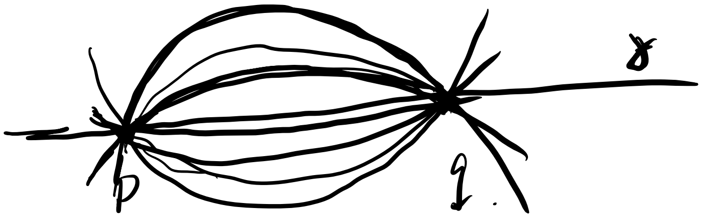
§7.2 Geodesics as shortest paths.



$\gamma_\tau(t) : (-\varepsilon, \varepsilon) \to S$. $\quad \tau \underline{\in (-\delta, \delta)}$.

$-\varepsilon < a < b < \varepsilon$

(1) $\gamma_\tau(a) = p$, $\gamma_\tau(b) = q$, $\forall \tau \in (-\delta, \delta)$

(2) $\gamma : (-\delta, \delta) \times (-\varepsilon, \varepsilon) \to S$.

$(\tau, t) \mapsto \gamma_\tau(t)$, (<u>smooth</u>).

$$L(\gamma_\tau) = \underline{\int_a^b \|\dot\gamma_\tau\| \, dt}. \qquad \underline{\tau = 0, \ \gamma_0 = \gamma(t)}.$$

Thm 7.2.1. The unit speed curve <u>γ is a geodesic</u> if and only if

$$\underline{\frac{d}{d\tau} L(\gamma_\tau) \Big|_{\tau=0} = 0}. \qquad \underline{\text{variation}}$$

Pf.

$$\underline{\frac{d}{d\tau} L(\gamma_\tau) = \boxed{\frac{d}{d\tau}} \int_a^b \boxed{\|\dot\gamma_\tau\|} \, dt}$$

$$= \int_a^b \boxed{\frac{\partial}{\partial\tau}} (\overset{g(\tau,t)}{\underline{E\dot u^2 + 2F\dot u\dot v + G\dot v^2}})^{\frac{1}{2}} \, dt$$

$$= \frac{1}{2}\int_a^b g^{-\frac{1}{2}} \boxed{\frac{\partial g}{\partial \tau}}\, dt.$$

$$\frac{\partial g}{\partial \tau} = \underline{\frac{\partial E}{\partial \tau}\dot{u}^2 + 2\frac{\partial F}{\partial \tau}\dot{u}\dot{v} + \frac{\partial G}{\partial \tau}\dot{v}^2 + 2E\dot{u}\frac{\partial \dot{u}}{\partial \tau} + 2F\left(\frac{\partial \dot{u}}{\partial \tau}\dot{v} + \dot{u}\frac{\partial \dot{v}}{\partial \tau}\right) + 2G\dot{v}\frac{\partial \dot{v}}{\partial \tau}\Big)}$$

$$= \left(E_u\frac{\partial u}{\partial \tau} + E_v\frac{\partial v}{\partial \tau}\right)\dot{u}^2 + 2\left(F_u\frac{\partial u}{\partial \tau} + F_v\frac{\partial v}{\partial \tau}\right)\dot{u}\dot{v} + \left(G_u\frac{\partial u}{\partial \tau} + G_v\frac{\partial v}{\partial \tau}\right)\dot{v}^2$$

$$+ 2E\dot{u}\boxed{\frac{\partial^2 u}{\partial \tau \partial t}} + 2F\left(\boxed{\frac{\partial^2 u}{\partial \tau \partial t}}\dot{v} + \boxed{\frac{\partial^2 v}{\partial \tau \partial t}}\dot{u}\right) + 2G\dot{v}\boxed{\frac{\partial^2 v}{\partial \tau \partial t}}$$

$$\int_a^b g^{-\frac{1}{2}}\left\{(E\dot{u} + F\dot{v})\boxed{\frac{\partial^2 u}{\partial \tau \partial t}} + (F\dot{u} + G\dot{v})\boxed{\frac{\partial^2 v}{\partial \tau \partial t}}\right\} dt$$

$$= \boxed{g^{-\frac{1}{2}}(E\dot{u} + F\dot{v})\boxed{\frac{\partial u}{\partial \tau}} + (F\dot{u} + G\dot{v})\boxed{\frac{\partial v}{\partial \tau}}\Big|_{\boxed{a}}^{\boxed{b}}}$$

$$- \int \left(\frac{d}{dt}\left(g^{-\frac{1}{2}}(E\dot{u} + F\dot{v})\frac{\partial u}{\partial \tau} + \frac{d}{dt}g^{-\frac{1}{2}}(F\dot{u} + G\dot{v})\right)\frac{\partial v}{\partial \tau}\right) dt.$$

