PROBLEMS ON "RIEMANNIAN GEOMETRY"

In the following problem, (M^n, g) be a n-dimensional Riemannian manifold.

Problem 1. Let $B_r(p) \subset M$ be the geodesic ball of radius r centered at p. Show that

$$\mathrm{Vol}(B_r(p)) = \omega_n r^n \left(1 - \frac{\mathrm{Scal}(p)}{6(n+2)} r^2 + O(r^3)\right), (r \to 0)$$

where ω_n is volume of Euclidean unit ball, $\mathrm{Scal}(p)$ is scalar curvature at p.

Problem 2. Let $S_r(p) \subset M$ be the geodesic normal sphere of radius r centered at p. Let d_p be the distance function to p and H is the mean curvature of $S_r(p)$. Show that

$$\Delta d_p = H.$$

Compute H for $S_r(p) \subset M_K$, space form of constant curvature K.

Problem 3. Let M be a Riemannian manifold with $Ric \geq 0$. Let f be a subharmonic function on M, i.e. $\Delta f \geq 0$. Show that: for $p \in M$ and $r < \mathrm{inj}(p)$,

$$f(p) \leq \frac{1}{\omega_n r^n} \int_{B_r(p)} f dV_g.$$

Problem 4. Let M be a compact Riemannian manifold with boundary $\partial M = \Sigma$. Assume $Ric \geq (n-1)Kg$ and the mean curvature $H_\Sigma \geq (n-1)c$. In the case $K \leq 0$, we assume $c > \sqrt{-K}$. Let d_Σ denote the distance function to Σ. Show that

$$\max_{p \in M} d_\Sigma(p) \leq \begin{cases} \frac{1}{c}, & K = 0, \\ \frac{1}{\sqrt{-K}} \coth^{-1}(\frac{c}{\sqrt{-K}}), & K < 0, \\ \frac{1}{\sqrt{K}} \cot^{-1}(\frac{c}{\sqrt{K}}), & K > 0. \end{cases}$$