EXERCISE

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Problem 1. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^N$ and $f(x) = \|x\|^2 : \mathbb{R}^N \to \mathbb{R}$. Suppose that f is C^2 near $x = 0$. Prove there is an inner product $(\cdot, \cdot)$ on $\mathbb{R}^N$ such that $\|x\|^2 = (x, x)$, $x \in \mathbb{R}^N$.*

Proof. First, by the definition of f we know $f'(x_0) = 2\|x_0\|$ and $f''(x_0) = 2$. Hence by Taylor expansion we have

$$f(x+y) = f(x) + f'(x)y + \frac{1}{2}f''(\xi)(y,y),$$
$$f(x-y) = f(x) - f'(x)y + \frac{1}{2}f''(\eta)(y,y).$$

Then we have

$$f(x+y) + f(x-y) = 2f(x) + 2f(y),$$

which is

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Hence it is easy to show that there is an inner product $(\cdot, \cdot)$ on $\mathbb{R}^N$ such that $\|x\|^2 = (x, x)$, $x \in \mathbb{R}^N$. □

Problem 2. *Let $X = L^p(\Omega)$. Compute the Gateaux and Fréchet derivatives of the functional $f(u) = \int_\Omega |u|^p dx : X \to \mathbb{R}$ for $p > 1$ and the sub-differential $\partial f(0)$ if $p = 1$.*

Proof. First consider $p > 1$. For $\forall\, h \in L^p(\Omega)$, we have

$$f(u+th) = \int_\Omega |u+th|^p \, dx.$$

Hence the G-derivative is

$$df(u,h) = \frac{d}{dt} f(u+th)\Big|_{t=0} = \frac{d}{dt}\int_\Omega |u+th|^p \, dx\Big|_{t=0}$$
$$= p\int_\Omega |u|^{p-1} h \, dx.$$

We define $A(u) \in \mathscr{L}(X)$ as

$$A(u)h = df(u,h) = p\int_\Omega |u|^{p-1} h \, dx.$$

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We next show that $u \mapsto A(u)$ is continuous at every $u_0 \in L^p(\Omega)$. Indeed

$$\begin{aligned}
\|A(u) - A(u_0)\| &= \sup_{\|h\|_{L^p} \le 1} |A(u)h - A(u_0)h| \\
&\le \sup_{\|h\|_{L^p} \le 1} \left(\int_\Omega \left| |u|^{p-1} - |u_0|^{p-1} \right|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \left(\int_\Omega |h|^p \, dx \right)^{\frac{1}{p}} \\
&\le \left(\int_\Omega \left| |u|^{p-1} - |u_0|^{p-1} \right|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \\
&\le \left(\int_\Omega |u - u_0|^p \, dx \right)^{\frac{p-1}{p}} \to 0, \quad \text{as } \|u - u_0\|_{L^p} \to 0.
\end{aligned}$$

Hence f is F-differentiable at u_0 and

$$f'(u_0) = A(u_0).$$

For $p = 1$, we need to find all $u^* \in X^*$ such that

$$f(u) \ge f(0) + \langle u^*, u \rangle \ \forall \, u \in X.$$

Indeed, it is easy to see that for $|k| \le 1$

$$\langle u^*, u \rangle := \int_\Omega ku \, dx \le \int_\Omega |u| \, dx.$$

Hence we have

$$\partial f(0) = \left\{ u^* \in X^* | \langle u^*, u \rangle = \int_\Omega ku \, dx, \ |k| \le 1. \right\}.$$

□

Problem 3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $\phi(x, \xi), \frac{\partial \phi(x,\xi)}{\partial \xi} : \Omega \times \mathbb{R} \to \mathbb{R}$ be Carathéodory functions satisfying*

$$\left| \frac{\partial \phi(x, \xi)}{\partial \xi} \right| \le b(x) + a|\xi|^r, x \in \Omega, \xi \in \mathbb{R}$$

