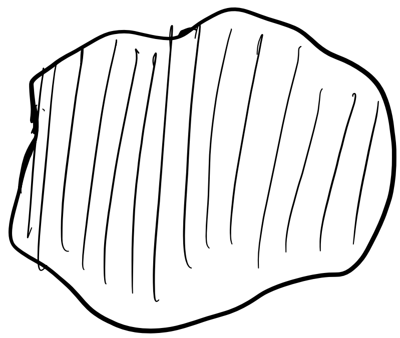
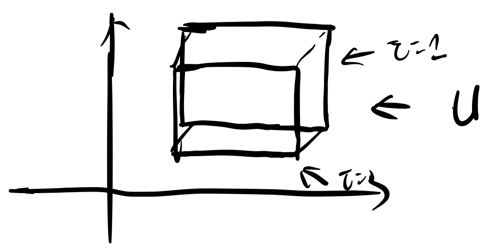
§8 Minimal surfaces.

§8.1 Plateau's problem.

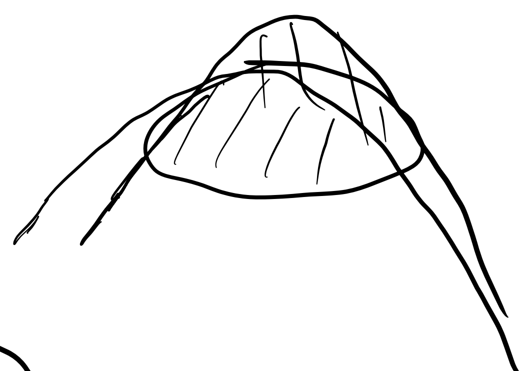


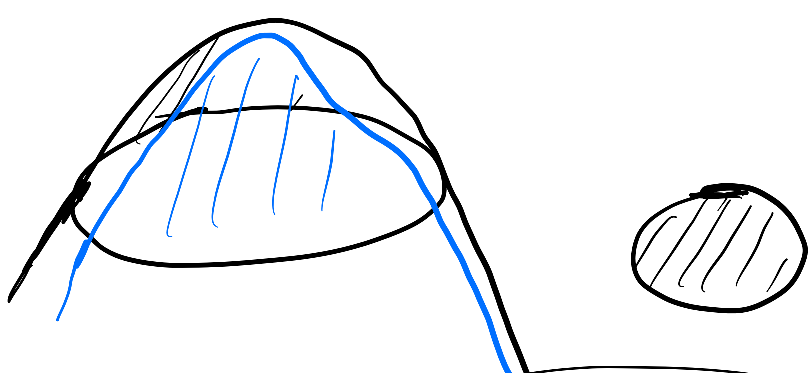
$\alpha^\tau : U \to \mathbb{R}^3$. $\alpha^\tau(u,v)$

$\alpha^\tau : (u,v,\tau) \to \mathbb{R}^3$



$\varphi = \frac{d}{d\tau}\alpha^\tau\Big|_{\tau=0}$





$$A(\tau) = \int_{int(\gamma)} dA_{a^\tau} = \int_{int(\gamma)} \| a^\tau_u \times a^\tau_v \| \, du\,dv$$

$$\frac{d}{d\tau} A(\tau) \Big|_{\tau=0} = \boxed{\int \dots \, dv} = 0$$

Thm 8.1.

$$\dot{A}(0) = -2 \int_{int(\gamma)} H (EG - F^2)^{\frac{1}{2}} \alpha \, du\,dv. = 0$$

$$\alpha = \varphi \cdot \vec{N} \quad , \quad \varphi = \frac{d}{d\tau} a^\tau \Big|_{\tau=0}$$

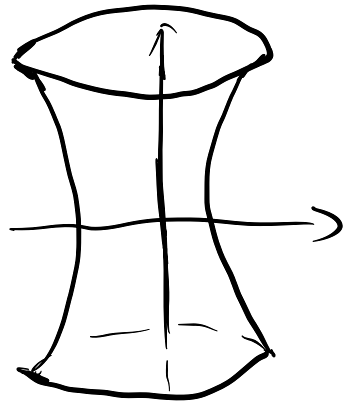
Def 8.2. A minimal surface is a surface whose mean curvature is zero everywhere.

exam. plane. $H = 0 \Rightarrow$ minimal surface.

