}\right)\dot u^2 + 2\left(F_u\frac{\partial u}{\partial \tau} + F_v\frac{\partial v}{\partial \tau}\right)\dot u\dot v + \left(G_u\frac{\partial u}{\partial \tau} + G_v\frac{\partial v}{\partial \tau}\right)\dot v^2$$

$$+ 2E\dot u\frac{\partial^2 u}{\partial\tau\partial t} + 2F\left(\frac{\partial^2 u}{\partial\tau\partial t}\dot v + \frac{\partial^2 v}{\partial\tau\partial t}\dot u\right) + 2G\dot v\frac{\partial^2 v}{\partial\tau\partial t}$$

$$\int_a^b g^{-\frac{1}{2}}\left\{(E\dot u + F\dot v)\frac{\partial^2 u}{\partial\tau\partial t} + (F\dot u + G\dot v)\frac{\partial^2 v}{\partial\tau\partial t}\right\} dt$$

$$= g^{-\frac{1}{2}}(E\dot u + F\dot v)\frac{\partial u}{\partial \tau} + (F\dot u + G\dot v)\frac{\partial v}{\partial \tau}\Big|_a^b$$

$$- \int \left(\frac{d}{dt}\left(g^{-\frac{1}{2}}(E\dot u + F\dot v)\right)\frac{\partial u}{\partial \tau} + \frac{d}{dt}\left(g^{-\frac{1}{2}}(F\dot u + G\dot v)\right)\frac{\partial v}{\partial \tau}\right) dt.$$

$\gamma_\tau(a) = p, \quad \gamma_\tau(b) = q.$

$\frac{\partial \gamma_\tau}{\partial \tau}(a) = 0, \quad \frac{\partial \gamma_\tau}{\partial \tau}(b) = 0$

$\Rightarrow \frac{\partial u}{\partial \tau}\sigma_u + \frac{\partial v}{\partial \tau}\sigma_v = 0 \Rightarrow \frac{\partial u}{\partial \tau}(a) = 0, \quad \frac{\partial v}{\partial \tau}(a) = 0$

$\frac{\partial u}{\partial \tau}(b) = 0 \quad \frac{\partial v}{\partial \tau}(b) = 0.$

$$\frac{d}{d\tau} L(\gamma_\tau) = \int_a^b U \frac{\partial u}{\partial \tau} + V \frac{\partial v}{\partial \tau} \, dt \quad \cdots, \ \tau = 0$$

$$U = \frac{1}{2} g^{-\frac{1}{2}} (E_u \dot{u}^2 + 2F_u \dot{u}\dot{v} + G_u \dot{v}^2) - \frac{d}{dt} (g^{-\frac{1}{2}} (E\dot{u} + F\dot{v}))$$

$$V = \frac{1}{2} g^{-\frac{1}{2}} (E_v \dot{u}^2 + 2F_v \dot{u}\dot{v} + G_v \dot{v}^2) - \frac{d}{dt} (g^{-\frac{1}{2}} (F\dot{u} + G\dot{v}))$$

$\gamma_0 = \gamma$ unit speed. $\| g(0,t) \| = 1$

$\Rightarrow$ γ is a geodesic.

$\Leftarrow$ $\| \dot{\gamma} \| = \text{const}$

$\boxed{\phi \quad \text{cut-off}}$