$a > 0$ be a constant, $b \in L^{\frac{2n}{n+2}}(\Omega), 1 \le r \le \frac{n+2}{n-2}$. Prove the functional

$$f(u) = \int_\Omega \phi(x, u(x)) dx, \quad H^1(\Omega) \to \mathbb{R}$$

is F-differentiable and

$$< f'(u), h >= \int_\Omega \frac{\partial \phi(x, \xi)}{\partial \xi}(x, u(x)) h(x) dx, \quad h \in H^1(\Omega).$$

Proof. We first calculate the G-derivative of f. For $\forall \, h \in H^1(\Omega)$, we have

$$f(u + th) = \int_\Omega \phi(x, u(x) + th(x)) \, dx.$$

Then

$$df(u,h) = \frac{d}{dt} f(u+th)\bigg|_{t=0} = \int_\Omega \frac{d}{dt}\phi(x, u(x)+th(x))\bigg|_{t=0} dx$$

$$= \int_\Omega \frac{\partial \phi}{\partial \xi}(x, u(x))h(x)\, dx.$$

We define $A(u) : H^1(\Omega) \to \mathbb{R}$ by

$$A(u)h = df(u,h) = \int_\Omega \frac{\partial \phi(x,\xi)}{\partial \xi}(x,u(x))h(x)dx, \quad h \in H^1(\Omega).$$

Next, we show that $A(u) \in \mathscr{L}(H^1(\Omega))$ and it is continuous respect to u, hence f is F-differentiable and $f'(u) = A(u)$. Indeed, it is clearly that $A(u)h$ is linear respect to h. Estimate

$$|A(u)h| = \left| \int_\Omega \frac{\partial \phi(x,\xi)}{\partial \xi}(x,u(x))h(x)\, dx \right|$$

$$\leq \int_\Omega \left| \frac{\partial \phi(x,\xi)}{\partial \xi}(x,u(x)) \right| |h(x)|\, dx$$

$$\leq \int_\Omega b(x)|h(x)|\, dx + a\int_\Omega |u|^r |h(x)|\, dx.$$

By Sobolev embedding, we have

$$\int_\Omega |bh|\, dx \leq \|b\|_{L^{\frac{2n}{n+2}}} \|h\|_{H^1},$$

and

$$\int_\Omega |u|^r |h|\, dx \leq \|u\|_{H^1} \|h\|_{H^1}.$$

Hence, $A(u) \in \mathscr{L}(H^1(\Omega))$. By the Theorem 1.1.5 of course book, we have $\frac{\partial \phi(x,\xi)}{\partial \xi}(\cdot,\cdot) : L^{\frac{2n}{n-2}} \to L^{\frac{2n}{n+2}}$ is continuous, then we know $A(u)$ is continuous respect to u. □

Problem 4. *Let X, Y be Banach spaces and $t \to A(t) : [0,1] \to \mathscr{L}(X,Y)$ be continuous. Suppose that for all $t \in [0,1]$, $A(t)$ is a Fredholm operator from X to Y, prove the Fredholm index* $\operatorname{ind}(A(t))$ *is independent of $t \in [0,1]$.*

Proof. We denote $\mathscr{F} = \mathscr{F}(X,Y)$ be the Fredholm operator from X to Y. Then we know $\operatorname{ind} : \mathscr{F} \to \mathbb{Z}$ is continuous. This can be proved by Theorem 4.6.7 in Kung-Ching Chang's *functional analysis*, which precisely state that if $T \in \mathscr{F}$, then there exists a $\varepsilon > 0$ such that when $S \in \mathscr{L}$ and $\|S\| < \varepsilon$, we have

$$T + S \in \mathscr{F},$$

and

$$\operatorname{ind}(T+S) = \operatorname{ind}(T).$$

Then by the continuity of $A(t)$ we know $\mathrm{ind}(A(t))$ is continuous. Since $[0,1]$ is connected, we have $\mathrm{ind}(A([0,1]))$ is connected in $\mathbb{Z}$, then $\mathrm{ind}(A([0,1]))$ is a constant, which implies that $\mathrm{ind}(A(t))$ is independent of $t \in [0,1]$. □