$\gamma_\tau(a) = p, \quad \gamma_\tau(b) = q.$

$\frac{\partial \gamma_\tau}{\partial \tau}(a) = 0, \quad \frac{\partial \gamma_\tau}{\partial \tau}(b) = 0$

$$\Rightarrow \frac{\partial u}{\partial \tau}\sigma_u + \frac{\partial v}{\partial \tau}\sigma_v = 0 \Rightarrow \boxed{\begin{array}{ll} \frac{\partial u}{\partial \tau}(a) = 0, & \frac{\partial v}{\partial \tau}(a) = 0 \\ \frac{\partial u}{\partial \tau}(b) = 0 & \frac{\partial v}{\partial \tau}(b) = 0. \end{array}}$$

$$\frac{d}{d\tau} L(\gamma_\tau) = \int_a^b U \frac{\partial u}{\partial \tau} + V \frac{\partial v}{\partial \tau} \; dt \; . \quad , \tau = 0$$

$$U = \frac{1}{2} g^{-\frac{1}{2}} \left(\boxed{E_u \dot u^2 + 2F_u \dot u \dot v + G_u \dot v^2} \right) - \frac{d}{dt} \left(g^{-\frac{1}{2}} \boxed{(E \dot u + F \dot v)} \right)$$

$$V = \frac{1}{2} g^{-\frac{1}{2}} \left(E_v \dot u^2 + 2F_v \dot u \dot v + G_v \dot v^2 \right) - \frac{d}{dt} \left(g^{-\frac{1}{2}} (F \dot u + G \dot v) \right)$$

① $\gamma_0 = \gamma$ unit speed. $\| g(0,t) \| = 1$

$\Rightarrow$ γ is a geodesic.

$\Leftarrow$ $\|\dot\gamma\| = \text{const}$

$\boxed{\phi \;\; \text{cut-off}}$

It suffices to show

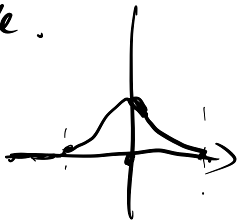
$$\int_a^b U \frac{\partial u}{\partial \tau} + V \frac{\partial v}{\partial \tau} \; dt \Big|_{\tau = 0} = 0 \Rightarrow U = 0, \; V = 0 \Big|_{\tau = 0}$$

$\exists \; t_0 \in (a,b)$ s.t. $U(0, t_0) > 0$. $\exists \; \eta > 0$, s.t. $\forall \; t \in (t_0 - \eta, t_0 + \eta)$

↑ continuity

$U(0,t)>0$. Let $\phi(t)$ be a smooth function such that

$\phi>0$ on $(t_0-\eta, t_0+\eta)$, $\phi=0$, else.



$u(\tau,t)=u(t)+\tau\phi(t)$. $\quad v(\tau,t)=v(t)$

$\frac{\partial u}{\partial \tau}=\phi(t)$. $\quad \frac{\partial v}{\partial \tau}=0$.

$$\Rightarrow \int_a^b U\phi(t)\,dt=0, \quad \tau=0$$

$$\Rightarrow \int_{t_0-\eta}^{t_0+\eta} \boxed{U}\phi(t)\,dt \boxed{=0}, \quad \tau=0.$$

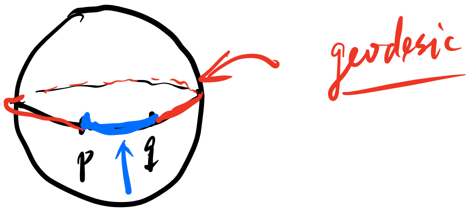
>0, contradiction.

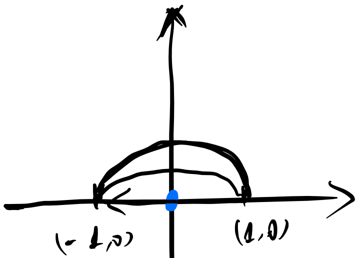
$\Rightarrow U(0,t)=0$.
Similarly, $V(0,t)=0$. $\Big\} \Rightarrow$ γ is a geodesic. ✓

$$\boxed{\theta(t)=\begin{cases} e^{-\frac{1}{t^2}}, & t>0 \\ 0, & t\le 0\end{cases}} \Leftarrow \text{smooth}$$

- If γ is shortest path, then γ is a geodesic.

