BOUNDARY HÖLDER REGULARITY OF UNIFORMLY ELLIPTIC EQUATIONS WITH DRIFTS

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1. Boundary regularity

In this short notes, we will use the method of barrier function to prove the boundary Hölder regularity of uniformly elliptic equations with drifts. More precisely, we consider the following Dirichlet problem

$$\begin{cases} Lu := a_{ij}D_{ij}u + b_iD_iu + cu = f & \text{in} \quad \Omega, \\ \qquad\qquad\qquad\qquad\qquad\quad u = \varphi & \text{on} \quad \partial\Omega. \end{cases} \tag{1.1}$$

We shall assume throughout the notes that the operator L, as ginven in (1.1), is uniformly elliptic with bounded coefficients in the domain Ω, i.e. there are fixed positive constants λ, Λ and M such that

$$0 < \lambda I \leq (a_{ij}) \leq \Lambda I; \tag{1.2}$$

$$\|b\|_{L^\infty(\Omega)} + \|c\|_{L^\infty(\Omega)} \leq M; \tag{1.3}$$

$$c \leq 0 \quad \text{in } \Omega. \tag{1.4}$$

There are many methods to obtain boundary Hölder estimates, barrier function is one of those which can be applied to other type of equations. Hence we will use this technique to prove it. Firstly, we give a definition of uniform sphere condition.

Definition 1.1. *We say that a domain $\Omega \subset \mathbb{R}^n$ satisfies the exterior (interior, resp.) sphere condition if for any $x \in \partial\Omega$, there exist a sphere $B_r(y) \subset \mathbb{R}^n \setminus \Omega$ ($B_r(y) \subset \Omega$, resp.) and $x \in \partial B_r(y)$. If we can take a radius $r > 0$ independent of point x, we say that Ω satisfies the uniform sphere condition.*

Secondly, we state a useful maximum principle for non-divergence type elliptic equations. Denote $D^* := (\det(a_{ij}))^{1/n}$.

Theorem 1.2 (Aleksandrov-Bakelman-Pucci Theorem)**.** *Suppose that $u \in C(\overline{\Omega}) \cap W^{2,n}_{loc}(\Omega)$ satisfies $Lu \geq f$ in Ω with the following conditions*

$$0 < \lambda \leq D^* \leq \Lambda, \quad \frac{|b|}{D^*}, \frac{f}{D^*} \in L^n(\Omega) \quad \textit{and} \quad c \leq 0 \textit{ in } \Omega.$$

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Then there holds

$$sup_{\Omega} u \le \sup_{\partial\Omega} u^+ + C \left\| \frac{f^-}{D^*} \right\|_{L^n(\Gamma^+)}$$

where Γ^+ *is the upper contact set of* u *and* C *is a constant depending only on* n*,* $\mathrm{diam}(\Omega)$ *and* $\left\| \frac{b}{D^*} \right\|_{L^n(\Gamma^+)}$*. In fact,* C *can be written as*

$$\mathrm{diam}(\Omega) \cdot \left\{ \exp\left[\frac{2^{n-2}}{\alpha_n n^n} \left(\left\| \frac{b}{D^*} \right\|^n_{L^n(\Gamma^+)} + 1 \right) \right] - 1 \right\}$$

with α_n *as the volume of the unit ball in* $\mathbb{R}^n$*.*

Remark 1.3. *Note that* $c \le 0$ *in Theorem 1.2 is crucial, that's also the reason we assume condition* (1.4) *in* (1.1)*.*

Then we derive a key lemma.

Lemma 1.4. *Assume that* Ω *is bounded domain in* $\mathbb{R}^n$ *which satisfies uniform exterior ball condition. Let* $u \in C(\overline{\Omega}) \cap W^{2,n}_{loc}(\Omega)$ *be the solution to* (1.1)*, where* $\varphi \in C^\alpha(\partial\Omega)$ *for some* $\alpha \in (0,1)$*,* $f \in L^n(\Omega)$ *and the operator* L *satisfies* (1.2)*-*(1.4)*. Then there exist* δ*,* C *depending only on* λ*,* Λ*,* M*,* n*,* α*,* $\mathrm{diam}(\Omega)$ *and the radius of uniform sphere such that for any* $x_0 \in \partial\Omega$*, we have*

$$|u(x) - u(x_0)| \le C|x - x_0|^{\frac{\alpha}{2+\alpha}} \left(\|\varphi\|_{C^\alpha(\partial\Omega)} + \|f\|_{L^n(\Omega)} \right), \quad \forall\, x \in \Omega \cap B_\delta(x_0).$$

