§8 Symmetry Transformations and Dihedral groups.

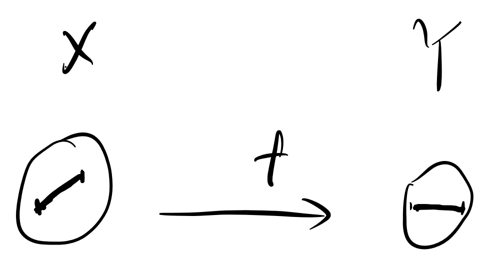
Def 8.1. A symmetry Transformation is an action on a set that leaves the set as a whole unaltered.

- even function. $f(-x) = f(x)$. $\quad x \mapsto -x$

Def 8.2. Let X and Y be two vector spaces equipped with distance functions D_X and D_Y. An isometry between X and Y is a distance preserving map $f: X \to Y$.

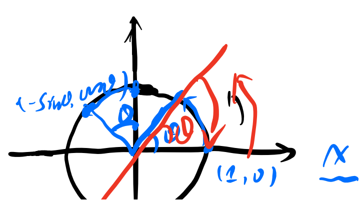
$$D_X(x_1, x_2) = D_Y(y_1, y_2)$$

$y_i = f(x_i)$



$X = Y = \mathbb{R}^2$, $\quad D_{\mathbb{R}^2}(x, y) = |x-y| = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$.

$\{(x,y) : x^2+y^2 = 1\}$

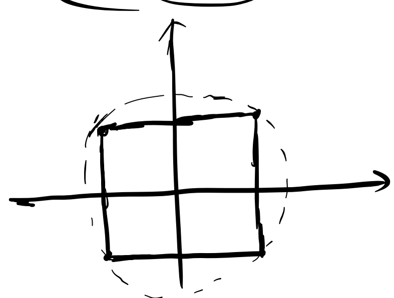


$$A(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$B(\theta) = A(\theta) B A(-\theta)$$

- $A(\theta) A(\theta') = A(\theta + \theta')$, $A(2\pi) = I$, $B^2 = I$
- $SO(2)$ $\quad \det M = \pm 1$ $\quad \boxed{M^T M = M M^T = I}$ $O(n)$



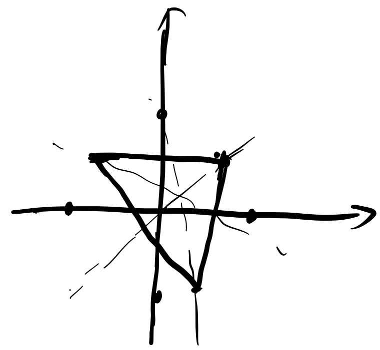
Def 8.3. Let $n \geq 2$ be an integer. The set of rotations and reflections that preserve the regular polygon P_n, form by

$$\left(\cos\frac{2k\pi}{n}, \sin\frac{2k\pi}{n}\right), \quad k = 0, 1, 2, \ldots, n-1.$$

is called the dihedral group D_n.

$|D_3| = 6$ $\qquad S_3$, $\mathbb{Z}_6$

$n=4$.



- $|D_n| = 2n$ $\qquad S_n$

Exercise. Is D_n isomorphic to S_n, $n \geq 4$?

$|D_4| = 8$.

$D_n = \{e, a, \cdots, a^{n-1}, b, ab, \cdots, a^{n-1}b\}$,

$\boxed{a^n = e, \quad b^2 = e, \quad a^k b = b a^{-k}, \quad k = 1, \cdots, n-1}$

S_n $\quad D_n$

§9 Conjugation, Normal subgroups, Quotient Groups.

§9.1 Conjugation.

Def. Given a group G, we say that a is conjugate to b if $\boxed{\exists\, g \in G}$ s.t. $a = gbg^{-1}$.

Thm 9.1 The conjugacy relation is an equivalence relation.

Exercise.

- S_3

$$(23)(123)(23) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (132).$$

$(123) \sim (132)$

- All elements of a conjugacy class have the same order.

$a = gbg^{-1}$ $\quad o(a) = o(b)$. $\quad o(a) = k$. $\quad b = g^{-1}ag$.

$b^k = (g^{-1}ag)^k = g^{-1}a^kg = g^{-1}g = e$.

$o(b) \mid k$ $\quad k \mid o(b)$ $\quad \Leftrightarrow \quad o(a) = o(b)$.

- If G is abelian, then $[a]_c = \{a\}$ for all $a \in G$.

Thm 9.2. On any subgroup $H \leq G$, the conjugation map

$$M_g : H \to gHg^{-1}, \quad h \mapsto ghg^{-1}$$

associated to $g \in G$ is bijective.

- injective. $\boxed{M(h_1) = M(h_2) \Rightarrow h_1 = h_2}$

$$gh_1g^{-1} = gh_2g^{-1} \Rightarrow h_1 = h_2.$$

- surjective. $ghg^{-1} \in gHg^{-1}$, $m(h) = ghg^{-1}$.

$$H = gHg^{-1}, \quad \forall g \in G$$

§ 9.2 Normal subgroups.

Def 9.1. A subgroup $H \leq G$ is called normal if $gHg^{-1} \subset H$ for all $g \in G$.

- H is normal $\Rightarrow gHg^{-1} = H. \Leftrightarrow gH = Hg, \forall g \in G$.

- H is normal $\Leftrightarrow \forall g \in G, h \in H, ghg^{-1} \in H$

- $\{e\}$, G

- Every subgroup of an abelian group is normal.

Def 9.2. A group is simple if has no proper normal group.

A semi-simple group if it has no proper abelian normal subgroup.

Def 9.3. The center $Z(G)$ of a group G is the set of all elements which commute with all elements of G.

$$Z(G) = \{a \in G : ab = ba, \forall b \in G\}.$$

Thm 9.3 The $Z(G)$ is a normal subgroup.

Exercise. Prove that $Z(G)$ is a subgroup.

$\forall g \in G$, $\forall a \in Z(G)$ $\quad gag^{-1} \in Z(G)$.

$$gag^{-1} = gg^{-1}a = a \in Z(G).$$

$Z(G)$ is a normal subgroup.

- If G is simple, then $Z(G) = \{e\}$ or $Z(G) = G$

Exercise. $H = \{e, a, a^2\} \le S_3$. Prove that H is normal.