Problem 5. *(1) Let $a_i, x \in \mathbb{R}$ with $a_n \neq 0$. Suppose $R > 0$ such that all real roots of $f(x) = a_0 + a_1 x + \cdots a_n x^n = 0$ are contained in $(-R, R)$. Compute the degree $\deg(f, [-R,R], 0)$. We can also consider f as a continuous map from $S^1 = \mathbb{R} \cup \{\infty\} = \mathbb{R}P^1$ into itself. Compute $\deg(f, S^1)$.*

(2) Let $a_i, z \in \mathbb{C}$ with $a_n \neq 0$. Suppose $R > 0$ such that all complex roots of $f(z) = a_0 + a_1 z + \cdots a_n z^n = 0$ are contained in $D(R) = \{z \in \mathbb{C} \| z \| < R\}$. Compute the degree $\deg(f, D(R), 0)$. We can also consider f as a continuous map from $S^2 = \mathbb{C} \cup \{\infty\} = \mathbb{C}P^1$, the Riemannian sphere, into itself. Compute $\deg(f, S^2)$.

Solution of (1). We define the following homotopy of f:

$$F(x,t) : [-\widetilde{R}, \widetilde{R}] \times [0,1] \longrightarrow \mathbb{R}, \quad (x,t) \mapsto a_n x^n + t(a_{n-1}x^{n-1} + \cdots + a_0),$$

where $\widetilde{R} > R$ is large enough such that $F(\pm \widetilde{R}, t) \neq 0$. Hence by the homotopy invariance of degree and all real roots of $f(x)$ are contained in $(-R,R)$., we have

$$\begin{aligned} \deg(f, [-R,R], 0) &= \deg(f, [-\widetilde{R}, \widetilde{R}], 0) \\ &= \deg(a_n x^n, [-\widetilde{R}, \widetilde{R}], 0) \\ &= \begin{cases} \mathrm{sign}(a_n), & \text{n is odd,} \\ 0, & \text{n is even.} \end{cases} \end{aligned}$$

When we consider f as a continuous map from S^1 into itself, it no needs to choose large $\widetilde{R}$, hence we have

$$\deg(f, S^1) = \deg(a_n x^n, S^1) = \begin{cases} \mathrm{sign}(a_n), & \text{n is odd,} \\ 0, & \text{n is even.} \end{cases}$$

□

Solution of (2). Similar to (1), and note that complex Jacobi of holomorphic function is the square of real Jacobi one, we have

$$\deg(f, D(R), 0) = \deg(a_n z^n, D(R), 0) = n.$$

Also,

$$\deg(f, S^2) = \deg(a_n z^n, S^2) = n.$$

□

Problem 6. *Let $B = \left\{ x = (x_1, \cdots, x_n) \in \mathbb{R}^n \,|\sum_1^n | x_i|^2 \leq 1 \right\}$. Assume that $f : B \to \mathbb{R}$ be a C^2 function such that*

$$\nabla f(x) \cdot x = \sum_1^n \frac{\partial f}{\partial x_i} x_i \neq 0, \quad x \in \partial B$$

Determine the degree

$$\deg(\nabla f, B, 0), \quad \nabla f = \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right) : B \to \mathbb{R}^n$$

Solution. Since $\nabla f(x) \cdot x \neq 0$, $\forall\, x \in \partial B$, we know

$$t(\nabla f(x) \cdot x)x + (1-t)\nabla f \neq 0, \quad \forall\, (x,t) \in \partial B \times [0,1].$$

Then by the homotopy invariance of degree, we have

$$\deg(\nabla f, B, 0) = \deg((\nabla f(x) \cdot x)x, B, 0).$$

Since f is C^2 on ∂B, we know $\nabla f(x) \cdot x$ is continuous on ∂B. Then by $\nabla f(x) \cdot x \neq 0$, $\forall\, x \in \partial B$, we get either $\nabla f(x) \cdot x > 0$ or $\nabla f(x) \cdot x < 0$, $\forall\, x \in \partial B$.

