NOTES FOR “AFFINE BERNSTEIN PROBLEM”

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Abstract. In this note, I'll go through the proof of affine Bernstein problem given by Trudinger and Wang. First, I'll make some detail calculations for conclusion stated in the original paper, and then restate the theorems based on my own understanding. Finally, I'll show that they produced a (non-smooth) counterexample for $n \geq 10$.

1. Introduction

This note is about a talk I give on the seminar, *Geometric PDEs*, held by CAS. In this note, I'll go through the proof of affine Bernstein problem given by Trudinger and Wang [TW], and I mainly introduce the idea about how to get the all dimensional conclusions under the assumption of uniform, “strict convexity” and just mention the proof of dimension two, which is the Chern's conjecture [Ch]. It is because their method for dimension two can not be extended to higher dimensions, even for dimension 3 and there are also other proofs of Chern's conjecture in dimension two. In the end, it is also worthy to mention that they produced a (non-smooth) counterexample for $n \geq 10$.

Next, I will state the main theorem of this note.

Theorem 1.1 ([TW, Theorem 1.1 & 5.2])**.** *An entire, affine maximal, locally uniformly convex C^4 graph in $\mathbb{R}^3$ must be an elliptic paraboloid.*

Before I give the proof of Theorem 1.1, I'll introduce some backgrounds of Bernstein problems. As all we known, the Bernstein problem for minimal surfaces is a fundamental problem in differential geometry and PDEs, ever since Bernstein proved that an entire, two dimensional, minimal graph must be a hyperplane [Be]. And the theorem was extended to $n = 3$ by De Giorgi [De], $n = 4$ by Almgren [Al] and $n \leq 7$ by Simons [Si]. Finally Bombieri, De Giorgi, and Giusti [BDG] gave an example showing that the result fails for $n \geq 8$, and this is one of the results reported by Professor Sun in the past lectures. The Bernstein (or Liouville) theorem for Monge-Ampère equation is also important. A celebrated result of Jörgens($n = 2$) [Jo], Calabi($2 \leq n \leq 5$) [Ca] and Pogorelov($n \geq 2$) [Po] stated that any entire classical convex solution to the Monge-Ampère equation must be a quadratic polynomial. Later, Caffarelli [Caf1] extended this result to viscosity solutions. And an interesting thing is that in dimension two, the Bernstein theorem for minimal surfaces can be deduced from Liouville theorem for Monge-Ampère equation. Indeed, we can rewrite the minimal surface equation in the form

$$(1 + u_x^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_y^2)u_{yy} = 0 \quad \text{in } \mathbb{R}^2.$$

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Let

$$\phi_{xx} = \frac{1+u_x^2}{\sqrt{1+u_x^2+u_y^2}},$$

$$\phi_{xy} = \frac{u_x u_y}{\sqrt{1+u_x^2+u_y^2}},$$

$$\phi_{yy} = \frac{1+u_y^2}{\sqrt{1+u_x^2+u_y^2}}.$$

Such a function satisfies

$$\phi_{xx}\phi_{yy} - \phi_{xy}^2 = 1.$$

Then following by the Liouville theorem for Monge-Ampère equation that ϕ is a quadratic function, and hence u_x, u_y are constants, which implies u is a linear function.

In 1977, Theorem 1.1 was proposed by Chern [Ch] as the Bernstein problem for affine maximal hypersurfaces, and it was proved in 2000 by Trudinger and Wang [TW]. In the following, I will explain what affine maximal means. Suppose $\mathcal{M}$ is given by

$$x_{n+1} = u(x), \quad x = (x_1, \cdots, x_n), \tag{1.1}$$

where $u \in C^2(\Omega)$ is convex. On $\mathcal{M}$ we can introduce a metric, called the affine metric (also called the Berwald-Blaschke metric), given by

$$g_{ij} = \frac{u_{ij}}{[\det D^2u]^{1/(n+2)}},$$

where $D^2u = [u_{ij}]$ is the Hessian matrix of the second derivatives of u. If u is locally uniformly convex in Ω, then $\det D^2u > 0$ and g is well defined. From the metric, we introduce the affine area A by defining

$$A(u, \Omega) = \int_\Omega \left[\det D^2u\right]^{1/(n+2)}$$

Definition 1.2. *A hypersurface $\mathcal{M}$, given by* (1.1)*, is called* <u>*affine maximal*</u> *if the function u is a critical point of the affine area functional A.*

