A CALCULATION FOR THE THIRD FUNDAMENTAL FORM

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In this short notes, we give a detailed calculation for the third fundamental form of a regular surface in $\mathbb{R}^3$. The following technique is very basic, but the calculation is very painful. Our main result is as follows:

Theorem 1. *Define the* third fundamental form *of a surface to be*

$$\mathrm{III} = d\mathbf{N} \cdot d\mathbf{N},$$

where $d\mathbf{N} = \mathbf{N}_u du + \mathbf{N}_v dv$*, for a surface* $\mathbf{x}(u, v)$*. Show that*

$$\mathrm{III} = 2H\mathrm{II} - K\mathrm{I},$$

where H *is the mean curvature,* K *is the Gauss curvature,* I *is the first fundamental form, and* II *is the second fundamental form.*

Proof. By definition, we have

$$\mathrm{I} = E(du)^2 + 2F du dv + G(dv)^2,$$

and

$$\mathrm{II} = L(du)^2 + 2M du dv + N(dv)^2.$$

Hence, the mean curvature H and the Gauss curvature K are the form of

$$H = \frac{LG + NF - 2MF}{2(EG - F^2)}, \quad K = \frac{LN - M^2}{EG - F^2}.$$

Since $\mathbf{N} \cdot \mathbf{N} = 1$, we know that $\mathbf{N}_u$, $\mathbf{N}_v$ are elements of the tangent space to the surface. Hence they can be written as linear combinations of $\mathbf{x}_u$ and $\mathbf{x}_v$, i.e. there are a, b, c, d such that

$$-\mathbf{N}_u = a\mathbf{x}_u + b\mathbf{x}_v, \quad -\mathbf{N}_v = c\mathbf{x}_u + d\mathbf{x}_v.$$

Hence, by the definition of E, F, G, L, M, and N, we have

$$\begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix},$$

which gives us that

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \\ &= \frac{1}{EG - F^2} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \end{aligned}$$

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$$= \frac{1}{EG - F^2} \begin{pmatrix} LG - MF & -LF + ME \\ MG - NF & -MF + NE \end{pmatrix}.$$

Now, we compute the coefficients of III.

The first term is

$$\begin{aligned}
\mathbf{N}_u \cdot \mathbf{N}_u &= (a\mathbf{x}_u + b\mathbf{x}_v) \cdot (a\mathbf{x}_u + b\mathbf{x}_v) \\
&= a^2E + 2abF + b^2G \\
&= \frac{1}{(EG - F^2)^2} \big[(LG - MF)^2E + 2(LG - MF)(-LF + ME)F \\
&\qquad + (-LF + ME)^2G\big] \\
&= \frac{1}{(EG - F^2)^2} \big(L^2G^2E - 2MLEFG + M^2F^2E - 2L^2GF^2 + 2MLEFG \\
&\qquad + 2MLF^3 - 2M^2EF^2 + L^2F^2G - 2MLEFG + M^2E^2G\big) \\
&= \frac{1}{(EG - F^2)^2} (L^2G - 2MLF + M^2E)(EG - F^2) \\
&= \frac{1}{EG - F^2} (L^2G - 2MLF + NLE - NLE + M^2E) \\
&= 2HL - KE.
\end{aligned}$$

The second term is

$$\begin{aligned}
\mathbf{N}_u \cdot \mathbf{N}_v &= (a\mathbf{x}_u + b\mathbf{x}_v) \cdot (c\mathbf{x}_u + d\mathbf{x}_v) \\
&= acE + (ad + bc)F + bdG \\
&= \frac{1}{(EG - F^2)^2} \big[(LG - MF)(MG - NF)E + (LG - MF)(-MF + NE)F \\
&\qquad + (-LF + ME)(MG - NF)F + (-LF + ME)(-mF + NE)G\big] \\
&= \frac{1}{(EG - F^2)^2} \big(MLG^2E - NLEFG - M^2EFG + MNF^2E - MLGF^2 \\
&\qquad + NLEFG + M^2F^3 - MNEF^2 - MLGF^2 + NLF^3 + M^2EFG \\
&\qquad - MNEF^2 + MLF^2G - NLEFG - M^2EFG + MNE^2G\big) \\
&= \frac{1}{(EG - F^2)^2} (MLG + MNE - M^2F - NLF)(EG - F^2) \\
&= \frac{1}{EG - F^2} (MLG + MNE - 2M^2F + M^2F - NLF) \\
&= 2HM - KF.
\end{aligned}$$

The third term is

$$\mathbf{N}_v \cdot \mathbf{N}_v = (c\mathbf{x}_u + d\mathbf{x}_v) \cdot (c\mathbf{x}_u + d\mathbf{x}_v)$$

$$
\begin{aligned}
&= c^2E + 2cdF + d^2G \\
&= \frac{1}{(EG-F^2)^2}\big[(MG-NF)^2E + 2(MG-NF)(-MF+NE)F \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad + (-MF+NE)^2G\big] \\
&= \frac{1}{(EG-F^2)^2}\big(M^2G^2E - 2MNEFG + N^2F^2E - 2M^2GF^2 + 2MNEFG \\
&\qquad\qquad + 2MNF^3 - 2N^2EF^2 + M^2F^2G - 2MNEFG + N^2E^2G\big) \\
&= \frac{1}{(EG-F^2)^2}(M^2G - 2MNF + N^2E)(EG-F^2) \\
&= \frac{1}{EG-F^2}(NLG - 2MNF + N^2E - NLG + M^2G) \\
&= 2HN - KG.
\end{aligned}
$$

Hence, we obtain

$$\mathrm{III} = 2H\mathrm{II} - K\mathrm{I}.$$

□

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