Quotient Group.

$|G| = n$. $H \trianglelefteq G$, $|H| = k$

$\underline{|G/H| = \{gH : g \in G\}}$ is a group.

$\Rightarrow$ $|G/H| = \frac{n}{k}$ $\leftarrow$ $< n$

$$\boxed{G \cong H \times G/H}$$

The Homomorphism theorem. G, G'. $\phi : G \to G'$ homomorphism

$$G/\ker\phi \cong \mathrm{Im}\,\phi$$

$$\boxed{\ker\phi = \{g \in G : \phi(g) = e'\}}$$

$$\phi : (\mathbb{R}^*, \cdot) \to (\mathbb{R}^+, \cdot)$$
$$x \mapsto |x|.$$

ϕ is homomorphism. $\ker\phi = \{1, -1\} \cong \mathbb{Z}_2$.

$$\mathbb{R}^*/\mathbb{Z}_2 \cong \mathbb{R}^+.$$

$\boxed{x\mathbb{Z}_2} = \{x, -x\}$. $\sim |x|$.

$$\phi : GL(n,\mathbb{R}) \to \mathbb{R}^*$$
$$A \mapsto \det A$$

ϕ $\quad \ker\phi = \{ A \in GL(n,\mathbb{R}) : \det A = 1 \} \triangleq SL(n,\mathbb{R})$.

special

$\phi(A) = \det A = 1$

$$GL(n,\mathbb{R}) / SL(n,\mathbb{R}) \cong \mathbb{R}^*$$

3.1. $\quad \mathbb{C}^\times = \mathbb{C} \setminus \{0\} \qquad \mathbb{C}^1 = \{ |z| = 1 \}$.

$$\phi : \mathbb{C}^\times \longrightarrow \mathbb{R}^+$$
$$z \mapsto |z|$$

$z = x+iy \quad |z| = \sqrt{x^2+y^2}$

$\phi(z) = 1 \Leftrightarrow |z| = 1$

$\ker\phi = \mathbb{C}^1$

$\Rightarrow \mathbb{C}^\times / \mathbb{C}^1 \cong \mathbb{R}^+$

$\mathbb{R}^+ \times \mathbb{C}^1 \cong \mathbb{C}^1 \times \mathbb{R}^+$

$\mathbb{C}^\times \cong \mathbb{C}^1 \times \mathbb{C}^\times / \mathbb{C}^1 \cong \mathbb{R}^+ \times \mathbb{C}^1 = \{ (x, z) : \}$

$= \boxed{\mathbb{R}^+\mathbb{C}^1}$ $\quad \cong$ $\quad xz$

~~$x+iy$~~ $\qquad z = |z| e^{i\theta}$

§11 Automorphism

Def 11.1 An automorphism is an isomorphism of G onto itself.

ϕ

Exam 11.1 Let $a \in G$. Define

$H \trianglelefteq G$

$\iff gHg^{-1} \subset H, \ \forall g \in G$

$$\phi_a : G \to G$$
$$g \mapsto aga^{-1}$$

conjugation

Then ϕ_a is an automorphism.

Exercise: prove it.

Def 11.2 For any fixed $a \in G$, define $\phi_a : G \to G$, $\phi_a(g) = aga^{-1}$

An inner automorphism is an automorphism ϕ such that $\phi = \phi_a$ for some $a \in G$. All inner automorphisms is call $\mathrm{Inn}(G)$.

Def 11.3 All automorphisms of G is denoted by $\mathrm{Aut}(G)$

$$\boxed{\mathrm{Aut}(G) = \{\phi : \phi \text{ is an automorphism}\}}$$

Thm 11.1 The set of all automorphisms $\mathrm{Aut}(G)$ is a group under composition. $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$

Prove $\mathrm{Aut}(G)$ is a group

Pf. Closure. $\boxed{\forall \phi_1, \phi_2 \in \mathrm{Aut}(G) \Rightarrow \phi_1\phi_2 \in \mathrm{Aut}(G)}$

$\phi_1\phi_2 : G \to G$

$\phi_1\phi_2(g_1g_2) = \phi_1\circ\phi_2(g_1g_2) = \phi_1(\phi_2(g_1g_2))$

$= \phi_1(\phi_2(g_1)\,\phi_2(g_2))$

$= \phi_1\phi_2(g_1)\,\phi_1\phi_2(g_2).$

$Inn(G) \trianglelefteq Aut(G).$

$\forall \phi_a \in Inn(G),\ \forall \phi \in Aut(G) \ \overset{?}{\Rightarrow}\ \phi\phi_a\phi^{-1} \in Inn(G)$