Calabi [Ca1] proved that if $u \in C^4(\Omega)$ is a critical point of the functional A, the second variation of A at u is non-positive, that is, the affine area of $\mathcal{M}$ reaches a maximum under smooth interior perturbations. Accordingly he proposed that $\mathcal{M}$ be called an affine maximal hypersurface. The Euler-Lagrange equation of the functional A is a fourth order, nonlinear partial differential equation, given by

$$H_A[\mathcal{M}] =: D_{ij}\left(U^{ij}w\right) = 0, \tag{1.2}$$

where

$$w = \left[\det D^2u\right]^{-(n+1)/(n+2)},$$

and $[U^{ij}]$ denotes the cofactor matrix of $[u_{ij}]$. Noting that

$$D_jU^{ij} = 0.$$

In fact, since (refer to [Fi, Lemma A.1 & A.2] for detail calculations)

$$\frac{d}{d\varepsilon}\det(A+\varepsilon B)\bigg|_{\varepsilon=0} = \det A\,\mathrm{tr}(A^{-1}B),$$

and

$$\frac{d}{d\varepsilon}(A+\varepsilon B)^{-1}\bigg|_{\varepsilon=0} = -A^{-1}BA^{-1},$$

we have

$$D_j(\det D^2u) = \frac{d}{d\varepsilon}\det(D^2u+\varepsilon D^2u_j)\bigg|_{\varepsilon=0} = \det D^2u\,\mathrm{tr}([D^2u]^{-1}D^2u_j) = \det D^2u\cdot u^{kl}u_{klj}$$

and

$$D_j([u_{kl}]^{-1}) = \frac{d}{d\varepsilon}([u_{kl}]+\varepsilon[u_{klj}])^{-1}\bigg|_{\varepsilon=0} = -[u^{kl}][u_{stj}][u^{mn}].$$

Hence

$$D_ju^{ij} = -u^{ik}u_{klj}u^{lj} = -u^{kl}u_{klj}u^{ij},$$

which yields

$$D_jU^{ij} = D_j(\det D^2u\cdot u^{ij}) = D_j(\det D^2u)u^{ij} + \det D^2uD_ju^{ij} = 0.$$

By this divergence free property, we know (1.2) can be written as

$$H_A[\mathcal{M}] = U^{ij}D_{ij}w = 0. \tag{1.3}$$

The quantity $H_A[\mathcal{M}]$ on the left hand side of equations (1.2) and (1.3) represents the affine mean curvature of the hypersurface $\mathcal{M}$. Denoting

$$\begin{aligned} h = g^{1/2} &= (\det[g_{ij}])^{1/2} \\ &= \left(\det D^2u\right)^{1/(n+2)} = w^{-(n+1)}, \end{aligned}$$

equation (1.2) can also be written as

$$\Delta_{\mathcal{M}}\left(\frac{1}{h}\right) = 0,$$

where $\Delta_{\mathcal{M}}$ is the Laplace-Beltrami operator with respect to the affine metric, given by

$$\Delta_{\mathcal{M}} = \frac{1}{\sqrt{g}}D_i\left(\sqrt{g}g^{ij}D_j\right) = \frac{1}{h}D_i\left(h^2u^{ij}D_j\right),$$

and $[g^{ij}],[u^{ij}]$ are the inverses of $[g_{ij}],[u_{ij}]$. In fact, note that

$$\begin{aligned} D_{ij}w = D_{ij}\left(h^{-1}\right)^{n+1} = {}& n(n+1)\left(h^{-1}\right)^{n-1}D_ih^{-1}D_jh^{-1} \\ &+(n+1)\left(h^{-1}\right)^{n}D_{ij}h^{-1}. \end{aligned}$$

Then

$$\begin{aligned}\Delta_{\mathcal{M}}\left(h^{-1}\right) &= \frac{1}{h}D_i\left(h^2u^{ij}D_j\left(h^{-1}\right)\right) = \frac{1}{h}D_i\left(\left(h^{-1}\right)^nU^{ij}D_j\left(h^{-1}\right)\right)\\ &= \frac{1}{h}U^{ij}D_i\left(\left(h^{-1}\right)^nD_j\left(h^{-1}\right)\right)\\ &= \frac{1}{h}U^{ij}\left(n\left(h^{-1}\right)^{n-1}D_ih^{-1}D_jh^{-1}+\left(h^{-1}\right)^nD_{ij}h^{-1}\right)\\ &= \frac{1}{(n+1)h}U^{ij}D_{ij}w=0.\end{aligned}$$

Therefore the hypersurface $\mathcal{M}$ is affine maximal if and only if $1/h$ is harmonic on $\mathcal{M}$.