Cor 8.3. If a surface S has least area among all surfaces with boundary curve, then S is minimal surface.

Exam 8.6 catenoid

$$x(u,v) = (\cosh u \cos v,\ \cosh u \sin v,\ u).$$



$$E = G = \cosh^2 u,\quad F = 0,\qquad L = -1,\ M = 0,\ N = 1$$

$$\Rightarrow\quad H = \frac{LG - 2MF + NE}{2(EG - F^2)} = \frac{-\cosh^2 u + \cosh^2 u}{2\cosh^4 u} = 0.$$

Prop.

$$x(u,v) = (f(u)\cos v,\ g(u)\sin v,\ u).$$

$$\mathrm{I} = du^2 + f(u)^2 dv^2 \qquad \mathrm{II} = (\dot f \ddot g - \ddot f \dot g)\, du^2 + f \dot g\, dv^2$$

$$\Rightarrow\quad H = \frac{1}{2}\left(\dot f \ddot g - \ddot f \dot g + \frac{\dot g}{f}\right) = 0$$

$$\dot f^2 + \dot g^2 = 1 \Rightarrow \dot f \ddot f + \dot g \ddot g = 0.$$

$$\Rightarrow \dot f \ddot f = -\dot g \ddot g$$

$$\Rightarrow \dot f \ddot g - \ddot f \dot g = -\frac{\dot g}{f}$$

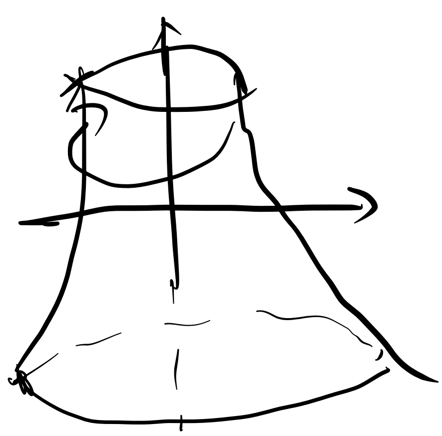
$$(\dot f \ddot g \dot g - \ddot f \dot g \dot g)\frac{1}{\dot g} = -\frac{\ddot f}{g}$$

$$-\dot f \dot f \ddot f$$

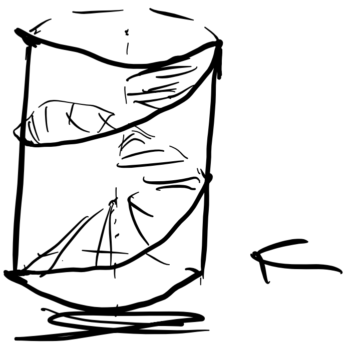
$$\Rightarrow \frac{\ddot f}{g} = \frac{\dot g}{f} \Rightarrow f \ddot f = \dot g^2 = 1 - \dot f^2$$

$$\Rightarrow f = 0 \quad \text{or} \quad f = \cosh u$$

$$\dot f^2 + \dot g^2 = 1$$



ruled surface.



$$\alpha(u,v) = (u\cos v,\ u\sin v,\ v)$$

$$\alpha(u,v) = \gamma(u) + v\,\gamma'(u).$$

Bernstein theorem. PDE.

$S:\ (x, y, f(x,y))$ $(x,y) \in \mathbb{R}^2$

$\Rightarrow$ $f(x,y) = Ax + By + C$.

Geometric measure theory. De Giorgi.

$$\alpha(x,y) = (x, y, f(x,y))$$

$$\sigma_x = (1,\ 0,\ f_x)$$
$$\sigma_y = (0,\ 1,\ f_y)$$

$$\begin{vmatrix} i & j & k \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix}$$

$$E = 1+f_x^2, \quad F = f_x f_y, \quad G = 1+f_y^2.$$

$$\sigma_{xx} = (0,\ 0,\ f_{xx})$$
$$\sigma_{xy} = (0,\ 0,\ f_{xy})$$
$$\sigma_{yy} = (0,\ 0,\ f_{yy})$$

$$\vec{N} = \frac{(-f_x,\ -f_y,\ 1)}{\sqrt{1+f_x^2+f_y^2}}$$

$$L = \frac{f_{xx}}{\sqrt{1+f_x^2+f_y^2}}, \quad M = \frac{f_{xy}}{\sqrt{1+f_x^2+f_y^2}}, \quad N = \frac{f_{yy}}{\sqrt{1+f_x^2+f_y^2}}$$

$$\Rightarrow \quad H = \frac{LG - 2MF + NE}{2(EG-F^2)} = 0$$

$$\Rightarrow \quad (1+f_y^2) f_{xx} - 2 f_x f_y f_{xy} + (1+f_x^2) f_{yy} = 0$$

$$\Leftrightarrow \quad \boxed{\operatorname{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) = 0} \quad \text{in } \mathbb{R}^2$$

