THE HESSIAN OF SUPPORT FUNCTIONS ON ROUND SPHERE

LING WANG

1. Introduction

In this short note, we are going to calculate the Hessian of the support functions in the local coordinate system on the round sphere. Firstly, we define the support function of a convex body, M, as a function on $\mathbb{S}^n$, i.e.

$$H(z) := z \cdot y(z), \quad z \in \mathbb{S}^n,$$

where $y : \mathbb{S}^n \to M$ is inverse Gauss map. Extending this definition to $\mathbb{R}^{n+1} \backslash \{0\}$ by the equation $H(z) = |z| H(z/|z|)$, we obtain a homogeneous function of degree one. It is easy to see that we can recover the coordinate functions of M by the equations

$$y_i = \frac{\partial H}{\partial z_i}, \quad \text{for} \quad i = 1, 2, \cdots, n+1.$$

Since y_i are functions of homogeneity of degree zero, they are completely determined by their values on the hyperplanes $z_i = -1$ for $i = 1, 2, \cdots, n+1$. In the following, we will restrict $H(z)$ to the plane $z_{n+1} = -1$ and compute its Hessian.

In the celebrated work of Cheng-Yau [CY], they claimed that for an orthonormal frame of the sphere $\mathbb{S}^n$, restricting $H(z)$ to $\mathbb{S}^n$ and taking its Hessian $\{H_{ij}\}$ with respect to this frame, we have

$$\begin{aligned} &\left(1+|x|^2\right)^{\frac{n}{2}+1} \det\left(\frac{\partial^2 H}{\partial x_i \partial x_j}\right)(x_1, \cdots, x_n, -1) \\ &= \det\left(H_{ij} + H\delta_{ij}\right)\left(\frac{x_1}{(1+|x|^2)^{1/2}}, \cdots, \frac{x_n}{(1+|x|^2)^{1/2}}, \frac{-1}{(1+|x|^2)^{1/2}}\right). \end{aligned} \tag{1.1}$$

In the following, we will give a detail calculation to (1.1).

2. Calculation

We consider the following local parameterized of S^n:

$$\phi : \mathbb{R}^n \to \mathbb{S}^n,$$

with

$$(x_1, \cdots, x_n) \mapsto \left(\frac{x_1}{(1+|x|^2)^{1/2}}, \cdots, \frac{x_n}{(1+|x|^2)^{1/2}}, \frac{-1}{(1+|x|^2)^{1/2}}\right).$$

This can be seen as the reverse of the stereographic projection at the center of $\mathbb{S}^n$. Note that

$$z_i = \frac{x_i}{(1+|x|^2)^{1/2}}, \quad i = 1, 2, \cdots, n,$$

$$z_{n+1} = \frac{-1}{(1+|x|^2)^{1/2}}.$$

Hence, we have

$$\mathrm{d}z_i = \sum_{j=1}^{n} \frac{(1+|x|^2)\delta_{ij} - x_i x_j}{(1+|x|^2)^{3/2}} \mathrm{d}x_j, \quad i = 1, 2, \cdots, n,$$

$$\mathrm{d}z_{n+1} = \sum_{j=1}^{n} \frac{x_j}{(1+|x|^2)^{3/2}} \mathrm{d}x_j.$$

Then the induced metric on $\mathbb{S}^n$ is

$$\begin{aligned} g &= \sum_{k=1}^{n} (\mathrm{d}z_k)^2 + (\mathrm{d}z_{n+1})^2 \\ &= \sum_{i,j,k=1}^{n} \left(\frac{(1+|x|^2)\delta_{ki} - x_k x_i}{(1+|x|^2)^{3/2}} \mathrm{d}x_i \right) \left(\frac{(1+|x|^2)\delta_{kj} - x_k x_j}{(1+|x|^2)^{3/2}} \mathrm{d}x_j \right) + \sum_{i,j=1}^{n} \frac{x_i x_i}{(1+|x|^2)^3} \mathrm{d}x_i \mathrm{d}x_j \\ &= \sum_{i,j,k=1}^{n} \frac{(1+|x|^2)^2 \delta_{ki}\delta_{kj} - 2(1+|x|^2)\delta_{ki} x_k x_j + x_k^2 x_i x_j}{(1+|x|^2)^3} \mathrm{d}x_i \mathrm{d}x_j + \sum_{i,j=1}^{n} \frac{x_i x_i}{(1+|x|^2)^3} \mathrm{d}x_i \mathrm{d}x_j \\ &= \sum_{i,j=1}^{n} \frac{1}{1+|x|^2} \left(\delta_{ij} - \frac{x_i x_j}{1+|x|^2} \right) \mathrm{d}x_i \mathrm{d}x_j, \end{aligned}$$

i.e.

$$g = \sum_{i,j=1}^{n} \frac{1}{1+|x|^2} \left(\delta_{ij} - \frac{x_i x_j}{1+|x|^2} \right) \mathrm{d}x_i \mathrm{d}x_j. \tag{2.1}$$