Proof. We assume that the radius of uniform exterior sphere is R. For any $x_0 \in \partial\Omega$, considering the exterior ball $B_r(y)$, where $0 < r \le R$. We consider the barrier function

$$w(x) = \frac{1}{r^p} - \frac{1}{|x-y|^p},$$

where $p > 0$ is a large constant to be determined later. Clearly, $w(x) \ge 0$ in Ω. A direct calculation yields

$$\begin{aligned} Lw =& a_{ij} \left[-p(p+2) \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^{p+4}} + p \frac{\delta_{ij}}{|x-y|^{p+2}} \right] \\ &+ p\, b_i \frac{x_i - y_i}{|x-y|^{p+2}} + cw \\ \le& \frac{p\lambda}{|x-y|^{p+2}} \left(-p - 2 + \frac{\Lambda}{\lambda} n + \frac{M}{\lambda r} \right), \end{aligned}$$

where conditions (1.2)-(1.4) are used. Set

$$\mathcal{N}_r = \Omega \cap \{ r < |x-y| < 3r \}.$$

If we choose $p > \frac{\Lambda}{\lambda} n + \frac{M}{\lambda r}$, then there is

$$Lw < 0 \quad \text{in } \mathcal{N}_r.$$

Note that by the ABP estimate (Theorem 1.2), we know

$$|u(x)| \le \|\varphi\|_{L^\infty(\Omega)} + C_n\, \mathrm{diam}(\Omega) \|f\|_{L^n(\Omega)}, \quad \forall\, x \in \Omega.$$

Now, we set the auxiliary function in $\mathcal{N}_r$ as

$$v_\pm(x) = (u(x) - u(x_0)) \pm (6r)^\alpha \|\varphi\|_{C^\alpha(\partial\Omega)} \pm Kw,$$

where $K = 4r^p \|u\|_{L^\infty(\Omega)}$. Note that on $\partial\mathcal{N}_r \cap \Omega$, there is

$$Kw(x) \geq K\frac{1}{r^p}\left(1 - \frac{1}{3^p}\right) \geq K\frac{1}{2r^p} = 2\|u\|_{L^\infty(\Omega)}.$$

Hence, we have

$$v_- \leq 0,\ v_+ \geq 0 \quad \text{on } \partial\mathcal{N}_r.$$

On the other hand, we have

$$Lv_- = f - cu(x_0) - c(6r)^\alpha \|\varphi\|_{C^\alpha(\partial\Omega)} - KLw \geq f - M\|u\|_{L^\infty(\Omega)},$$

and

$$Lv_+ = f - cu(x_0) + c(6r)^\alpha \|\varphi\|_{C^\alpha(\partial\Omega)} + KLw \leq f + M\|u\|_{L^\infty(\Omega)},$$

where (1.3) and (1.4) are used. Then the ABP estimate gives us that

$$v_- \leq C_n \operatorname{diam}(\mathcal{N}_r) \|f - M\|u\|_{L^\infty(\Omega)}\|_{L^n(\mathcal{N}_r)} \leq Cr\|f\|_{L^n(\Omega)}, \quad \text{in } \mathcal{N}_r,$$

and

$$v_+ \geq -C_n \operatorname{diam}(\mathcal{N}_r) \|f + M\|u\|_{L^\infty(\Omega)}\|_{L^n(\mathcal{N}_r)} \geq -Cr\|f\|_{L^n(\Omega)}, \quad \text{in } \mathcal{N}_r.$$

Hence, we conclude that

$$\begin{aligned}
|u(x) - u(x_0)| &\leq Kw + (6r)^\alpha \|\varphi\|_{C^\alpha(\partial\Omega)} + Cr\|f\|_{L^n(\Omega)} \\
&\leq \frac{Kp}{r^{p+1}}(|x-y| - r) + (6r)^\alpha \|\varphi\|_{C^\alpha(\partial\Omega)} + Cr\|f\|_{L^n(\Omega)} \\
&\leq 4p\|u\|_{L^\infty(\Omega)} \frac{|x - x_0|}{r} + (6r)^\alpha \|\varphi\|_{C^\alpha(\partial\Omega)} + Cr\|f\|_{L^n(\Omega)} \\
&\leq C\frac{|x - x_0|}{r^2} + (6r)^\alpha \|\varphi\|_{C^\alpha(\partial\Omega)} + Cr\|f\|_{L^n(\Omega)},
\end{aligned}$$

where we used $|x - y| - r \leq |x - x_0| + |x_0 - y| - r = |x - x_0|$.