$(n \leq 7)$

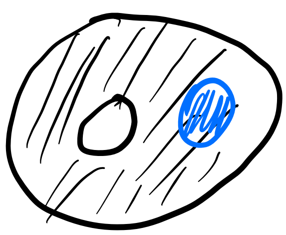
$\nabla f = c$

divergence elliptic PDE

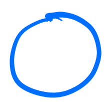
complex analysis.

§9. The Gauss-Bonnet theorem.

- simple closed curves



$\gamma(t) = \gamma(t+T)$



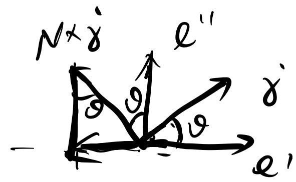
$\gamma(t) = (\cos t, \sin t)$

$t \in \mathbb{R}$

Thm 9.1. Let $\gamma(s)$ be a unit-speed simple closed curve, $l(\gamma)$. Then

$$\int_0^{l(\gamma)} k_g \, ds = 2\pi - \int_{\mathrm{int}(\gamma)} K \, dA_\sigma$$

Pf. $\{e', e'', \vec{N}\}$. $\vec{N} = e' \times e''$.



$$\dot\gamma = \cos\theta\, e' + \sin\theta\, e''$$

$$N\times\dot\gamma = -\sin\theta\, e' + \cos\theta\, e''$$

$$\ddot\gamma = \cos\theta\, \dot e' + \sin\theta\, \dot e'' + (-\sin\theta\, e' + \cos\theta\, e'')\,\dot\theta$$

- $\ddot\gamma = \kappa_n \vec N' + \kappa_g \vec N' \times \dot\gamma$

$e'\cdot e''=0$ $\quad \|e'\|=1,\ \|e''\|=1$

$$\kappa_g = (\vec N' \times \dot\gamma)\cdot\ddot\gamma$$

$$= \dot\theta - e'\cdot\dot e''$$

- $e'_u\cdot e''_v - e'_v\cdot e''_u = \dfrac{LN-M^2}{(EG-F^2)^{\frac12}}$

$$\int_0^{l(\gamma)} \kappa_g\, ds = \int_0^{l(\gamma)} \dot\theta\, ds - \int_0^{l(\gamma)} e'\cdot\dot e''\, ds$$

$$\int_0^{l(\gamma)} e'\cdot\dot e''\, ds = \int_0^{l(\gamma)} e'\cdot(e''_u\,\dot u + e''_v\,\dot v)\, ds$$

$$= \int_\gamma (e'\cdot e''_u)\,du + (e'\cdot e''_v)\,dv$$

Line integral.

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Green theorem.

$$= \int_{int(\gamma)} [(e'\cdot e_v'')_u - (e'\cdot e_u'')_v]\, du\,dv.$$

$$= \int_{int(\gamma)} (e_u'\cdot e_v'' - e_v'\cdot e_u'')\, du\,dv$$

$$= \int_{int(\gamma)} \frac{LN-M^2}{(EG-F)^{\frac{1}{2}}}\, du\,dv.$$

$\|\sigma_u \times \sigma_v\|\, du\,dv$

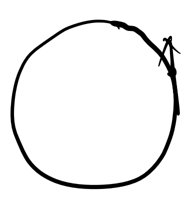
$$= \int_{int(\gamma)} K\, (EG-F^2)^{\frac{1}{2}}\, du\,dv$$