In [Ch], Chern conjectured that, in the two dimensional case, any entire solution to (1.3) must be quadratic. From Bernstein [Be], if the function $w = o(|x|)$, as $x \to \infty$, then w is constant and Chern's conjecture follows from Jörgens' theorem [Jo], that an entire convex solution of the Monge-Ampère equation

$$\det D^2u = \text{constant}$$

is a quadratic function, (which is true in all dimensions). Calabi [Ca1] verified the Chern conjecture under the hypothesis that the affine metric of the graph of the solution is complete. For if $n = 2$, the Ricci tensor under the affine metric is non-negative definite, and by a result of Blanc and Fiala [BF], (see [Yau] for the higher dimensional case), that a positive harmonic function on a complete Riemannian manifold with non-negative Ricci curvature is constant, the result follows again from Jörgens' theorem. Li [Li] proved that if all the affine principal curvatures are bounded, then Euclidean completeness implies affine completeness, so that in the two dimensional case, the Chern conjecture is valid if the affine Gauss curvature is bounded from below (see also [MM]). It is worthy to know that instead of Chern conjecture, Calabi asked whether affine completeness alone is sufficient for the Bernstein theorem. This question was answered affirmatively in [TW2], see also [LJ] for a different treatment based on the result in [MM]. In [TW2] the authors proved a much stronger result. That is *an affine complete, locally uniformly convex hypersurfaces in* $\mathbb{R}^{n+1}$*,* $n \geq 2$*, is also Euclidean complete.*

What I will do in the following content is to give the proof of Theorem 1.1.

2. Proof of Theorem 1.1

First, we note that the fourth order equation (1.3) can be viewed as a system of two second order PDEs. The first is a linearized Monge-Ampère equation

$$U^{ij}w_{ij} = 0, \tag{2.1}$$

and the other equation in the system is the standard Monge-Ampère equation

$$\det D^2u = w^{-\frac{n+2}{n+1}}. \tag{2.2}$$

For this system, if we know, for example, $w \in C^\alpha$ in (2.1), then put this information into (2.2) and by classical interior $C^{2,\alpha}$ estimate for Monge-Ampère equation derived by Caffarelli [Caf], we have $u \in C^{2,\alpha}$. After updating this information in (2.1), we know the equation now is a uniformly elliptic equation with C^α coefficients, hence by classical Schauder estimate we know again $w \in C^{2,\alpha}$, and then by (2.2) we can know $u \in C^{4,\alpha}$, and bootstrap, we obtain $u \in C^\infty$ and all derivatives estimates of u. Once we get higher derivative estimates, we scale the equation to define it between two balls, and then blow up to get the conclusion.

Nowadays, the above method is a standard way to investigate regularities of 4th order equations, also for some other nonlinear equations. Clearly, it is the key point that if we can begin the first step. Luckly, in the remarkable paper [CG], Caffarelli and Gutiérrez established the Harnack inequality for positive solutions to linearized Monge-Ampère equation (2.1) under the assumption of $\mathcal{A}_\infty$ condition, which implies the Hölder regularity of w. And it is easy to show that $\mathcal{A}_\infty$ condition can be implied by the pinching of the $\det D^2u$, that is there are two positive numbers λ, Λ such that $0 < \lambda \le \det D^2u \le \Lambda$. This theory of Caffarelli and Gutiérrez is an affine invariant version of the classical Harnack inequality for uniformly elliptic equations with measurable coefficients [KS1, KS2].

In the following, we derive upper bounds and lower bounds for the Hessian determinant, $\det D^2u$, of solutions u of (1.3). It should be noted that there are also no pure interior estimates for Monge-Ampère type 4th order equations, which is very similar to classical Monge-Ampère equations, so we derive a Pogorelov type estimate in the following. Since we will normalize the sections of u, we derive the estimates in a normalized domain.

Lemma 2.1 ([TW, Lemma 3.1]). *Let Ω be a bounded convex domain in $\mathbb{R}^n$ and $u \in C^4(\Omega) \cap C^{0,1}(\bar{\Omega})$ a locally uniformly convex solution of equation* (1.3) *in Ω, satisfying*

$$u = 0 \quad \text{on } \partial\Omega \quad \inf_\Omega u = -1.$$

Then, for $y \in \Omega$,

$$\det D^2u(y) \le C,$$

where C depends on $n, \mathrm{dist}(y, \partial\Omega)$, and $\sup_\Omega |Du|$.