If we choose $\delta < R$, then for any $x \in \Omega \cap B_\delta(x_0)$, let $r = |x - x_0|^{\frac{1}{2+\alpha}} R^{\frac{\alpha}{2+\alpha}}$, there is

$$|u(x) - u(x_0)| \leq C|x - x_0|^{\frac{\alpha}{2+\alpha}} \left(\|\varphi\|_{C^\alpha(\partial\Omega)} + \|f\|_{L^n(\Omega)}\right),$$

where C depending only on λ, Λ, M, n, α, $\operatorname{diam}(\Omega)$ and R. □

Next, we will apply Lemma 1.4 to the main theorem of this notes. Before that, we first state Krylov-Safonov's interior Hölder estimates.

Theorem 1.5 (Krylov-Safonov)**.** *Given a bounded solution of the following elliptic PDE*

$$a_{ij}D_{ij}u + b_iD_iu + cu = f \quad \text{in } B_1,$$

where repeated indices denotes summation and we assume $0 < \lambda I \leq (a_{ij}) \leq \Lambda I$; $b, c, f \in L^n(B_1)$. *Then the function* $u \in C^\alpha(B_{1/2})$ *for some small* $\alpha > 0$ *with the estimate*

$$|u(x) - u(y)| \leq C|x - y|^\alpha \left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^n(B_1)}\right), \quad \forall\, x, y \in B_{1/2},$$

where constant C *depends only on* λ, Λ, n, b *and* c.

Theorem 1.6. *Assume that Ω is bounded domain in $\mathbb{R}^n$ which satisfies uniform exterior ball condition. Let $u \in C(\overline{\Omega}) \cap W^{2,n}_{loc}(\Omega)$ be the solution to (1.1), where $\varphi \in C^\alpha(\partial\Omega)$ for some $\alpha \in (0,1)$, $f \in L^n(\Omega)$ and the operator L satisfies (1.2)-(1.4). Then, $u \in C^\beta(\overline{\Omega})$ with estimate*

$$\|u\|_{C^\beta(\overline{\Omega})} \leq C\left(\|\varphi\|_{C^\alpha(\partial\Omega)} + \|f\|_{L^n(\Omega)}\right),$$

where β depending only on λ, Λ, n, α, and C depending only on λ, Λ, M, n, α, $\operatorname{diam}(\Omega)$ *and the radius of uniform sphere.*

Proof. Let $x \in \Omega$, denote $r_x := \operatorname{dist}(x, \partial\Omega)$. Considering the rescaling $\tilde{u} : B_1 \to \mathbb{R}$ of u

$$\tilde{u}(z) = u(x + r_x z).$$

It's easy to see that $\tilde{u}$ solves

$$\tilde{a}_{ij} D_{ij}\tilde{u} + \tilde{b}_i D_i \tilde{u} + \tilde{c}\tilde{u} = \tilde{f} \quad \text{in } B_1,$$

where $\tilde{a}_{ij}(z) = a_{ij}(x + r_x z)$, $\tilde{b}_i(z) = r_x b_i(x + r_x z)$, $\tilde{c}(z) = r_x^2 c(x + r_x z)$ and $\tilde{f}(z) = r_x^2 f(x + r_x z)$. By (1.2) and (1.3), we know $0 < \lambda I \leq (\tilde{a}_{ij}) \leq \Lambda I$ and $\|\tilde{b}\|_{L^\infty(B_1)} + \|\tilde{c}\|_{L^\infty(B_1)} \leq \tilde{M}$. Then, we apply Krylov-Safonov's interior Hölder estimates (Theorem 1.5) to $\tilde{u}$ in B_1 to obtain

$$|\tilde{u}(z_1) - \tilde{u}(z_2)| \leq C|z_1 - z_2|^\gamma \left(\|\tilde{u}\|_{L^\infty(B_1)} + \|\tilde{f}\|_{L^n(B_1)}\right), \quad \forall\, z_1, z_2 \in B_{1/2},$$

for some small constant $\gamma \in (0,1)$ depending only on n, λ, Λ and some constant C depending only on n, λ, Λ and M. Rescaling back to u yields

$$|u(x_1) - u(x_2)| \leq C r_x^{-\gamma} |x_1 - x_2|^\gamma \left(\|u\|_{L^\infty(\Omega)} + \|f\|_{L^n(\Omega)}\right), \quad \forall\, x_1, x_2 \in B_{r_x/2}(x). \tag{1.5}$$

Combing (1.5), ABP estimate and Lemma 1.4, it's easy to obtain that

$$\|u\|_{C^\beta(\overline{\Omega})} \leq C\left(\|\varphi\|_{C^\alpha(\partial\Omega)} + \|f\|_{L^n(\Omega)}\right),$$

for some β depending only on λ, Λ, n, α. For completeness, we include the details.

