$S_n = \{\sigma_0, \sigma_1, \dots\}$. $n!$ $\quad S_n \quad X=\{1,\dots,n\}$

$n=4$. $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = a \quad \{1,2,3,4\}$.

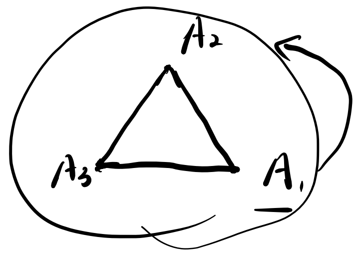
$a(1)=2$, $a(2)=3$, $a(3)=4$, $a(4)=1$.

$a\cdot 1=2$, $a\cdot 2=3$, $a\cdot 3=4$, $a\cdot 4=1$.

$\tau\cdot 1=i_1$, $\tau\cdot 2=i_2$, $\tau\cdot 3=i_3$, $\tau\cdot 4=i_4$.

$D_n = \{e, a, \dots a \dots, b, ab, \dots\}$. $2n$.

D_3



$\{A_1, A_2, A_3\}$.

$a \in D_3$

$a\cdot A_1 = A_2$, $a\cdot A_2 = A_3$, $a\cdot A_3 = A_1$.

a rotation $\boxed{\frac{2}{3}\pi}$

§13 G-sets, stabilisers and Orbits.

Def 13.1. For a group G a G-set is set X equipped with a rule assigning to each $g\in G$ and each element $x\in X$ an element $g\cdot x\in X$. satisfying

(i) $e\cdot x = x$, $\forall x\in X$

(ii) $(g_1g_2)\cdot x = g_1\cdot(g_2\cdot x)$, $\forall g_1, g_2 \in G$, $x \in X$.

Exam 13.1. If $X = G$. A group G is a G-set.

(i) $g\cdot x = gx$, $\forall g \in G$, $\forall x \in G$.

(ii) $g\cdot x = xg$. $\forall g \in G$. $\forall x \in G$.

(iii) $g\cdot x = gxg^{-1}$, $\forall g \in G$, $\forall x \in G$. (Conjugation or Adjoint action).

Pf. $e\cdot x = exe^{-1} = x$, $\forall x \in G$.

$$
\begin{aligned}
(g_1g_2)\cdot x &= (g_1g_2)x\boxed{(g_1g_2)^{-1}} \\
&= g_1(g_2 x g_2^{-1})g_1^{-1} \\
&= g_1(g_2\cdot x)g_1^{-1} \\
&= g_1\cdot(g_2\cdot x)
\end{aligned}
$$

Exam 13.2 Let X be the set of all subsets of elements of a finite group G. Define G-set action by $g\cdot S = gS$, i.e. $gS = \{gs_1, \cdots, gs_n\}$, where $S = \{s_1, \cdots, s_n\} \in X$.

Pf. (i) $e\cdot S = eS = \{s_1, \cdots, s_n\} = S$, $\forall S \in X$.

(ii) $(g_1g_2)\cdot S = (g_1g_2)S = \{g_1g_2s_1, \cdots, g_1g_2s_2\}$

$= g_1 \{ \underbrace{g_2 s_1, \cdots, g_2 s_2} \} = g_1 \cdot \{ \underbrace{g_2 s_1, \cdots, g_2 s_n} \}$.

$= g_1 \cdot (g_2 \cdot S)$.

- X be the set of all subgroups of G. $\quad g \cdot H = gHg^{-1}$ Exercise

(i) $e \cdot H = eHe^{-1} = H$, $\forall H \in X$.

(ii) $(g_1 g_2) \cdot H = g_1 \cdot (g_2 \cdot H)$, $\forall g_1, g_2 \in G$, $H \in X$.

Def 13.2. Given a G-set X, the stabiliser of the element $x \in X$ is the set of elements $g \in G$ such that $g \cdot x = x$, i.e.

$$\boxed{G_x} = \{ g \in G : \boxed{g \cdot x = x} \} \subset G.$$

Exam 13.3. If $X = G$, $g \cdot x = gx$. $x \in X = G$.

$$G_x = \{e\}.$$

Thm 13.1 G_x is a subgroup of G.