For $\nabla f(x) \cdot x > 0$, $\forall\, x \in \partial B$, we know

$$t + (1-t)\nabla f(x) \cdot x > 0, \quad \forall\, (x,t) \in \partial B \times [0,1].$$

Hence

$$\deg((\nabla f(x) \cdot x)x, B, 0) = \deg(x, B, 0) = 1.$$

For $\nabla f(x) \cdot x < 0$, $\forall\, x \in \partial B$, we know

$$-t + (1-t)\nabla f(x) \cdot x < 0, \quad \forall\, (x,t) \in \partial B \times [0,1].$$

Hence

$$\deg((\nabla f(x) \cdot x)x, B, 0) = \deg(-x, B, 0) = (-1)^n.$$

Combing above, we get

$$\deg(\nabla f, B, 0) = \begin{cases} 1, & \nabla f(x) \cdot x > 0, \\ (-1)^n, & \nabla f(x) \cdot x < 0. \end{cases}$$

□

Problem 7. *Let*

$$Q = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = 0, x_1^2 + x_2^2 \leq 1\right\},$$
$$\partial Q = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = 0, x_1^2 + x_2^2 = 1\right\},$$
$$S = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = 0, (x_2 - 1)^2 + x_3^2 = 1\right\}, \quad \phi : Q \to \mathbb{R}^3$$

be continuous with $\phi(x) = x, x \in \partial Q$. Prove $\phi(Q) \cap S \neq \emptyset$.

Proof. First by the homotopy invariance of degree, we have

$$\deg(\phi, Q, 0) = (\, x, Q, 0) = 1 \neq 0.$$

Then by Kronecker existence theorem, we know $\phi(Q) \cap S \neq \emptyset$. □

Problem 8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded regular domain. Given some conditions on $f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that the equation*

$$-\triangle u = f(x, u, \nabla u), \quad x \in \Omega$$
$$u(x) = 0, \quad x \in \partial\Omega$$

possesses a solution $u \in C^{2,\gamma}$.

Solution. Assume that $f \in C^1(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ and satisfies

(1) There exists an increasing function $c : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f(x, \eta, \xi)| \leq c(|\eta|)(1 + |\xi|^2), \quad \forall\, (x, \eta, \xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n.$$

(2)

$$\frac{\partial f}{\partial \eta}(x, \eta, \xi) \leq 0.$$

(3) Assume there exists a $M > 0$ such that

$$f(x, \eta, \xi) = \begin{cases} < 0, & \text{if } \eta > M, \\ > 0, & \text{if } \eta < -M. \end{cases}$$

Then the equation possesses a unique solution in $C^{2,\gamma}$. For proof we refer to Theorem 1.2.10 in Kung-Ching Chang's *Methods in Nonlinear Analysis.* □

Problem 9. *Let $\Omega \subset \mathbb{R}^n$ be a bounded regular domain with $n \geq 3$ and $2 < q < \frac{2n}{n-2}$. Suppose $u \in H_0^1(\Omega)$ such that*

$$\int_\Omega \nabla u \cdot \nabla \phi dx = \int_\Omega |u|^{q-2} u \phi dx, \quad \forall \phi \in H_0^1(\Omega).$$

Prove that u is C^2 via the L^p and C^α estimate of $-\triangle$.

