NOTES FOR "BERNSTEIN PROBLEM"

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1. Two-dimensional case

This ia a seminar note that I reported at the seminar, *Geometric analysis 3+X*, held by PKU in December, 2020. In this note, I'll mainly give the proof of Bernstein theorem [Be] in dimension two based on Colding and Minicozzi [CM], then I will briefly introduce the higher dimensions and half space cases, and in the last, I give a problem (Problem 3.3) related to Bernstein theorem. The main theorem stated as following:

Theorem 1.1 (Bernstein)**.** *If $u : \mathbb{R}^2 \to \mathbb{R}$ is an entire solution to the minimal surface equation, then $u(x, y) = ax + by + c$ for some constants $a, b, c \in \mathbb{R}$.*

Actually, Bernstein obtained Theorem 1.1 as an application of the so called Bernstein s geometric theorem:

Theorem 1.2. *If the Gauss curvature of the graph of $u \in C^\infty(\mathbb{R}^2)$ in $\mathbb{R}^3$ satisfies $K \leq 0$ everywhere and $K < 0$ at some point, then u cannot be bounded.*

In the original proof of Bernstein, there is a gap, and later it is filled by Hopf [Ho]. As a corollary, Bernstein proved a very general Liouville theorem:

Corollary 1.3. *Suppose u is a smooth solution to the elliptic equation*

$$\sum_{i,j=1}^{2} a_{ij} u_{ij} = 0 \quad \text{in } \mathbb{R}^2$$

such that

$$|u(x)| = o(|x|) \text{ as } |x| \to +\infty.$$

Then u is a constant.

In this note, I'll use a method belongs to differential geometry different from Berntsein's and Hopf's to prove Theorem 1.1.

Considering the Gauss map $N : \Sigma \to S^2 \subset \mathbb{R}^3$ is a continuous choice of a unit normal. Since the unit normal to S^2 at $N(x)$ is just $N(x)$ itself, the differential of the map N can be identified with the Weingarten map $\nabla . N : T_x\Sigma \to T_x\Sigma$. Hence, the differential dN is given by

$$\langle dN(E_i), E_j\rangle = \langle \nabla_{E_i} N, E_j\rangle = -\langle N, \nabla_{E_i} E_j\rangle = -A_{ij},$$

where A_{ij} are the second fundamental form. There is a property of Gauss map related to minimal surface.

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Proposition 1.4. *If Σ is a minimal surface, then the Gauss map is an (anti)conformal.*

Proof. We only need to check that under principle vector fields. Take principle vector fields $\{e_1, e_2\}$ for Σ, i.e. $\nabla_{e_1} N = -\kappa_1 e_1$, $\nabla_{e_2} N = -\kappa_2 e_2$. So

$$|\nabla_{e_1} N| = |\kappa_1| = |\kappa_2| = |\nabla_{e_2} N|$$

by $H = \kappa_1 + \kappa_2 = 0$. $\square$

Moreover, for minimal surface, we have

$$|dN|^2 = |A|^2 = \kappa_1^2 + \kappa_2^2 = -2\kappa_1\kappa_2 = -2K = -2\det(dN). \tag{1.1}$$

Note that (1.1) will be used to prove Theorem 1.1. In the following, I'll prove Theorem 1.1. It can be done by following two propositions.

Lemma 1.5. *If $u : \Omega \to \mathbb{R}$ is a solution to minimal surface equation, then for all nonnegative Lipschitz function η with support contained in $\Omega \times \mathbb{R}$, there is*

$$\int_{\mathrm{Graph}_u} |A|^2 \eta^2 \le C \int_{\mathrm{Graph}_u} |\nabla_{\mathrm{Graph}_u} \eta|^2.$$

Proof. Let ω denote the area two-form on the unit S^2. Since the upper hemisphere is contractible, we know $H^r_{dR}(S^2_+) = 0$, $r = 1, 2$. Then closed form $\omega = d\alpha$ also exact. Since Σ is minimal and the differential d commutes with pull-backs, we see that

$$|A|^2 dArea = -2K dArea = -2N^*\omega = -2dN^*\alpha.$$

Moreover, since α is a one-form, there is a constant C_α so that

$$|N^*\alpha| = |\alpha(dN)| \le C_\alpha |dN| = C_\alpha |A|.$$

Set $\Sigma = \mathrm{Graph}_u$. By Stokes theorem, we get

$$\begin{aligned}
\int_\Sigma \eta^2 |A|^2 dArea = 2\int_\Sigma -\eta^2 dN^*\alpha &= 4\int_\Sigma \eta d\eta \wedge N^*\alpha \\
&\le 4C_\alpha \int_\Sigma |\eta||\nabla_\Sigma \eta||A| dArea \\
&\le 4C_\alpha \left(\int_\Sigma \eta^2 |A|^2 dArea\right)^{\frac{1}{2}} \left(\int_\Sigma |\nabla_\Sigma \eta|^2 dArea\right)^{\frac{1}{2}}
\end{aligned}$$