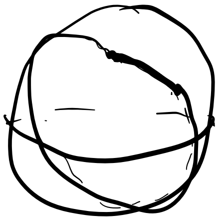
$\not\Leftarrow$



- 

<u>complete</u>. <u>完备</u>.

- $\mathbb{R}^3$ <u>closed subset</u>.



§7. Gauss's. Theorema Egregium.

<u>§7.1</u>. Gauss's remarkable theorem.

$$K = \underline{k_1 k_2} = \frac{\boxed{LN-M^2}}{EG-F^2}. \quad \underline{\det \mathrm{II}}.$$

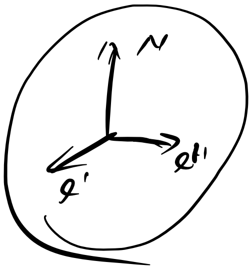
<u>Thm 7.1.1</u>. The gaussian curvature of a surface depends only on its first fundamental form, i.e. it is presserved by isometric.

intrinsic

$e' = \underline{e'(u,v)}$, $e'' = \underline{e''(u,v)}$

orthonormal basis $\{\underline{e', e''}\}$ of $\boxed{T_pS}$ $\leftarrow \{\boxed{a_u, a_v}\}$

$$\begin{cases} e' \cdot e'' = 0 \\ \|e'\| = 1 , \|e''\| = 1 . \end{cases}$$



$e'_u = a e' + N$

$e'_v = \beta e'' + \mu N$

$e''_u = -a' e' + \lambda' N$

$e''_v = -\beta' e' + \mu' N$

$\boxed{e' \cdot e'' = 0} \Rightarrow a = a' , b = b'$

$\underline{e'_u = a e'' + \lambda' N}$

$e'_v = \beta e'' + \mu' N$

$\underline{e''_u = -a e' + \lambda'' N}$

$e''_v = -\beta e' + \mu'' N$

Lem.

$e'_u \cdot e''_v - e''_u \cdot e'_v = \boxed{\lambda' \mu'' - \lambda'' \mu'}$

$$= \alpha_v - \beta_u$$

$$= \frac{LN - M^2}{(EG - F^2)^{\frac{1}{2}}} = \underline{K}\,(EG - F^2)^{\frac{1}{2}}$$

Pf. $\alpha_v - \beta_u = \frac{\partial}{\partial u}(e' \cdot e''_v) - \frac{\partial}{\partial v}(e' \cdot e''_u)$

$$= e'_u \cdot e''_v + e' \cancel{e''_{uv}} - e'_v \cdot e''_u - \cancel{e'} \cdot e''_{vu}$$

$$= e'_u e''_v - e'_v \cdot e''_u$$

$$\underline{N_u \times N_v = K\, a_u \times a_v}$$

$$N = \frac{a_u \times a_v}{\|a_u \times a_v\|}, \quad \|a_u \times a_v\| = (EG - F^2)^{\frac{1}{2}}$$

$$\Rightarrow \quad \underline{N_u \times N_v} = \frac{LN - M^2}{(EG - F^2)^{\frac{1}{2}}}\, N$$

$$(N_u \times N_v) \cdot N = \frac{LN - M^2}{(EG - F^2)^{\frac{1}{2}}}$$

- $\boxed{(a \times b) \cdot (c \times d) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c)}$ (exercise)

$\underline{a}, \underline{b}, \underline{c}, \underline{d}$

$$\Rightarrow \quad (N_u \times N_v) \cdot N = (N_u \times N_v) \cdot (e' \times e'')$$

$$= (N_u \cdot e')(N_v \cdot e'') - (N_u \cdot e'')(N_v \cdot e')$$

- $\underline{N\cdot e'=0} \Rightarrow N_u\cdot e' + N\cdot e'_u = 0$

$$= (N\cdot e'_u)\cdot(N\cdot e''_v) - (N\cdot e''_u)(N\cdot e'_v)$$

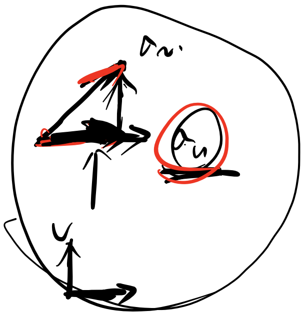
$$= \underline{\lambda'\mu'' - \lambda''\mu'}.$$

<u>pf of Th 7.1.1</u>. Note that

$$K = \frac{\alpha_v - \beta_u}{(EG-F^2)^{\frac{1}{2}}}.$$

$\{\sigma_u, \sigma_v\}$ is a basis of T_pS.