Proof. Choosing $\phi = u$ in the integral equation and combing $u \in H_0^1(\Omega)$ yield

$$\int_\Omega |u|^q\, dx = \int_\Omega |\nabla u|^2\, dx < +\infty,$$

i.e. $u \in L^q(\Omega)$. By definition, $u \in H_0^1(\Omega)$ is a weak solution to the Dirichlet problem

$$\begin{cases} -\Delta u = |u|^{q-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with right hand side $|u|^{q-2}u \in L^{\frac{q}{q-1}}(\Omega)$. Since $\frac{q}{q-1} > 1$, we have $u \in W^{2,\frac{q}{q-1}}(\Omega)$ by L^p estimate of elliptic equations. Then by Sobolev imbedding we know $u \in L^{\gamma q}(\Omega)$, where $\gamma = \frac{n}{nq-2q-n} > 1$ ($2 < q < \frac{2n}{n-2}$ used here). This means $|u|^{q-2}u \in L^{\frac{\gamma q}{q-1}}(\Omega)$, then by L^p estimate again we have $u \in W^{2,\frac{\gamma q}{q-1}}(\Omega)$. Also, by Sobolev imbedding we have $u \in L^{\gamma' \gamma q}(\Omega)$, where $\gamma' = \frac{n}{nq-2\gamma q-n}$. Since $\gamma > 1$, it's easy to verify that $\gamma' > \gamma$. Then using Hölder inequality we get $u \in L^{\gamma^2 q}(\Omega)$. Repeating the above way, we can obtain $u \in L^{\gamma^k q}(\Omega)$ for any $k \in \mathbb{N}$. We choose a $k_0 \in \mathbb{N}$ such that $\gamma^{k_0} q > \frac{n}{2}$, hence the L^p estimate and Sobolev embedding yield $u \in C^\alpha(\Omega)$, where $\alpha = 1 - \frac{n}{2\gamma^{k_0} q}$, which implies $|u|^{q-2}u \in C^\alpha(\Omega)$. Finally, by classical Schauder estimate we get $u \in C^{2,\alpha}(\Omega)$. (Actually, we can obtain $u \in C^\infty(\Omega)$.) □

Problem 10. *Let $\Omega \subset \mathbb{R}^n$ be a bounded regular domain and $X = W_0^{1,p}(\Omega)$ with $p > 1$, $f : \Omega \times \mathbb{R}$ be continuous satisfying*

$$|f(x,u)| \le C\left(1 + |u|^\alpha\right), \quad (x,u) \in \Omega \times \mathbb{R},$$

$\alpha < \frac{n+2}{n-2}$, $F(x,u) = \int_0^u f(x,s)ds$.
(1) Prove the functional

$$I(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \int_\Omega F(x, u(x))dx$$

is w.s.l.s.c. in X ;
(2) compute the Euler-Lagrange equation of $I(u)$.

Proof of (1). We refer to Remark 4.3.11 in Kung-Ching Chang's *Methods in Nonlinear Analysis*, and I think there should be $1 < p < n$ and $0 \le \alpha < \frac{np}{n-p} - 1$. □

Solution of (2). Choosing a test function $\varphi \in C_0^\infty(\Omega)$ yields

$$I(u + \varepsilon\varphi) = \frac{1}{p}\int_\Omega |\nabla u + \varepsilon \nabla \varphi|^p \, dx - \int_\Omega F(x, u + \varepsilon\varphi)\, dx.$$

Since u is the local minimizer, we have

$$\begin{aligned} 0 = \frac{d}{d\varepsilon} I(u + \varepsilon\varphi)\Big|_{\varepsilon=0} &= \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \, dx - \int_\Omega f(x,u)\varphi \, dx \\ &= -\int_\Omega \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)\varphi \, dx - \int_\Omega f(x,u)\varphi\, dx. \end{aligned}$$

Hence by the arbitrariness of $\varphi \in C_0^\infty(\Omega)$, we know the Euler-Lagrange equation of $I(u)$ is

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = f(x,u) \quad \text{in } \Omega.$$

□

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