Proof. Let

$$z = \ln w - \beta \ln(-u) - A|Du|^2, \quad w = (\det D^2u)^{-\frac{n+1}{n+2}},$$

where β and A are positive constants to be specified later. Since $z \to \infty$ on $\partial\Omega$, it attains a minimum at some point $x_0 \in \Omega$. At x_0, we then have

$$0 = z_i = \frac{w_i}{w} - \beta \frac{u_i}{u} - 2Au_k u_{ki}$$

and

$$0 \le [z_{ij}] = \left[\frac{w_{ij}}{w} - \frac{w_i w_j}{w^2} - \frac{\beta u_{ij}}{u} + \frac{\beta u_i u_j}{u^2} - 2Au_{ki}u_{kj} - 2Au_k u_{kij} \right].$$

Recalling $w = (\det D^2u)^{\theta - 1}, \theta = \frac{1}{n+2}$, we have

$$u^{ij}u_{kij} = \left(\log \det D^2u\right)_k = -\frac{1}{1-\theta}\frac{w_k}{w},$$

where $[u^{ij}] = (\det D^2 u)^{-1} [U^{ij}]$ is the inverse of $D^2 u$. Note that we have calculated this formula in the proof of divergence free of U^{ij}, hence we omit it here. Then we have

$$\frac{w_i w_j}{w^2} = \beta^2 \frac{u_i u_j}{u^2} + \frac{2\beta A}{u} (u_i u_k u_{kj} + u_j u_k u_{ki}) + 4A^2 u_k u_l u_{ki} u_{lj},$$

and hence, at x_0, we have

$$\begin{aligned}
0 \leq & u^{ij} z_{ij} \\
= & -\frac{\beta n}{u} - \frac{u^{ij} w_i w_j}{w^2} + \frac{\beta u^{ij} u_i u_j}{u^2} - 2Au^{ij} u_{kj} u_{kj} + \frac{2A}{1-\theta} \frac{u_k w_k}{w} \\
= & -\frac{\beta n}{u} - \beta^2 \frac{u^{ij} u_i u_j}{u^2} + \frac{\beta u^{ij} u_i u_j}{u^2} - \frac{2\beta A}{u} (u^{ij} u_i u_k u_{kj} + u^{ij} u_j u_k u_{ki}) - 4A^2 u^{ij} u_k u_l u_{ki} u_{lj} \\
& - 2Au^{ij} u_{ki} u_{kj} + \frac{2A}{1-\theta} \beta \frac{u_k^2}{u} + \frac{4A^2}{1-\theta} u_k u_l u_{kl} \\
= & -\frac{\beta n}{u} - \beta(\beta - 1) \frac{u^{ij} u_i u_j}{u^2} - 2A\Delta u + \frac{4A^2\theta}{1-\theta} u_{ij} u_i u_j - 2\beta A \frac{1-2\theta}{1-\theta} \frac{|Du|^2}{u} \\
\leq & -A\Delta u - \frac{\beta n}{u} + 2\beta A \frac{|Du|^2}{|u|},
\end{aligned}$$

with the choice

$$A = \frac{1-\theta}{4\theta \sup_\Omega |Du|^2},$$

where we used $u_{ij} u_i u_j \leq \Delta u |Du|^2$. Consequently, we obtain

$$-u\Delta u\,(x_0) \leq C(n, \beta) \sup_\Omega |Du|^2.$$

Setting $\beta = (1-\theta)n = n(n+1)/(n+2)$, we obtain

$$\begin{aligned}
z(x) & \geq z\,(x_0) \\
& = (\theta - 1) \log |u|^n \det D^2 u\,(x_0) - A|Du|^2\,(x_0) \\
& \geq (\theta - 1) n \log |u| \Delta u\,(x_0) - A|Du|^2\,(x_0) \\
& \geq -C\,(n, M_1)\,,
\end{aligned}$$

where $M_1 = \sup_\Omega |Du|$. Accordingly we estimate, for any $y \in \Omega$,

$$\det D^2 u \leq \frac{C\,(n, M_1)}{|u(y)|^n} \leq \frac{C\,(n, M_1)\,(\operatorname{diam}\Omega)^n}{(\operatorname{dist}(y, \partial\Omega))^n}$$

by boundary conditions and the convexity of u, and hence Lemma 2.1 is proved. □

Remark 2.2. *It is clear that Lemma 2.1 will hold for any $\theta \in (0,1)$ with constant C depending on θ.*

We next derive a lower bound for $\det D^2 u$ in terms of a modulus of strict convexity for the function u. For any $y \in \Omega$, $h > 0$, we define the section $S(y, h)$ by

$$S(y, h) = \{x \in \Omega \mid u(x) < u(y) + Du(y)(y - x) + h\}.$$

We then define the modulus of convexity of u at y, by

$$h_{u,y}(r) = \sup\left\{h \geq 0 \mid S(y,h) \subset B_r(y)\right\}, \quad r > 0$$

and the modulus of convexity of u on Ω, by

$$h(r) = h_{u,\Omega}(r) = \inf_{y\in\Omega} h_{u,y}(r), \quad r > 0.$$

Observe that a function u is strictly convex in Ω if and only if $h(r) > 0$ for all $r > 0$.