Exercise.

Def 13.3. Let $X = G$, $g \cdot x = gxg^{-1}$.

$$C_G(x) = \{ g \in G : gxg^{-1} = x \} = \{ g \in G : gx = xg \}.$$

$\rightarrow$ centerliser.

Exam 13.4. $D_3 = \{e, a, a^2, b, ab, a^2b\}$. $a^3 = e$, $b^2 = e$, $ab = ba^2$.

$$C_{D_3}(b) = \{g \in D_3 : gb = bg\}. \le D_3.$$

$|C_{D_3}(b)|$, $1, 2, 3, 6$.

2

$\langle b \rangle \le C_{D_3}(b)$

$\Rightarrow \quad C_{D_3}(b) = \{e, b\}$.

Exam 13.5. D_4 $\quad |C_{D_4}(b)| = 4$ $\quad$ (2值)

$|D_4| = 8$. $\quad \langle a, b \rangle \quad a^4 = e$, $b^2 = e$. $ab = ba^3$

$= \{e, a, a^2, a^3, b, ba, ba^2, ba^3\}$.

Def 13.4. Let X be the set of subgroups $H \subset G$. $g \cdot H = gHg^{-1}$.

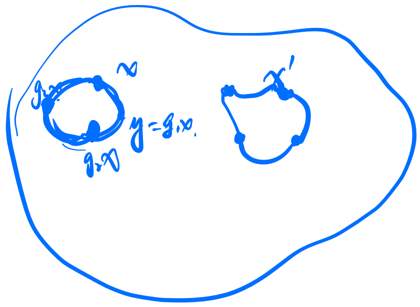
$$H \le N_G(H) = \{g \in G : gHg^{-1} = H\}. \trianglelefteq G$$

normaliser

Def 13.5. The orbit of x in G-set is given by

$$\mathrm{orb}(x) = \{g \cdot x : \forall g \in G\}. \subseteq X.$$

X .



$$X = |orb(x_1)| + |orb(x_2)| + \cdots + |orb(x_n)|.$$

Exercise. Prove $orb(x)$ is an equivalence class.

Exam 13.7. If $X = G$, $g \cdot x = gx$.

$$orb(x) = \{gx : \forall g \in G\} = G. \quad (G_x = \{e\}).$$

$\forall y \in G$, $\exists g = yx^{-1}$, s.t. $gx = y$

Exam 13.8. $X = \{\text{subgroups of } S_3\}$ $\quad g \cdot H = gH$. $\quad \langle a \rangle = \{e, a, a^2\}$. $\langle b \rangle$

$$orb(\langle a \rangle) = \{\langle a \rangle, \{b, ba, ba^2\}\}$$

Thm 13.2 (The orbit-stabiliser Theorem). Let G be a group and X be a G-set. For each $x \in X$, there is

$$|orb(x)| = \frac{|G|}{|G_x|} = i(G_x, G). \quad \text{index}$$

Pf. $\mu: \operatorname{orb}(x) \to \mathcal{C} = \{gG_x : \forall g \in G\}$.

$g\cdot x \mapsto gG_x$.

- μ is well-defined. $g_1\cdot x = g_2 \cdot x \Rightarrow g_1 G_x = g_2 G_x$.

$\Updownarrow \qquad \Updownarrow$

$(g_2^{-1}g_1)\cdot x = x \Leftrightarrow g_2^{-1}g_1 \in G_x$.

- μ is injective. $g_1 G_x = g_2 G_x \Rightarrow g_1\cdot x = g_2 \cdot x$.

- μ is surjective. $gG_x = \mu(g\cdot x)$.

$$|\operatorname{orb}(x)| = |\mathcal{C}| = i(G_x, G) = \frac{|G|}{|G_x|}$$

Example 13.9. $X = \{\text{subgroups of } G\}$. $g\cdot H = gH$.

$\operatorname{orb}(H) = \{gH : \forall g \in G\}$.

$G_H = \{g \in G : gH = H\}$

$= H$.

$$i(H,G) = |\operatorname{orb}(H)| = \frac{|G|}{|G_H|} = \frac{|G|}{|H|}$$