$\phi\phi_a\phi^{-1} = \phi_{\phi(a)} \in Inn(G).$

$\Rightarrow \ Inn(G) \trianglelefteq Aut(G)$

- $\psi\phi_a\psi^{-1} = \phi_{\psi(a)}$

$\Leftrightarrow \ \psi\phi_a\psi^{-1}(g) = \phi_{\psi(a)}(g),\ \forall g \in G.$

$\psi\phi_a(\psi^{-1}(g)) = \psi(a\psi^{-1}(g)a^{-1}).$

$= \psi(a)\,\psi(\psi^{-1}(g))\,\psi(a^{-1})$

$= \psi(a)\,g\,[\psi(a)]^{-1}.$

$= \phi_{\psi(a)}(g).$

- $\phi_a\phi_b = \phi_{ab}.$

$\boxed{\text{Exercise}}$

- $\phi(e_1) = e_2$.
- $(\phi(g))^{-1} = \phi(g^{-1})$

Thm 1.2 $G/Z(G) \cong \mathrm{Inn}(G)$.

pf. $\psi: G \to \mathrm{Inn}(G)$.

$a \mapsto \phi_a$.

$\mathrm{Im}\,\psi = \mathrm{Inn}(G)$

$\forall \phi_g \in \mathrm{Inn}(G)$, $\exists a \in G$

s.t. $\psi(a) = \phi_g$.

$a = g$

- ψ homomorphism. $\psi(g_1 g_2) = \psi(g_1)\psi(g_2)$.

$$\phi_{g_1 g_2} = \phi_{g_1}\phi_{g_2}$$

$\ker\psi = \{a \in G : \psi(a) = \phi_e\}$.

$\phi_a = \phi_e$

$\Leftrightarrow \phi_a(g) = g, \ \forall g \in G$.

$\Leftrightarrow aga^{-1} = g, \ \forall g \in G$

$\Leftrightarrow ag = ga, \ \forall g \in G$

$\Leftrightarrow a \in Z(G)$.

$\Rightarrow \ker\psi = Z(G)$.

$\Rightarrow G/Z(G) \cong \mathrm{Inn}(G)$.

§12. The Semi-Direct Product.

Recall. Direct product. — exterior direct product

$$G \times H = \{(g,h) : g \in G,\ h \in H\}.$$

$$(g_1,h_1)(g_2,h_2) = (g_1g_2,\ h_1h_2).$$

$$\boxed{|G\times H| = |G||H|}.$$

$\mathbb{Z}_2 \times \mathbb{Z}_2 \cong V_4$.

$\mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6$

$\mathbb{Z}_p \times \mathbb{Z}_q \cong \mathbb{Z}_{pq}$, $(p,q)=1$

$G \times \{e_2\} \cong G$ $\quad$ $\{e_1\} \times H \cong H$

- $\{e_1\} \times H \trianglelefteq G \times H$

$G\times H \cong H \times G$

$(g,h) \mapsto (h,g)$

$$(g',h')(e,h)(g',h')^{-1} = (g',h')(e,h)(g'^{-1},h'^{-1})$$

$$= (e,\ h'hh'^{-1}) \in \{e\} \times H.$$

$$(G\times\{e_2\}) \cap (\{e_1\}\times H) = \{(e_1,e_2)\}.$$

- ① $GH = HG$, ② $G \cap H = \{e\}$

$\Rightarrow$ $J \stackrel{\Delta}{=} GH$ Direct product

$\mathbb{C}^\times \cong \mathbb{R}^+ \times \mathbb{C}^1$

$\cong \mathbb{R}^+\mathbb{C}^1$

$z = |z| e^{i\theta}$

inner direct product

Def 12.1. The semi-direct product $G \ltimes H$ is the group whose elements are those of the set $G \times H$ and whose multiplication law is

$h' \in H$ $\quad$ $\phi_g(h') \in H$

$\phi_g(h') = h''$

$$(g, h)(g', h') = (gg', h\phi_g(h'))$$

$\phi_g = id.$

$\phi_g \in \mathrm{Aut}(H).$ $\quad$ $h' \xrightarrow{\phi_g} \phi_g(h') \in H$

$\phi_g(h') = h'$