Lemma 2.3 ([TW, Lemma 3.2])**.** *Let $u \in C^4(\Omega)$ be a locally uniformly convex solution of equation* (1.3) *in a domain $\Omega \subset \mathbb{R}^n$, satisfying $-1 \leq u \leq 0$ in Ω. Then, for $y \in \Omega$, there exists a positive constant C depending on n, dist $(y, \partial\Omega)$,* $\operatorname{diam}(\Omega)$*, and $h_{u,\Omega}$, such that*

$$C^{-1} \leq \det D^2u(y) \leq C$$

Proof. Since $u \in C^4(\Omega)$ is locally uniformly convex, so also is its Legendre transform, u^*, defined by

$$u^*(x) = \sup_{y\in\Omega}(x \cdot y - u(y)), \quad x \in \Omega^* = Du(\Omega),$$

with

$$Du^*(x) = y, \quad \det D^2u^*(x) = \left(\det D^2u(y)\right)^{-1},$$

whenever $x = Du(y), y \in \Omega$. Since u is maximal with respect to the functional A, it follows that u^* is maximal with respect to the functional A^* given by

$$\begin{aligned} A^*[u,\Omega] &= \int_\Omega [\det D^2u]^{1/(n+2)} \\ &= \int_{\Omega^*} \left[\det D^2u^*\right]^{(n+1)/(n+2)}. \qquad \text{(change of variable)} \end{aligned}$$

Therefore, if u satisfies (1.3), we see that u^* satisfies a similar equation

$$(U^*)^{ij}(w^*)_{ij} = 0,$$

where $\left[(U^*)^{ij}\right]$ is the cofactor matrix of $(u^*)_{ij}$ and

$$w^* = \left[\det D^2u^*\right]^{-1/(n+2)}$$

We cannot apply Lemma 2.1, with $\theta = \frac{n+1}{n+2}$, directly as the function u^* is not necessarily constant on $\partial\Omega^*$. However, for any point $y \in \Omega$, and $x = Du(y) \in \Omega^*$, we can see that the section $S^*(x,\delta)$ of the Legendre transform u^* lies in Ω^* for $\delta = h\left(\frac{1}{2}\operatorname{dist}(y,\partial\Omega)\right)$. Furthermore, we have, (for $0 \in \Omega$),

$$|Du^*| \leq \operatorname{diam}(\Omega)$$

and hence the ball $B_R(x) \subset S^*(x,\delta)$ for $R \leq \delta/\operatorname{diam}\Omega$. In fact, to compare the slop of a line connect boundary points of $S^*(x,\delta)$ and x with the gradient of u^*, we have

$$\frac{\delta}{\operatorname{dist}(x, S^*(x,\delta))} \leq \sup|Du^*| \leq \operatorname{diam}\Omega,$$

which implies that $\operatorname{dist}(x, S^*(x,\delta)) \geq \delta/\operatorname{diam}\Omega$. Accordingly, we may apply Lemma 2.1, with $\theta = \frac{n+1}{n+2}$, to the function u^* in the domain $S^*(x,\delta)$ to deduce the lower bound for $\det D^2 u$. The upper bound follows by applying Lemma 2.1 in the section $S(y,\delta)$ where we would have the gradient bound $|Du| \leq 2/\operatorname{dist}(y, \partial\Omega)$. (For this inequality, we can refer [Fi, Corollary A.23] or [Le, Lemma 2.6]) $\square$

Note that we can directly get the lower bound of $\det D^2 u$ without using Legendre transform, just performing the same method as Lemma 2.1.

Lemma 2.4 ([TW1, Lemma 4.2])**.** *Let $u \in C^4(\Omega) \cap C^{0,1}(\bar{\Omega})$ be a locally uniformly convex solution of* (1.3)*. Suppose there exists an open set $\omega \subset \Omega$ such that $x \cdot Du < u$ in ω and $x \cdot Du = u$ on $\partial\omega$. Then for any $y \in \omega$,*

$$\det D^2 u(y) \geq C,$$

where $C > 0$ depends on n, dist $(y, \partial\omega), \sup_\Omega |Du|, \inf_\Omega f$ and $\sup_\omega |u - x \cdot Du|$.