Hence, we know the coefficients matrix is

$$g_{ij} = \frac{1}{1+|x|^2} \left(\delta_{ij} - \frac{x_i x_j}{1+|x|^2} \right), \quad i, j = 1, 2, \cdots, n.$$

Denote $\partial_{x_k} := \frac{\partial}{\partial x_k}$. For any $i, j, k \in \{1, 2, \cdots, n\}$, we have

$$\begin{aligned} \partial_{x_k} g_{ij} &= \partial_{x_k} \left(\frac{1}{1+|x|^2} \left(\delta_{ij} - \frac{x_i x_j}{1+|x|^2} \right) \right) \\ &= \frac{-2x_k}{(1+|x|^2)^2} \left(\delta_{ij} - \frac{x_i x_j}{1+|x|^2} \right) + \frac{1}{1+|x|^2} \left(\frac{2x_i x_j x_k}{(1+|x|^2)^2} - \frac{\delta_{ik} x_j}{1+|x|^2} - \frac{\delta_{jk} x_i}{1+|x|^2} \right) \end{aligned}$$

$$= -\frac{2x_k\delta_{ij} + \delta_{ik}x_j + \delta_{jk}x_i}{(1+|x|^2)^2} + \frac{4x_ix_jx_k}{(1+|x|^2)^3}.$$

It is easy to see that the inverse matrix of the coefficients matrix is

$$g^{ij} = (1+|x|^2)(\delta_{ij} + x_ix_j), \quad i,j = 1,2,\cdots,n.$$

Indeed, a direct calculation yields

$$\begin{aligned} g^{ik}g_{kj} &= \sum_{k=1}^{n}(\delta_{ik} + x_ix_k)\left(\delta_{kj} - \frac{x_kx_j}{1+|x|^2}\right) \\ &= \sum_{k=1}^{n}\left(\delta_{ik}\delta_{kj} - \frac{\delta_{ik}x_kx_j}{1+|x|^2} + \delta_{kj}x_ix_k - \frac{x_k^2x_ix_j}{1+|x|^2}\right) \\ &= \delta_{ij} - \frac{x_ix_j}{1+|x|^2} + x_ix_j - \frac{|x|^2x_ix_j}{1+|x|^2} \\ &= \delta_{ij}. \end{aligned}$$

By definition, the Christoffel symbols of g are

$$\begin{aligned} \Gamma_{ij}^k &= \frac{1}{2}g^{kl}\left(\partial_{x_i}g_{lj} + \partial_{x_j}g_{il} - \partial_{x_l}g_{ij}\right) \\ &= \sum_{l=1}^{n}\frac{1}{2}(1+|x|^2)(\delta_{kl} + x_kx_l)\Bigg(- \frac{2x_i\delta_{lj} + \delta_{li}x_j + \delta_{ij}x_l}{(1+|x|^2)^2} + \frac{4x_ix_jx_l}{(1+|x|^2)^3} \\ &\qquad - \frac{2x_j\delta_{il} + \delta_{ij}x_l + \delta_{lj}x_i}{(1+|x|^2)^2} + \frac{4x_ix_jx_l}{(1+|x|^2)^3} \\ &\qquad + \frac{2x_l\delta_{ij} + \delta_{il}x_j + \delta_{jl}x_i}{(1+|x|^2)^2} - \frac{4x_ix_jx_l}{(1+|x|^2)^3}\Bigg) \\ &= \sum_{l=1}^{n}\frac{1}{2}(\delta_{kl} + x_kx_l)\left(\frac{-2x_i\delta_{jl} - 2x_j\delta_{il}}{1+|x|^2} + \frac{4x_ix_jx_l}{(1+|x|^2)^2}\right) \\ &= \frac{-x_i\delta_{jk} - x_j\delta_{ik}}{1+|x|^2} + \frac{2x_ix_jx_k}{(1+|x|^2)^2} + \frac{-2x_ix_jx_k}{1+|x|^2} + \frac{2x_ix_jx_k|x|^2}{(1+|x|^2)^2} \\ &= -\frac{x_i}{1+|x|^2}\delta_{jk} - \frac{x_j}{1+|x|^2}\delta_{ik}. \end{aligned}$$

Then the Levi-Civita connection in the local coordinate is

$$\nabla_{\partial_{x_i}}\partial_{x_j} = \Gamma_{ij}^k\partial_{x_k} = -\frac{x_i}{1+|x|^2}\partial_{x_j} - \frac{x_j}{1+|x|^2}\partial_{x_i}.$$