<u>Gram-Schmidt process</u>



$$e' = \frac{\sigma_u}{\|\sigma_u\|} = \varepsilon\sigma_u \qquad \varepsilon = \|\sigma_u\|^{-1}$$

$$e'' = \gamma\sigma_u + \delta\sigma_v. \qquad e'\cdot e'' = 0,\ \|e''\| = 1.$$

$$\Rightarrow \begin{cases} E^{-\frac{1}{2}}(\gamma E + \delta F) = 0 \\ \gamma^2 E + 2\gamma\delta F + \delta^2 G = 1 \end{cases}$$

$\Rightarrow$

$$\begin{cases} \gamma = \dfrac{FE^{-\frac{1}{2}}}{(EG-F^2)^{\frac{1}{2}}} \\ \delta = \dfrac{E^{\frac{1}{2}}}{(EG-F^2)^{\frac{1}{2}}} \end{cases}, \quad \varepsilon = E^{-\frac{1}{2}}$$

- $e' = \varepsilon a_u$, $e'' = \gamma a_u + \delta a_v$.

$$\alpha = e'_u \cdot e''$$

$$= (\varepsilon_u a_u + \varepsilon a_{uu}) \cdot (\gamma a_u + \delta a_v)$$

$$= \frac{\varepsilon_u}{\varepsilon} \cancel{e' \cdot e''} + \varepsilon\gamma\, a_{uu} \cdot a_u + \varepsilon\delta\, a_{uu} \cdot a_v .$$

$$= \frac{1}{2}\varepsilon\gamma\, (a_u \cdot a_u)_u + \varepsilon\delta\, \Big((a_u \cdot a_v)_u - \underbrace{a_u \cdot a_{uv}}_{\frac{1}{2}(a_u \cdot a_u)_v} \Big)$$

$$= \frac{1}{2}\varepsilon\gamma\, E_u + \varepsilon\delta\, \left(F_u - \frac{1}{2} E_v \right)$$

$$\beta = e'_v \cdot e''$$

$$= (\varepsilon_v a_u + \varepsilon a_{uv}) \cdot (\gamma a_u + \delta a_v)$$

$$= \frac{\varepsilon_v}{\varepsilon} \cancel{e' \cdot e''} + \varepsilon\gamma\, \underbrace{a_{uv} \cdot a_u}_{\frac{1}{2}(a_u \cdot a_u)_v} + \varepsilon\delta\, \underbrace{a_{uv} \cdot a_v}_{\frac{1}{2}(a_v \cdot a_v)_u} .$$

$$= \frac{1}{2}\varepsilon\gamma\, E_v + \frac{1}{2}\varepsilon\delta\, G_v .$$

$$\underline{K = \boxed{\frac{\alpha_v - \beta_u .}{(EG-F^2)^{\frac{1}{2}}}}} \qquad \underline{v}$$

- $\underline{K \sim I}$

<u>Cor 7.1.3</u>.

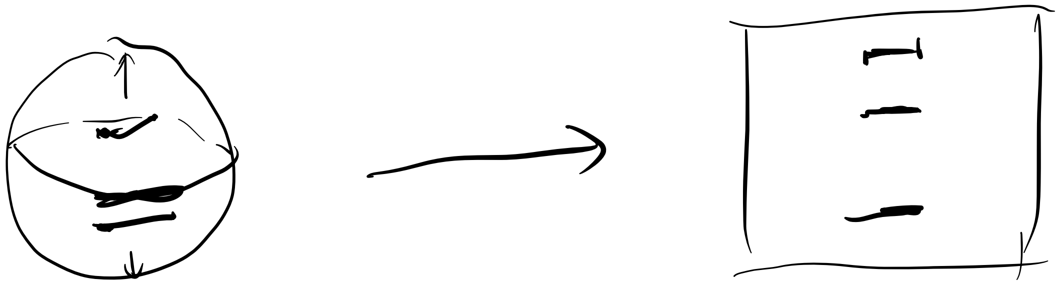
$$K = \frac{1}{(EG-F^2)^{\frac{1}{2}}} \left\{ \begin{vmatrix} -\frac{1}{2}E_{vv} + F_{uv} - \frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix} \right\}.$$

<u>Cor</u>. (a) $F=0$

$$K = -\frac{1}{2\sqrt{EG}} \left\{ \frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right) + \frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right) \right\}.$$

(b) $E=1$, $F=0$.

$$K = -G^{-\frac{1}{2}} \frac{\partial^2}{\partial u^2}(G^{\frac{1}{2}}).$$



Prop. Any map of any region of the earth's surface must distort distances.