where the last inequality used the Cauchy-Schwarz inequality. $\square$

Corollary 1.6. *If $u : \Omega \to \mathbb{R}$ is a solution to the minimal surface equation, $k > 1$, and Ω contains a ball of radius kR centered at the origin, then*

$$\int_{B_{\sqrt{k}R} \cap \mathrm{Graph}_u} |A|^2 \le \frac{C}{\ln k}.$$

Proof. Set $\Sigma = \mathrm{Graph}_u$. Define the cutoff function η on all of $\mathbb{R}^3$ and then restrict it to the graph of u as follows: Let r denote the distance to the origin in $\mathbb{R}^3$ and define η by

$$\eta = \begin{cases} 1 & r^2 \leq kR^2, \\ 2 - 2\dfrac{\ln(rR^{-1})}{\ln k} & kR^2 < r^2 \leq k^2R^2, \\ 0 & r^2 > k^2R^2. \end{cases}$$

Since $|\nabla_\Sigma r| \leq |\nabla r| = 1$, we have

$$|\nabla_\Sigma \eta| \leq \frac{2}{r \ln k}.$$

Applying Lemma 1.5 with this cutoff function η, we get

$$\begin{aligned} \int_{B_{\sqrt{k}R}\cap\Sigma} |A|^2 &\leq \int_\Sigma \eta^2|A|^2 \leq C\int_\Sigma |\nabla_\Sigma\eta|^2 \\ &\leq \frac{4C}{(\ln k)^2}\int_{(B_{kR}\setminus B_{\sqrt{k}R})\cap\Sigma} r^{-2} \\ &\leq \frac{4C}{(\ln k)^2}\sum_{l=\frac{\ln k}{2}}^{\ln k}\int_{(B_{l^lR}\setminus B_{l^{l-1}R})\cap\Sigma} r^{-2} \\ &\leq \frac{4C}{(\ln k)^2}\sum_{l=\frac{\ln k}{2}}^{\ln k} 2\pi e^2 \leq \frac{4\pi Ce^2}{\ln k}. \end{aligned}$$

□

Theorem 1.1 followed by Corollary 1.6 easily.

Proof of Theorem 1.1. By Corollary 1.6, we have

$$\int_{B_{\sqrt{k}R}\cap\mathrm{Graph}_u} |A|^2 \leq \frac{C}{\ln k}.$$

Letting $k \to +\infty$ yields

$$\int_{\mathbb{R}^2} |A|^2 = 0,$$

which means $|A| = 0$ in $\mathbb{R}^2$, i.e. the second fundamental form of graph of u is identically equal to zero, then we know it is a plane. Indeed, If the second fundamental form vanishes, we have

$$0 = r_u \cdot n_u = r_v \cdot n_u = r_u \cdot n_v = r_v \cdot n_v,$$

so that

$$n_u = n_v = 0.$$

Since n_u, n_v are orthogonal to n and hence linear combinations of r_u, r_v. Thus n is constant. This means

$$(r \cdot n)_u = r_u \cdot n = 0,$$

$$(r \cdot n)_v = r_v \cdot n = 0.$$

So $r \cdot n =$ const., which is the equation of a plane. Hence we complete the proof. □

It is not hard to extend the Bernstein theorem to complete minimal surfaces whose Gauss map omits an open set, this was a conjecture of Nirenberg and was proven by Osserman [Os]. Later, it is improved by Xavier [Xa].

Theorem 1.7 ([Os, Theorem])**.** *Every complete simply-connected minimal surface in 3-space whose normal mapping into the unit sphere omits a neighborhood of some point must be a plane.*

Theorem 1.8 ([Xa, Theorem])**.** *The complement of the image of the Gauss map of a non-flat complete minimal surface in $\mathbb{R}^3$ contains at most 6 points of S^2.*

2. Higher-dimensional case

In this section, I'll state some results in higher dimensions without proof. For details discussion we refer to related references.

In 1962, Fleming [Fl][1] gave a new proof of the two dimensional theorem, using a method independent of the number of dimensions and provided hope of proving the theorem in more than two variables (we should be note that it doesn't yield Osserman's or Xavier's results). The geometric measure theory technique described there led to yet another solution of Bernstein's theorem. The main idea in the proof is to construct a sequence of surface by blowing down the original surface about a point. It is shown that this sequence converges to a minimizing cone. The question is then reduced to the existence of singular cones in $\mathbb{R}^n$. Since no such cones exist in $\mathbb{R}^3$, Fleming's argument gives the new proof in 2-dimension.

In 1965, De Giorgi [De] improved the result showing that nonexistence of singular minimal k-cones in $\mathbb{R}^{k+1}$ would imply Bernstein's theorem for minimal graphs in $\mathbb{R}^{k+2}$. Hence De Giorgi proved Bernstein's theorem is true in $\mathbb{R}^4$.

In 1966, Almgren [Al] proved that there exist no singular cones in $\mathbb{R}^4$, which extend Bernstein theorem that four-dimensional minimal surface in $\mathbb{R}^5$.