For any x and y in Ω. Let $r_x := \operatorname{dist}(x, \partial\Omega)$ and $r_y := \operatorname{dist}(y, \partial\Omega)$. Without loss of generality, we assume $r_y \leq r_x$. We know there are $x_0 \in \partial\Omega$ and $y_0 \in \partial\Omega$ such that $r_x = |x - x_0|$ and $r_y = |y - y_0|$. From the interior Hölder estimats of Krylov-Safonov (Theorem 1.5), we only need to consider the case $r_y \leq r_x \leq c$ for some small enough $c > 0$.

Assume firstly that $|x - y| \leq r_x^2$. Then $y \in B_{r_x^2}(x) \subset B_{r_x/2}(x)$. By (1.5), we have

$$\begin{aligned} |u(x) - u(y)| &\leq C r_x^{-\gamma} |x - y|^\gamma \left(\|u\|_{L^\infty(\Omega)} + \|f\|_{L^n(\Omega)}\right) \\ &\leq C|x - y|^{\gamma/2} \left(\|u\|_{L^\infty(\Omega)} + \|f\|_{L^n(\Omega)}\right). \end{aligned}$$

Note by ABP estimate (Theorem 1.2) that

$$\|u\|_{L^\infty(\Omega)} \leq \|\varphi\|_{L^\infty(\Omega)} + C_n \operatorname{diam}(\Omega) \|f\|_{L^n(\Omega)}.$$

Hence when $|x - y| \leq r_x^2$, there is

$$|u(x) - u(y)| \leq C|x - y|^{\gamma/2} \left(\|\varphi\|_{C^\alpha(\partial\Omega)} + \|f\|_{L^n(\Omega)}\right).$$

Assume finally that $|x-y| \geq r_x^2$, we have

$$|x_0 - y_0| \leq |x - x_0| + |x - y| + |y - y_0| \leq |x-y|^{1/2} + |x-y|.$$

Hence, by Lemma 1.4, we know

$$\begin{aligned} |u(x) - u(y)| &\leq |u(x) - u(x_0)| + |u(x_0) - u(y_0)| + |u(y) - u(y_0)| \\ &\leq C(r_x^{\frac{\alpha}{2+\alpha}} + |x_0 - y_0|^\alpha + r_y^{\frac{\alpha}{2+\alpha}}) \left(\|\varphi\|_{C^\alpha(\partial\Omega)} + \|f\|_{L^n(\Omega)}\right) \\ &\leq C|x-y|^\beta \left(\|\varphi\|_{C^\alpha(\partial\Omega)} + \|f\|_{L^n(\Omega)}\right), \end{aligned}$$

for some β depending only on λ, Λ, n, α. Then we complete the proof. □

Those methods can be applied to fully nonlinear uniformly elliptic equations with minor revision. Also, we can use barrier function to obtain the boundary Hölder regularity of linearized Monge-Ampère equation without drifts, just note that it additional needs to use Savin's localization theorem. Now, the question is can we otain the boundary Hölder regularity of linearized Monge-Ampere equation with drifts. More precisely, can we have the following theorem?

Problem 1.7 (Global Hölder estimates for the linearized Monge-Ampère equation with a drift term)**.** *Assume that $\Omega \subset \mathbb{R}^n$ is a uniformly convex domain with boundary $\partial\Omega \in C^3$. Let $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ be a convex function satisfying*

$$\lambda \leq \det D^2 u \leq \Lambda \quad in \quad \Omega$$

for some positive constants λ and Λ. Moreover, assume that $u|_{\partial\Omega} \in C^3$. Let $(U^{ij}) = (\det D^2 u)(D^2 u)^{-1}$. Let $\mathbf{b} \in L^\infty(\Omega)$, $f \in L^n(\Omega)$ and $\varphi \in C^\alpha(\partial\Omega)$ for some $\alpha \in (0,1)$. Assume that $v \in C(\overline{\Omega}) \cap W^{2,n}_{loc}(\Omega)$ is a solution to the following linearized Monge-Ampère equation with a drift term

$$\begin{cases} U^{ij} D_{ij} v + \mathbf{b} \cdot Dv &= f \quad in \quad \Omega, \\ v &= \varphi \quad on \quad \partial\Omega. \end{cases}$$

Then, there exist constants $\beta, C > 0$ depending only on Ω, λ, Λ, n, α, and $\|\mathbf{b}\|_{L^\infty(\Omega)}$ such that

$$|v(x) - v(y)| \leq C|x-y|^\beta \Big(\|\varphi\|_{C^\alpha(\partial\Omega)} + \|f\|_{L^n(\Omega)} \Big), \ \forall x, y \in \Omega.$$

We failed to use barrier function to prove Problem 1.7, but we still believe that the conclusion of Problem 1.7 is right. It needs some others methods to work out this problem.

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