Proof. Let

$$z = \ln w + \beta \ln(u - x \cdot Du) + A|x|^2$$

for some positive constants β and A to be determined. Suppose z attains its maximum at $x_0 \in \omega$. Then at x_0,

$$\begin{aligned} 0 = z_i &= \frac{w_i}{w} - \beta \frac{x_k u_{ki}}{\phi} + 2Ax_i, \\ 0 \geq z_{ii} &= \frac{w_{ii}}{w} - \frac{w_i^2}{w^2} - \beta \frac{x_k u_{kii} + u_{ii}}{\phi} - \beta \frac{x_i^2 u_{ii}^2}{\phi^2} + 2A, \end{aligned}$$

where $\phi = u - x_i u_i$. By a rotation we may suppose $D^2 u$ is diagonal at x_0. Then

$$\begin{aligned} 0 &\geq u^{ii} z_{ii} \\ &= \frac{f}{d^\theta} - u^{ii}\left[\beta^2 \frac{x_i^2 u_{ii}^2}{\phi^2} - 4\beta A \frac{x_i^2 u_{ii}}{\phi} + 4A^2 x_i^2\right] - \frac{\beta x_k}{\phi} u^{ii} u_{kii} - \frac{\beta n}{\phi} - \frac{\beta x_i^2 u_{ii}}{\phi^2} + 2Au^{ii} \\ &= \frac{f}{d^\theta} - \beta(\beta+1)\frac{x_i^2 u_{ii}}{\phi^2} + \frac{4\beta A x_i^2}{\phi} + 2A\left(1 - 2Ax_i^2\right) u^{ii} - \frac{\beta n}{\phi} + \frac{\beta x_i}{\phi(1-\theta)} \frac{w_i}{w} \\ &= \frac{f}{d^\theta} - \frac{\beta n}{\phi} + \frac{4\beta A x_i^2}{\phi} + Au^{ii} - \beta(\beta+1)\frac{x_i^2 u_{ii}}{\phi^2} + \frac{\beta x_i}{\phi(1-\theta)}\left(\beta \frac{x_i u_{ii}}{\phi} - 2Ax_i\right) \\ &\geq \frac{f}{d^\theta} - \frac{C}{\phi} + Au^{ii} + \beta\left(\frac{\beta}{1-\theta} - \beta - 1\right)\frac{x_i^2 u_{ii}}{\phi^2} \\ &\geq \frac{f}{d^\theta} - \frac{C}{\phi} + Au^{ii} \end{aligned}$$

if β is large and A is sufficiently small. It follows that $|\phi| u^{ii} \leq C$ and hence the desired lower bound. $\square$

Hence, we also have Lemma 2.3 as follows. If u is a strictly convex solution of (1.3), we can characterize the open set ω_y ($y \in \Omega$) in the following way. Let $\varepsilon > 0$ be any given constant. Let $\mathcal{P}_\varepsilon$ denote the set of linear functions g such that $g < u$ in Ω and $g(y) = u(y) - \varepsilon$. Let $\bar{g}(x) = \sup\{g(x) \mid g \in \mathcal{P}_\varepsilon\}$. Then $\bar{g} \le u$ and the graph of $\bar{g}$ is a convex cone. Let ω denote the component of $\{\bar{g} < u\}$ containing y. Then if $\varepsilon < h_u\left(\frac{1}{2}r\right)$, where $r = \operatorname{dist}(y, \partial\Omega)$, we have $\bar{\omega} \subset \Omega$.

In the above discussing, we have established bounds for Hessian determinant of solutions to (1.3) in bounded convex domain $\Omega \subset \mathbb{R}^n$, namely, for any subdomain $\Omega' \subset\subset \Omega$, there is

$$0 < \lambda \le \det D^2 u \le \Lambda \quad \text{in } \Omega',$$

where λ and Λ are positive constants depending only on n, $\operatorname{diam}\Omega$, $\operatorname{dist}(\Omega', \Omega)$ and modulus of convexity of u, i.e. h_u. Next, we use the Hölder estimate from Caffarelli and Gutiérrez [CG] to get following theorem.

Theorem 2.5 ([TW, Theorem 4.2]). *Let Ω be a bounded convex domain in $\mathbb{R}^n$ and $u \in C^4(\Omega)$ a locally uniformly convex solution of equation* (1.3) *in Ω satisfying $-1 \le u \le 0$ in Ω. Then $u \in C^\infty(\Omega)$ and for any subdomain $\Omega' \subset\subset \Omega, k \ge 2$, we have the estimates*

$$D^2 u \ge C_1 I, \quad \left|D^k u\right| \le C_2,$$

where C_1 depends on n, dist $(\Omega', \partial\Omega)$, diam Ω, and the modulus of convexity $h_{u,\Omega}$, and C_2 depends additionally on k. Moreover, u is also analytic in Ω.

