PDE EXAM FOR THE 2024 SUMMER SCHOOL AT PKU

Problem 1.

(i) Let $u \in H^1(\mathbb{R}^n)$ be a solution of $D_j(a_{ij}(x)D_i u) = 0$ in $\mathbb{R}^n$, with $\|u\|_{L^\infty(\mathbb{R}^n)} \leq C$, where a_{ij} is measurable, bounded, and uniformly elliptic. Prove that u must be a constant.

(i) Let $u \in C(\mathbb{R}^n)$ be a solution of $a_{ij}(x)D_{ij}u = 0$ in $\mathbb{R}^n$, with $\|u\|_{L^\infty(\mathbb{R}^n)} \leq C$, where a_{ij} is measurable, bounded, and uniformly elliptic. Prove that u must be a constant.

Problem 2. Let $u \in C^4(\mathbb{R}^n)$ be a strictly convex solution of det $D^2u = 1$ in $\mathbb{R}^n$. Prove that there is a constant matrix (a_{ij}) with $\det a_{ij} = 1$ such that

$$u(x) = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}x_i x_j + Du(0)\cdot x + u(0).$$

Problem 3. Prove that the operator $\det^{1/n}(\cdot)$ is concave in the set of non-negative definite matrices, namley

$$\det{}^{1/n}(M+N) \geq \det{}^{1/n}(M) + \det{}^{1/n}(N) \qquad \forall\, M, N \geq 0.$$

Problem 4. Let $u : \mathbb{R}^n \to \mathbb{R}$ be a convex function satisfying $u \geq 0$ and $u(0) = 0$. Define the *sub-level set* of u by $S_h^0 := \{x \in \mathbb{R}^n : u(x) < h\}$. Assume that:

(i) $S_1^0 \approx B_1(0)$, namely there are two universal constants $r_1, r_2 > 0$ such that $B_{r_1}(0) \subset S_1^0 \subset B_{r_2}(0)$;

(ii) There exists a constant $\theta \in (0,1)$ such that for any small $h > 0$, $S_{h/2}^0 \subset \theta S_h^0$;

(iii) There exists a small constant $\sigma > 0$ such that for any small $h > 0$, $S_h^0 \subset 2S_{(1-\sigma)h/2}^0$.

Prove that there exist positive constants α, β, and C such that

$$C^{-1}|x|^{1+\beta} \leq u(x) \leq C|x|^{1+\alpha} \quad \text{near the origin.}$$