Pf. $f : S^2 \to \mathbb{R}^2$

$$\underline{r \mapsto \underset{\uparrow}{k} r}.$$

$$\underline{f \circ k^{-1}} : S^2 \to \mathbb{R}^2$$

$$\underline{r \mapsto r}.$$

$$\underline{K(S^2) = K(\mathbb{R}^2)} \quad \times.$$

$$K(S^2) = \frac{1}{R^2}, \qquad K(\mathbb{R}^2) = 0.$$

Prop. (Gauss equation). $\underline{\sigma(u,v)}$

$$\begin{cases} a_{uu} = \Gamma_{11}^{1} a_u + \Gamma_{11}^{2} a_v + L\vec{N}. \\ a_{uv} = \Gamma_{12}^{1} a_u + \Gamma_{12}^{2} a_v + M\vec{N} \\ a_{vv} = \Gamma_{22}^{1} a_u + \Gamma_{22}^{2} a_v + N\vec{N} \end{cases}$$

$$\Gamma_{11}^{1} = \frac{GE_u - 2FE_u + FE_v}{2(EG-F^2)} \quad , \quad \Gamma_{11}^{2} = \frac{2EF_u - EE_v - FE_u}{2(EG-F^2)}$$

$$\Gamma_{12}^{1} = \frac{GE_v - FG_u}{2(EG-F^2)} \quad . \quad \Gamma_{12}^{2} = \frac{EG_u - FE_v}{2(EG-F^2)}$$

$$\Gamma_{22}^{1} = \frac{2GF_v - GG_u - FG_u}{2(EG-F^2)} \quad , \quad \Gamma_{22}^{2} = \frac{EG_v - 2FF_v + FG_u}{2(EG-F^2)}$$

Pf.

$$\begin{cases} a_{uu} = \alpha_1 a_u + \alpha_2 a_v + \alpha_3 \vec{N} \\ a_{uv} = \beta_1 a_u + \beta_2 a_v + \beta_3 \vec{N} \\ a_{vv} = \gamma_1 a_u + \gamma_2 a_v + \gamma_3 \vec{N} \end{cases}$$

- $a_{uu} \cdot \vec{N} = \alpha_3. \quad \Rightarrow \quad \alpha_3 = L.$

  $a_{uv} \cdot \vec{N} = \beta_3 \quad \Rightarrow \quad \beta_3 = M$

  $a_{vv} \cdot \vec{N} = \gamma_3 \quad \Rightarrow \quad \gamma_3 = N.$

- $a_{uu} \cdot a_u = \alpha_1 E + \alpha_2 F = \frac{1}{2}(a_u \cdot a_u)_u = \frac{1}{2}E_u.$

  $a_{uu} \cdot a_v = \alpha_1 F + \alpha_2 G = F_u - \frac{1}{2}E_v$

$$\boxed{\Gamma^1_{11} = \alpha_1}, \quad \boxed{\Gamma^2_{11} = \alpha_2}$$

- $a_{uv} \cdot a_u = \beta_1 E + \beta_2 F = \frac{1}{2} E_v .$

  $a_{uv} \cdot a_v = \beta_1 F + \beta_2 G = \frac{1}{2} G_u$

$\Gamma^1_{12} = \beta_1 , \quad \Gamma^2_{12} = \beta_2$

Γ^k_{ij} $\quad 1 \leq i, j, k \leq 2 .$ $\quad$ six $\quad$ Christoffel symbols.

- $g_{11} = E , \quad g_{12} = g_{21} = F , \quad g_{22} = G .$

$$\mathcal{G}_1 = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

$$\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \mathcal{G}_1^{-1}$$

$$\boxed{\Gamma^k_{ij} = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})}$$

Riemannian geometry

Prop (Codazzi - Mainard equation) $\quad a(u,v) \quad \Gamma^k_{ij}$

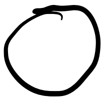
$$\begin{cases} L_v - M_u = L\Gamma_{12}^1 + M(\Gamma_{12}^2 - \Gamma_{11}^1) - N\Gamma_{11}^2 \\ M_v - N_u = L\Gamma_{22}^1 + M(\Gamma_{22}^2 - \Gamma_{12}^1) - N\Gamma_{12}^2 . \end{cases}$$

- constant Gaussian curvature
- plane . $\underline{K = 0}$. space form.

  sphere $K = 1$

•• Every $\underset{\Uparrow}{\underline{\text{compact}}}$ surface whose gaussian curvature is constant is sphere. closed, bounded closed subset of $\mathbb{R}^3$

open set



$\underline{B_R = \{(x,y) : (x)^2 + y^2 \le 1\}}$. closed

Poincaré disc. $K = -1$