Next, we apply Theorem 2.5 to prove the Bernstein problem provided uniform, strict convexity.

Theorem 2.6 ([TW, Corollary 4.3 & Theorem 2.1]). *Let $u \in C^4(\Omega)$ be a locally uniformly convex solution of equation* (1.3) *in a comvex domain $\Omega \subset \mathbb{R}^n$, satisfying $\lim\limits_{x \to \partial\Omega} u(x) = +\infty$. Then, if u satisfies the uniform strict convexity condition in Ω, i.e. there is a nondecreasing positive function h on $(0, \infty)$, independent of u, such that*

$$h_{u,x}(r) \ge h(r) \quad \textit{for } x \in \Omega.$$

Then it follows that u is a quadratic function and $\Omega = \mathbb{R}^n$.

Proof. By subtracting a linear function, we may suppose

$$u(0) = D_i u(0) = 0, \quad i = 1, \cdots, n.$$

Let $T_t = \left[a_t^{ij}\right]$ be a linear transformation which normalizes (John's lemma [Fi, Lemma A.13]) the section

$$S_t = \{x \in \Omega \mid u < t\}, \quad (t > 0),$$

and define u_t and Ω_t by

$$u_t(x) = \frac{1}{t} u\left(T^{-1}(x)\right), \quad \Omega_t = \{x \mid u_t < 1\} = T_t\left(S_t\right)$$

By the assumption of Theorem 2.6, $u_t \in C^4\left(\bar{\Omega}_t\right)$ is uniformly convex and satisfies the affine invariant (it's easy to check) equation (1.3) in Ω_t. Furthermore, we have

$$D^2 u_t(x) \ge C_1 I \tag{2.3}$$

for any $t \geq 1$ and $x \in \gamma B$. Let Λ_t denote the maximum eigenvalue of T_t. We claim there exists a positive constant Λ_0 such that

$$\overline{\lim}_{t\to\infty} t\Lambda_t^2 \leq \Lambda_0$$

In fact, we observe from (2.3),

$$u(x) = tu_t\left(T_t(x)\right) \geq C_1 t\left|T_t(x)\right|^2$$

and hence

$$\sup_{x\in rB} u(x) \geq \sup_{x\in rB} C_1 t\left|T_t(x)\right|^2 = C_1 r^2 t\Lambda_t^2,$$

where r is chosen small enough to ensure $rB \subset \Omega$. Next for $x \in \Omega$, we estimate

$$\begin{aligned}\left|D^3 u(x)\right| &\leq C\Lambda_t^3 t\left|D^3 u_t\left(T_t(x)\right)\right| \\ &\leq C\Lambda_0^{3/2} t^{-1/2}\end{aligned}$$

for $T_t(x) \in \gamma B$. Hence letting $t \to \infty$, we conclude $D^3 u = 0$, whence u is quadratic and $\Omega = \mathbb{R}^n$. □

In the following, we state that in the two dimensional case ($n = 2$) a solution to (1.3) with zero boundary condition satisfies a modulus of convexity estimate. We say $p \in \partial D$ is an extreme point of D if there is a supporting hyperplane of D such that D lies on one side of the plane and D touches the plane only at p. If D is convex, then any point in D can be represented as a linear combination of extreme points of D.

Lemma 2.7 ([TW, Lemma 5.1])**.** *Let Ω be a normalized convex domain in $\mathbb{R}^n$ and $u \in C^4(\Omega)$ be a locally uniformly convex solution of (1.4), satisfying (3.1). Then there exists a nondecreasing positive function h on $(0, \infty)$, independent of u, such that*

$$h_{u,x}(r) \geq h(r) \text{ for } x = (x_1, x_2) \in \frac{1}{2}\alpha_n B, r > 0.$$

Proof. We refer to [TW, Page 410-413] or [Zh, Page 35-39] for detail discussions. □

Remark 2.8. *In dimension 2, there is another proof of Theorem 1.1 without using Caffarelli and Gutiérrez's theory. Actually, it is original from Bernstein. Let's state it below.*

Theorem 2.9 ([Zh, Proposition 5.2])**.** *Suppose u is a solution to the elliptic equation*

$$\sum_{i,j=1}^{2} a_{ij}u_{ij} = 0 \quad \text{in } \mathbb{R}^2$$

such that

$$|u(x)| = o(|x|) \quad \text{as } x \to +\infty.$$

Then u is a constant.

