ELLIPTIC PDE EXERCISE

1. Suppose $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, show that the trace operator

$$T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

is compact.

2. Assume u satisfies $\sum_{i,j=1}^n a_{ij}u_{ij} = 0$ in viscosity sense in B_4, where $\lambda I \leq (a_{ij}) \leq \Lambda I$ for some positive constant λ, Λ. For $\epsilon > 0$ small, let

$$u_\epsilon(x) := \inf_{y \in B_4} \left(u(y) + \frac{1}{\epsilon}|y - x|^2 \right)$$

for $x \in B_1$.
1) Show that u_ϵ converges to u uniformly
2) Suppose $\phi \in C^2(B_4)$ is tangent to u_ϵ from below at $x_0 \in B_1$, show that if we slide ϕ down it will touch u from below at some point in B_2
3) Let ϕ be as in 2), suppose the eigenvalues of $D^2\phi(x_0)$ satisfy

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m < 0 \leq \lambda_{m+1} \leq \cdots \leq \lambda_n$$

then $\sum_{i=1}^m \Lambda\lambda_i + \sum_{i=m+1}^n \lambda\lambda_i \leq 0$.

3. Suppose u is a convex function defined on B_4. Let $v = u + \frac{1}{2}|x|^2$. Show that $|\partial v(B_1)| \geq |B_1|$.

4. Given a smooth uniformly convex body $K \subset \mathbb{R}^n$ with the origin in its interior. Let h_K be its support function, prove that the Gauss curvature of K is $\frac{1}{\det(\nabla^2 h_K + h_K I)}$.