In 1968, Simons [Si] extend the result to $\mathbb{R}^7$, which is seven-dimensional minimal surface in $\mathbb{R}^8$. The exciting discovery in the paper is the example of the cone

$$C = \{x \in \mathbb{R}^8 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2\}.$$

Simions' cone is not only stable but even absolutely area minimizing as shown by Bombieri, De Giorgi, Giusti in [BDG]. They also constructed a complete minimal graph over $\mathbb{R}^n, n \geq 8$ [BDG, Theorem B], different from hyperplane.

Combing all of results, we know Bernstein theorem now solved, but new problem arised. *Are there any additional conditions on the function $u(x, y)$ which guarantee for the solution to be a plane even in higher dimensions?* In the book of Giusti and Williams [GW], we can find the answers.

[1] The plateau problem is to find in $\mathbb{R}^3$ a minimal surface bounded by a given system of closed curve.

Theorem 2.1 ([GW, Theorem 17.5]). *Let u be a solution of minimal surface equation in $\mathbb{R}^n$, if u has bounded gradient in $\mathbb{R}^n$, then u is an affine function.*

Theorem 2.2 ([GW, Theorem 17.6]). *Let u be a solution of minimal surface equation in $\mathbb{R}^n$. Suppose that for every $x \in \mathbb{R}^n$, $u(x) \le K(1 + |x|)$ for some constant K. Then u is an affine function.*

3. Other interesting things

In this section, I'll briefly introduce some results of Bernstein theorem in half space and a conjecture made by myself. The first part comes from the work of Jiang, Wang and Zhu [JWZ]. The second part is a question motivated by seeing Mooney's notes: *The Monge-Ampère equations.*

Let $n \ge 2$ be an integer and $\mathbb{R}^n_+ = \{(x', x_n) | x' \in \mathbb{R}^{n-1}, x_n > 0\}$. We have

Theorem 3.1. *Let $n \ge 2$ be an integer and $u \in C^2(\mathbb{R}^n_+) \cap C(\partial \mathbb{R}^n_+)$ be a solution of*

$$\begin{cases} div\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 & in\ \mathbb{R}^n_+, \\ \qquad\qquad\qquad\quad u = l & on\ \partial\mathbb{R}^n_+, \end{cases}$$

where $l : \mathbb{R}^n \to \mathbb{R}$ is an affine function. Assume that $u : \overline{\mathbb{R}^n_+} \to \mathbb{R}$ has at most a linear growth, which means there exists a constant $K > 0$ such that

$$|u(x)| \le K(1 + |x|) \quad \forall x \in \overline{\mathbb{R}^n_+}. \tag{3.1}$$

Then u is an affine function.

Hence here comes two questions. The first is whether the assumption (3.1) is necessary? In dimension two, we know Theorem 3.1 is still true without (3.1), but in higher dimension it is still not clear. The other question is whether the affine boundary is necessary? The answer is yes. Following is the counterexample:

$$f(x) = \int_1^{|x|} \frac{dt}{\sqrt{1+t^2}} \qquad \text{in } P_+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 2\}.$$

From this point of view, it is interesting to know whether Liouville type theorem will be valid for Neumann boundary condition. Luckly, the answer is positive.

Theorem 3.2. *Let $n \ge 2$ be an integer and $u \in C^2(\mathbb{R}^n_+) \cap C^1(\overline{\mathbb{R}^n_+})$ be a solution of minimal surface equation with Neumann boundary condition*

$$\partial_{x_n} u = \tau \qquad on\ \partial\mathbb{R}^n,$$

where τ is a constant. If u satisfies (3.1), then u is an affine function.

In the following, I'll state a conjecture made by myself when I was reading Mooney's notes: The Monge-Amp re equations. We first note that in Corollary 1.3, the equation doesn't need to be uniformly elliptic, hence it is a very powerful result. What I want to know if this result has a half space version, which is like harmonic functions. More precisely, I want to obtain the following proposition:

Problem 3.3. *Suppose $u \geq 0$ is a solution to the elliptic equation*

$$\begin{cases} \sum_{i,j=1}^{2} a_{ij} u_{ij} = 0 & \textit{in } \mathbb{R}^2_+, \\ u(x,0) = 0 & \textit{on } \mathbb{R}. \end{cases}$$

Then u is a linear function of form

$$u(x,y) = Ay, \quad A \geq 0.$$

Note that in the question, a_{ij} could be degenerate or sigular at ∞. This question is motivated by seeing Mooney's notes: The Monge-Amp re equations. He used partial Legendre transform to investigate the Liouville theorem for Monge-Amp re equation in half space, and one of steps in his proof used the similar proposition for harmonic functions, and which can be proved by boundary Harnack inequality and odd extension of u. But it is failed for the case without uniform ellipticity.

Generally, if we don't assume any regularity condition on a_{ij}, Problem 3.3 is wrong. There is a counterexample given by Mooney [Mo]. And also I can construct a solution satisfies a equation degenerate on $\{y = 0\}$. Indeed, $u(x,y) = e^{-x} \sinh y$ is an example. But I still believe that Problem 3.3 maybe true if we assume a_{ij} is smooth. I have no idea how to prove it and I didn't find any references about this problem, either.

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