3. Example

In this section, we provide an example of affine maximal, convex graphs which does not satisfy the Bernstein property, and which violates the uniform strict convexity in high dimensions. Specifically we take $n = 10$ and define

$$u(x) = \sqrt{|x'|^9 + x_{10}^2},$$

where $x' = (x_1, \cdots, x_9)$. It is readily shown that $u \in W^{2,1}_{\text{loc}}(\mathbb{R}^{10})$ so that $D^2u = \partial^2 u$ and we need to show that u is affine maximal. For $x \neq 0$, we consider the transformation

$$\begin{cases} y' &= x' \\ y_{10} &= x_{10} + u \\ v &= u - x_{10} \end{cases}$$

so that the function v is given by

$$v(y) = \frac{|y'|^9}{y_{10}}$$

for $y_{10} > 0$. In fact,

$$\begin{aligned} v(y) &= \sqrt{|x'|^9 + x_{10}^2} - x_{10} \\ &= \frac{|y'|^9}{\sqrt{|x'|^9 + x_{10}^2} + x_{10}} \\ &= \frac{|y'|^9}{y_{10}}. \end{aligned}$$

To show that v satisfies the affine maximal surface equation, we consider, more generally, functions of the form,

$$u = \frac{r^{2\alpha}}{t}$$

where $\alpha \geq 1, r = |y'|, t = |y_n|, y' = (y_1, \cdots, y_{n-1})$. Then

$$\begin{aligned} u_r &= \frac{2\alpha r^{2\alpha-1}}{t}, \\ u_t &= -\frac{r^{2\alpha}}{t^2}, \\ u_{rr} &= 2\alpha(2\alpha - 1)\frac{r^{2\alpha-2}}{t}, \\ u_{rt} &= -\frac{2\alpha r^{2\alpha-1}}{t^2}, \\ u_{tt} &= \frac{2r^{2\alpha}}{t^3}. \end{aligned}$$

Denote

$$\begin{aligned}\Delta &= u_{rr}u_{tt} - u_{rt}^2 = 4\alpha(\alpha-1)\frac{r^{4\alpha-2}}{t^4},\\ \mathscr{D} &= \det D^2u = \left(\frac{u_r}{r}\right)^{n-2}\Delta = C\frac{r^{2n(\alpha-1)+2}}{t^{n+2}},\\ w &= \mathscr{D}^{\frac{1}{n+2}-1} = C'\frac{t^{n+1}}{r^\theta},\end{aligned}$$

where

$$\begin{aligned}C &= 2^n\alpha^{n-1}(\alpha-1), \quad C' = C^{-(n+1)/(n+2)}\\ \theta &= \frac{2(n+1)}{n+2}(n\alpha - n + 1).\end{aligned}$$

Also, denote

$$\begin{aligned}\tilde{\Delta} &= u_{tt}w_{rr} + u_{rr}w_{tt} - 2u_{rt}w_{rt}\\ &= \frac{C't^{n-2}}{r^{\theta-2\alpha+2}}(2\theta(\theta+1) + n(n+1)2\alpha(2\alpha-1) - 4\alpha(n+1)\theta).\end{aligned}$$

Then we have (see [li])

$$L[u] := u^{ij}w_{ij} = (n-2)\frac{r}{u_r}\frac{w_r}{r} + \frac{1}{\Delta}\widetilde{\Delta} = \frac{t^{n+2}}{r^{\theta+2\alpha}}K,$$

where

$$\begin{aligned}K =& C'\left[-\frac{n-2}{2\alpha}\theta\right.\\ &\left.+\frac{1}{2\alpha(\alpha-1)}(\theta(\theta+1) + n(n+1)\alpha(2\alpha-1) - 2(n+1)\alpha\theta)\right].\end{aligned}$$

For u to be affine maximal, we need $K=0$, i.e.,

$$\theta(\theta+1) + n(n+1)\alpha(2\alpha-1) - 2(n+1)\alpha\theta - (n-2)(\alpha-1)\theta = 0.$$

Substituting for θ, we obtain the equivalent quadratic equation for α,

$$8\alpha^2 - \left(n^2 - 4n + 12\right)\alpha + 2(n-1)^2 = 0,$$

which is solvable for $n \geq 10$. In particular for $n = 10, \alpha = \frac{9}{2}$ and we conclude that the function u satisfies (1.3) for $x \neq 0$. Consequently u is affine maximal in $\mathbb{R}^{10}\setminus\{0\}$. If $n > 10$, it is easy to verify that the function u, given by

$$u(x) = \sqrt{|x'|^9 + |x_{10}|^2} + |\widetilde{x}|^2,$$

is affine maximal in $\mathbb{R}^{10}\setminus\{0\}$, where $\tilde{x} = (x_{11}, \cdots, x_n)$.

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