1. On a Riemannian manifold (M, g), prove the second Bianchi identity of the Riemannian curvature tensor:

$$(\nabla_Z \mathcal{R})(X, Y)W + (\nabla_X \mathcal{R})(Y, Z)W + (\nabla_Y \mathcal{R})(Z, X)W = 0.$$

   Then prove the contracted second Bianchi identity:

$$dS = 2div(Ric),$$

   where S denotes the scalar curvature.

2. On a Riemannian manifold (M, g), a geodeic $\gamma : [0, \infty) \to M$ is called a geodesic ray starting from $\gamma(0)$, if for any $t > 0$, $\gamma|_{[0,t]}$ is a minimizing geodesic from $\gamma(0)$ to $\gamma(t)$.

   Assume that (M, g) is a complete and non-compact Riemannian manifold, let $p \in M$. Prove that M contains a geodesic ray starting from p.

3. Let (M, g) be a complete n-dim Riemannian manifold. Let $a > 0$ and $c \geq 0$ be given constants.

   Assume that, for any pair of points $p, q \in M$, for any minimizing geodesic $\gamma : [0, \ell] \to M$, parameterized by unit length with $\ell = d(p, q)$ and connecting p to q, we have

$$Ric(\dot{\gamma}(t), \dot{\gamma}(t)) \geq a + \frac{df}{dt}, \quad along \quad \gamma,$$

   where f is a function of t satisfying $|f(t)| \leq c$ for all $t \in [0, \ell]$. Prove that

$$Diam(M, g) \leq \frac{\pi^2}{\sqrt{c^2 + \pi^2 a} - c},$$

   then show (M, g) is compact.

4. A Riemannian manifold (M, g) is said to be homogeneous if the isometry group acts transitively. Prove that homogeneous manifolds are all complete.

5. Let (M, g) be a Riemannian manifold with non-positive sectional curvature. Show that, for all $p \in M$, there is no conjugate point of p on M.