Recall.

- H is normal $\Leftrightarrow$ $\forall g \in G$, $gHg^{-1} \subset H$.
- center $Z(G) = \{a \in G : ab = ba, \forall b \in G\}$.

$$Z(G) \trianglelefteq G$$

§9.3 Quotients

Def 9.4. The quotient $G/H = \{gH : g \in G\}$ is set of left cosets. $H\backslash G = \{Hg : g \in G\}$ is set of all right-cosets.

Def 9.5. Given A and B of G, the multiplication of A, B is defined by element-wise multiplication.

$$AB = \{ab : a \in A, b \in B\}.$$

Thm 9.4. If H is normal, then the quotient G/H with the above multiplication law on subsets, is a group.

Pf. • Closure:

normal

$$(g_1H)(g_2H) = g_1Hg_2H = g_1g_2(g_2^{-1}Hg_2)H = g_1g_2HH$$
$$= g_1g_2H \in G/H.$$

- Associativity: clearly.
- Identity. H.
- Inverse. $(gH)^{-1} = g^{-1}H$

We call G/H the quotient group of G with respect to H.

gH

Exam 9.1 $S_3 = \{e, a, a^2, b, ab, a^2b\}$. $a^3=e$, $b^2=e$, $ab=ba^2$.

$H = \{e, a, a^2\}$ is normal.

$g\,h\,g^{-1} \subset H$.

$g=e$. $g=a$, a^2.

$g=b$, $\boxed{g=ab}$ $g=a^2b$ $(a^2b)^{-1} = b^{-1}a^{-2} = ba$.

$bab = a^2 \in H$. $ba^2b = a \in H$

$(ab)a(ba^2) = ababa^2 = a\,a^2a^2 = a^2 \in H$.

$(ab)a^2(ba^2) = a \in H$.

$(a^2b)a(ba) = a^2baba = a^2 \cdot a^2 \cdot a = a^2 \in H$

$(a^2b)a^2(ba) = a \in H$

$\Rightarrow \quad gHg^{-1} \subset H \quad \Rightarrow \quad H \trianglelefteq S_3.$

$H = \{e, a, a^2\}$, $bH = \{b, ab, a^2 b\}$

$G/H = \{H, bH\} \cong \mathbb{Z}_2.$

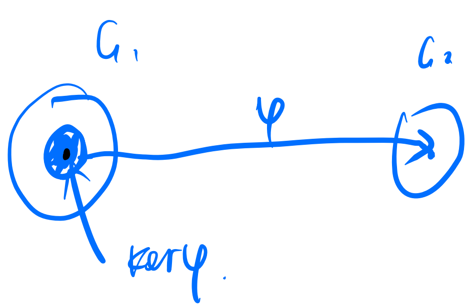
$\Rightarrow \quad S_3/H \cong \mathbb{Z}_2$

§10. Kernel, Image and the Homomorphism Theorem

Def 10.1. Let ϕ be a homomorphism of G_1 onto G_2. Then the kernel of ϕ is

$$\ker\phi = \{g \in G_1 : \phi(g) = e_2\}.$$

核



Thm 10.1. A homomorphism $\phi: G \to G'$ is an isomorphism if and only if it is onto and $\ker\phi = \{e\}$.

Pf. "$\Rightarrow$" ϕ is bijective $\Rightarrow$ ϕ is onto, one to one.

injective

$\forall g \in G$, $\phi(g) = e' \overset{?}{\Rightarrow} g = e$. $\Rightarrow \ker\phi = \{e\}$.

$\phi(e) = e'$

"$\Leftarrow$" $\phi(g_1) = \phi(g_2) \overset{?}{\Rightarrow} g_1 = g_2$

$\phi(g_1) = \phi(g_2) \Rightarrow \phi(g_1)\phi(g_2)^{-1} = e'$

$\Rightarrow \phi(g_1)\phi(g_2^{-1}) = e'$

$\Rightarrow \phi(g_1 g_2^{-1}) = e'$

$\Rightarrow g_1 g_2^{-1} \in \ker\phi = \{e\}$.

$\Rightarrow g_1 g_2^{-1} = e \Rightarrow g_1 = g_2$.

$\Rightarrow \phi$ is injective.

Thm 10.2. The kernel is a normal subgroup.

Pf. Prove the kernel is a group.

$\phi : G \to G'$ $\ker\phi = \{g \in G : \phi(g) = e'\}$.

closure. $\forall g_1, g_2 \in \ker\phi, \Rightarrow g_1 g_2 \in \ker\phi$.

$\phi(g_1 g_2) = e'$

• Normal. $g \ker\phi\, g^{-1} \subset \ker\phi$.

$\forall a \in \ker\phi,\ \forall g \in G,\quad \phi(gag^{-1}) = \phi(g)\underbrace{\phi(a)}_{e'}\phi(g)^{-1} = e'$

$\Rightarrow\ gag^{-1} \in \ker\phi \ \Rightarrow\ \ker\phi \trianglelefteq G.$

Thm 10.3. The image of $\phi: G_1 \to G_2$ homomorphism is a subgroup of G_2.

$\mathrm{Im}\,\phi = \{g_2 \in G_2 : \phi(g_1) = g_2,\ \exists\, g_1 \in G_1\}$

Exercise. $= \phi(G_1)$

§ 10.1 The Isomorphism Theorem.