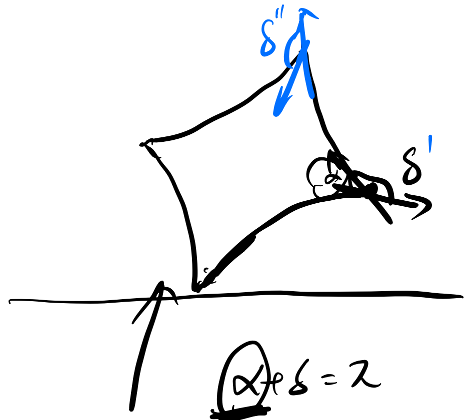
$$= \boxed{\int_{int(\gamma)} K\, dA_\sigma}$$

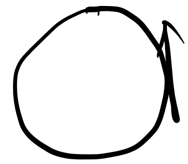
$$\Rightarrow \int_0^{l(\gamma)} \kappa_g\, ds = \int_0^{l(\gamma)} \dot\theta\, ds - \int_{int(\gamma)} K\, dA_\sigma$$

2π $\quad$ θ

- $\int_{int(\gamma)} K\, dA_\sigma$ total curvature







$\alpha + \delta = \pi$

curvilinear polygons

Thm 4.2. γ be a unit-speed curvilinear polygons with n edges on a surface σ. $\alpha_1, \cdots, \alpha_n$ be the interior angles.

Then

$$\int_0^{l(\gamma)} \kappa_g \, ds = \sum_{k=1}^{n} \alpha_k - (n-2)\pi - \int_{int(\gamma)} K \, dA_\sigma$$

π

$n=0$

$$\int_0^{l(\gamma)} 0 \, ds$$

closed . bounded

Gauss-Bonnet for compact surfaces

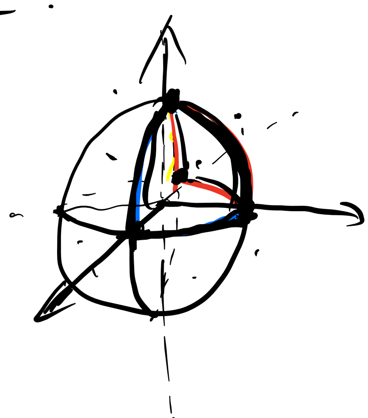
Def 4.4.

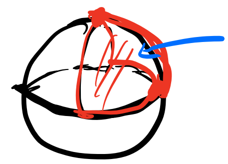
(1) Every point of S in at least one of curvilinear polygons

(2) Two curvilinear polygons are either disjoint, or their intersection is a common edge or a common vertex.

(3) Every edge is an edge of exactly two polygons.

Δ triangulation.





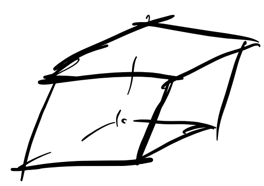
n dimension manifold

Def 9.5. The Euler number χ of a triangulation of a compact surface S with finitely many polygons is

$$\chi = V - E + F.$$

V = # vetices

topology



E = # edges.

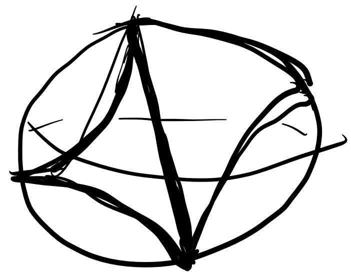
F = # faces.

• $\boxed{\chi = 6 - 12 + 8 = 2}$

Thm 9.6 (Gauss-Bonnet Thm)

Let S be a compact surface. Then

$$\int_S K \, dA = 2\pi \chi$$



- $\chi = 2$, $K = 1$.

$$\int_{S^2} K \, dA = \int_{S^2} dA = 4\pi .$$

$$2\pi\chi = 2\pi \cdot 2 = 4\pi$$

genus. $g=0$

$g=0$ $g=1$ $g=2$

Thm 9.7

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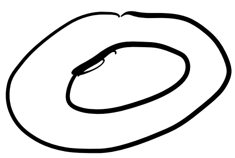
The Euler number of the compact surface T_g of genus g

is $2-2g$.

Cor 9.8

$$\int_{T_g} K\,dA = 4\pi(1-g) \Rightarrow \int_{T^2} K\,dA = 0.$$



$2-2g=0$

- $K = \text{const} > 0$ compact surface

$$c\int_{T_g} dA = c\,\text{Area}(T_g) = 4\pi(1-g) > 0$$

$$\Rightarrow g=0$$