Next, we denote $u(x_1,\cdots,x_n) := H(x_1,\cdots,x_n,-1)$. We differentiate H on $\mathbb{S}^n$ respect to $\{\partial_{x_1},\cdots,\partial_{x_n}\}$. By the definition of vector fields, we know that

$$H_i := \nabla_{\partial_{x_i}}H = \frac{\partial}{\partial x_i}\left(H\circ\phi(x_1,\cdots,x_n)\right)$$

$$
\begin{aligned}
&= \frac{\partial}{\partial x_i}\left(H\left(\frac{x_1}{(1+|x|^2)^{1/2}},\cdots,\frac{x_n}{(1+|x|^2)^{1/2}},\frac{-1}{(1+|x|^2)^{1/2}}\right)\right)\\
&= \frac{\partial}{\partial x_i}\Big((1+|x|^2)^{-1/2}H(x_1,\cdots,x_n,-1)\Big)\\
&= \frac{\partial}{\partial x_i}\Big((1+|x|^2)^{-1/2}u(x_1,\cdots,x_n)\Big)\\
&= (1+|x|^2)^{-1/2}u_i-(1+|x|^2)^{-3/2}x_iu,
\end{aligned}
$$

where $u_i := \dfrac{\partial u}{\partial x_i}$. Then the Hessian of H is

$$
\begin{aligned}
H_{ij} := \mathrm{Hess}_H(\partial_{x_i},\partial_{x_j}) &= \nabla_{\partial_{x_i}}\nabla_{\partial_{x_j}}H-\left(\nabla_{\partial_{x_i}}\partial_{x_j}\right)H\\
&= \frac{\partial}{\partial x_i}\Big((1+|x|^2)^{-1/2}u_j-(1+|x|^2)^{-3/2}x_ju\Big)\\
&\quad+\frac{x_i}{1+|x|^2}\Big((1+|x|^2)^{-1/2}u_j-(1+|x|^2)^{-3/2}x_ju\Big)\\
&\qquad+\frac{x_j}{1+|x|^2}\Big((1+|x|^2)^{-1/2}u_i-(1+|x|^2)^{-3/2}x_iu\Big)\\
&= (1+|x|^2)^{-1/2}u_{ij}-(1+|x|^2)^{-3/2}x_iu_j-(1+|x|^2)^{-3/2}x_ju_i\\
&\quad+3(1+|x|^2)^{-5/2}x_ix_ju-(1+|x|^2)^{-3/2}u\delta_{ij}\\
&\qquad+(1+|x|^2)^{-3/2}x_iu_j-(1+|x|^2)^{-5/2}x_ix_ju\\
&\qquad+(1+|x|^2)^{-3/2}x_ju_i-(1+|x|^2)^{-5/2}x_ix_ju\\
&= (1+|x|^2)^{-1/2}u_{ij}+(1+|x|^2)^{-5/2}x_ix_ju-(1+|x|^2)^{-3/2}u\delta_{ij},
\end{aligned}
$$

where $u_{ij} := \dfrac{\partial^2 u}{\partial x_i\partial x_j}$. Note that the Riemann metric (2.1), we have

$$
\begin{aligned}
H_{ij}+Hg_{ij} &= (1+|x|^2)^{-1/2}u_{ij}+(1+|x|^2)^{-5/2}x_ix_ju-(1+|x|^2)^{-3/2}u\delta_{ij}\\
&\quad+(1+|x|^2)^{-1/2}u\cdot(1+|x|^2)^{-1}\left(\delta_{ij}-\frac{x_ix_j}{1+|x|^2}\right)\\
&= (1+|x|^2)^{-1/2}u_{ij}.
\end{aligned}
$$

Hence, there is

$$\det(H_{ij}+Hg_{ij}) = (1+|x|^2)^{-n/2}\det D^2u.$$

Let $\{e_1,\cdots,e_n\}$ be an orthonormal frame obtained by $\{\partial_{x_1},\cdots,\partial_{x_n}\}$, and note that

$$
\begin{aligned}
\det(g_{ij}) &= \det\left(\frac{1}{1+|x|^2}\left(\delta_{ij}-\frac{x_ix_j}{1+|x|^2}\right)\right)\\
&= (1+|x|^2)^{-n}\det\left(\delta_{ij}-\frac{x_ix_j}{1+|x|^2}\right)
\end{aligned}
$$

$$= (1+|x|^2)^{-n}\left(1 - \frac{|x|^2}{1+|x|^2}\right) = (1+|x|^2)^{-n-1},$$

where we used the fact that $\det(I + aa^T) = 1 + a^T a$. Hence the Hessian of H respect to $\{e_1, \cdots, e_n\}$ satisfies

$$\det(H_{ij} + H\delta_{ij}) = \det(g_{ij})^{-1} \det(H_{ij} + Hg_{ij}) = (1+|x|^2)^{n/2+1} \det D^2 u,$$

which is just (1.1).

References

[CY] Cheng, S.Y.; Yau, S.T.: On the regularity of the solution of the n-dimensional Minkowski problem. *Comm. Pure Appl. Math.* **29** (1976), no. 5, 495–516.

School of Mathematical Sciences, Peking University, Beijing 100871, China.
Email address: `lingwang@stu.pku.edu.cn`