Exam 10.1. $\mathbb{R}^* = (\mathbb{R}\setminus\{0\}, \cdot)$

$$\phi: \mathbb{R}^* \to \mathbb{R}^+ = \{x>0\}.$$
$$x \mapsto |x|.$$

• ϕ is a homomorphism.

$\forall x_1, x_2 \in \mathbb{R}^*,\quad \phi(x_1x_2) = |x_1x_2| = |x_1||x_2| = \phi(x_1)\phi(x_2)$

$\ker\phi = \{x \in \mathbb{R}^* : |x| = 1\} = \{-1, 1\} \cong \boxed{\mathbb{Z}_2}$

$$\mathbb{R}^*/\ker\phi = \{x\mathbb{Z}_2 : x \in \mathbb{R}^*\} = \{x\mathbb{Z}_2 : x \in \mathbb{R}^+\}$$

$$\cong \mathbb{R}^+$$

$$\boxed{x\mathbb{Z}_2} = \{x, -x\} \quad x \in \mathbb{R}^*$$

$$\Rightarrow \quad \mathbb{R}^*/\ker\phi \cong \mathbb{R}^+ = \mathrm{Im}\,\phi.$$

$$\phi: \mathbb{R}^* \to \mathbb{R}^+ \qquad x \mapsto |x|$$

$$G/\ker\phi \cong \mathrm{Im}\,\phi$$

Thm 10.4 (The homomorphism theorem) Let G and G' be groups, and $\phi: G \to G'$ be homomorphism. Then

$$G/\ker\phi \cong \mathrm{Im}\,\phi.$$

Pf. Let $H = \ker\phi$, $\tilde{G} = G/H$.

$$\tilde{\phi}: \tilde{G} \to \mathrm{Im}\,\phi$$
$$gH \mapsto \phi(g) \quad \Rightarrow \quad \tilde{\phi}(gH) = \phi(g)$$

- $\tilde{\phi}$ well-defined. $g_1H = g_2H \Leftrightarrow \tilde{\phi}(g_1H) = \tilde{\phi}(g_2H)$.

$$\hat{\phi}(g_1H) = \phi(g_1) = \hat{\phi}(g_2H) = \phi(g_2)$$

$$\Leftrightarrow \phi(g_1) = \phi(g_2) \Leftrightarrow \phi(g_1^{-1}g_2) = e_2$$

$$\Leftrightarrow g_1^{-1}g_2 \in H \Leftrightarrow g_1H = g_2H.$$

$$Id = g_1^{-1}g_2H$$

- $\hat{\phi}$ is homomorphism.

$$\hat{\phi}(g_1H\, g_2H) = \hat{\phi}(g_1g_2H) = \phi(g_1g_2)$$

$$= \phi(g_1)\phi(g_2) = \hat{\phi}(g_1H)\,\hat{\phi}(g_2H).$$

- Injective.

$$\boxed{\hat{\phi}(g_1H) = \hat{\phi}(g_2H) \Rightarrow g_1H = g_2H}$$

$$\hat{\phi}(g_1H) = \hat{\phi}(g_2H) \Rightarrow \phi(g_1) = \phi(g_2)$$

$$\Rightarrow \phi(g_1^{-1}g_2) = e_2 \Rightarrow g_1^{-1}g_2 \in H \Rightarrow g_1H = g_2H.$$

- Surjective.

$$\forall g' \in \operatorname{Im}\phi,\ \exists g \in G,\ s.t.\ \phi(g) = g'.$$

$\tilde{\phi}(gH) = \phi(g) = g'$.

$\Rightarrow$ $\tilde{\phi}$ is an isomorphism. $\tilde{G} \cong \operatorname{Im}\phi$.

$$G/\ker\phi \cong \operatorname{Im}\phi.$$

Exam 10.3 $M \in GL(N, \mathbb{R})$ General linear group

M^{-1} exists.

$$\phi : GL(N, \mathbb{R}) \to \mathbb{R}^*$$
$$A \mapsto \det(A)$$

- ϕ is homomorphism.

$$\phi(AB) = \phi(A)\,\phi(B)$$

$$\phi(AB) = \det(AB) = (\det A)(\det B) = \phi(A)\,\phi(B).$$

$$\Rightarrow \quad GL(N, \mathbb{R}) / \ker\phi \cong \mathbb{R}^*$$

$$SL(N, \mathbb{R}) := \ker\phi = \{A : \det A = 1\}.$$

special linear group

$$\Rightarrow \quad GL(N,\mathbb{R})/SL(N,\mathbb{R}) \cong \mathbb{R}^*$$

Thm 10.5. Given a group G and a normal subgroup H, there exists a homomorphism $\phi: G \to G/H$ (onto) such that $\ker\phi = H$.

Pf. $\phi: G \to G/H = \{gH : g \in G\}$.
$g \mapsto gH$

- well-defined. ✓
- $\phi(g_1g_2) = g_1g_2H = g_1H\, g_2H = \phi(g_1)\phi(g_2)$.
- $\ker\phi = \{g : \phi(g) = H\} \subset H$

$gH = H \Rightarrow g \in H$

$\forall g \in H, \ \phi(g) = gH = H \Rightarrow H \subset \ker\phi$ ✓

$g \in \ker\phi$.