$C^{1,\alpha}$ REGULARITY OF VARIATIONAL PROBLEMS WITH A CONVEXITY CONSTRAINT

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ABSTRACT. In this paper, we establish the interior $C^{1,\alpha}$ regularity of minimizers of a class of functionals with a convexity constraint, which includes the principal-agent problems studied by Figalli-Kim-McCann (*J. Econom. Theory* **146** (2011), no. 2, 454-478). The $C^{1,1}$ regularity was previously proved by Caffarelli-Lions in an unpublished note when the cost is quadratic, and recently extended to the case where the cost is uniformly convex with respect to a general preference function by McCann-Rankin-Zhang(*arXiv:2303.04937v3*). Our main result does not require the uniform convexity assumption on the cost function. In particular, we show that the solutions to the principal-agent problems with q-power cost are $C^{1,\frac{1}{q-1}}$ when $q > 2$ and $C^{1,1}$ when $1 < q \leq 2$. Examples can show that this regularity is optimal when $q \geq 2$.

1. INTRODUCTION

In this paper, we will investigate the regularity of minimizers of the functional

$$\int_X F(x, u, Du) \, \mathrm{d}x, \tag{1.1}$$

over the set of b-convex functions, where X is a bounded, smooth domain in $\mathbb{R}^n$, $F(x, z, \mathbf{p}) : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a smooth function that is convex in each of the variables $z \in \mathbb{R}$ and $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{R}^n$. Here b-convex functions refer to admissible functions with respect to a function $b(x, y)$(see Definition 1.1).

Unlike the unconstrained case, the regularity of (1.1) is very subtle, since the typical techniques in calculus of variations and partial differential equations are no longer applicable. Indeed, due to the convexity constraint, it is generally challenging to write down a tractable Euler-Lagrange equation for the minimizers of (1.1) [2, 4, 14]. There are some efforts on constructing approximations of the minimizers satisfying explicit equations for practical purposes [5, 11, 13, 12], but it is still difficult to obtain the regularity of the minimizers of (1.1) for general $F(x, z, \mathbf{p})$.

A typical example of (1.1) arises from the principle-agent problems in economics. Principal-agent problems are a class of economic models with applications in tax policy,

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2020 *Mathematics Subject Classification.* 35B65, 49N60.

This research is partially supported by National Key R&D Program of China SQ2020YFA0712800, 2023YFA009900 and NSFC Grant 11822101.

regulation of public utilities, product line design, and contract theory [9]. We give a brief introduction as follows.

A monopolist wants to assign the prices of products to gain the maximal profit. Denote by X, $Y \subset \mathbb{R}$ the sets of buyers and products, respectively. Let $c(y)$ be the cost of the product of $y \in Y$ and $b(x, y)$ be function that measures the preference of the buyer $x \in X$ to $y \in Y$. Let $\overline{X}$ be the closure of $X \subset \mathbb{R}^n$. In order to investigate the strategy of pricing products to maximize the profit, Figalli, Kim and McCann [9] introduced the following conditions for each fixed $(x_0, y_0) \in \overline{X} \times \overline{Y}$(See also [6, 7, 16]):

(B0) $b \in C^4(\overline{X} \times \overline{Y})$, where $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^n$ are open and bounded;

(B1) (bi-twist) both $x \in X \mapsto D_y b(x, y_0)$ and $y \in Y \mapsto D_x b(x_0, y)$ are diffeomorphisms onto their ranges;

(B2) (bi-convexity) both $X_{y_0} := D_y b(X, y_0)$ and $Y_{x_0} := D_x b(x_0, Y)$ are convex subsets of $\mathbb{R}^n$.

(B3) (non-negative cross-curvature)

$$\frac{\partial^4}{\partial s^2 \partial t^2}\bigg|_{(s,t)=(0,0)} b(x(s), y(t)) \geq 0$$

whenever either of the two curves $s \in [-1, 1] \mapsto D_y b(x(s), y(0))$ and $t \in [-1, 1] \mapsto D_x b(x(0), y(t))$ forms an affinely parameterized line segment (in $\overline{X}_{y_0}$, or in $\overline{Y}_{x_0}$, respectively).

Now we consider the utility function

$$u(x) := \sup_{y \in Y}\{b(x, y) - v(y)\}, \tag{1.2}$$

instead of the price function $v : Y \to \mathbb{R}$. To formulate the profit functional and admissible functions, we need the definitions of b-convexity and b-exponential map.

Definition 1.1 (b-convexity)**.** *A function $u : X \to \mathbb{R}$ is called b-convex if $u = (u^{b^*})^b$, where*

$$u^b(x) = \sup_{y \in \overline{Y}}\{b(x, y) - u(y)\}, \quad \text{and} \quad u^{b^*}(y) = \sup_{x \in \overline{X}}\{b(x, y) - u(x)\}.$$

Definition 1.2 (b-exponential map)**.** *For each $\mathbf{p} \in \overline{Y}_x$ we define $y_b(x, \mathbf{p})$ as the unique solution to*

$$D_x b(x, y_b(x, \mathbf{p})) = \mathbf{p},$$

where the uniqueness is guaranteed by **(B1)**.

Remark 1.3. *For the classical convexity, i.e. $b(x, y) = x \cdot y$, it is easy to see that $y_b(x, \mathbf{p}) = \mathbf{p}$.*

By **(B1)**, $u(x) = b(x, y_b(x, Du(x))) - v(y_b(x, Du(x)))$ for any differentiable point x of u. Then the monopolist's profit is $-L(u)$, where

$$L(u) = \int_X \left[c(y_b(x, Du(x))) - b(x, y_b(x, Du(x))) + u\right] \eta_0(x)\ \mathrm{d}x. \tag{1.3}$$

Here η_0 is the nonnegative relative frequency of buyers in the population. Equivalently, the principal-agent problem is to minimization problem

$$\min_{u\in U_0} L(u), \tag{1.4}$$

where the admissible set

$$U_0 := \{u : X \to \mathbb{R} \,|\, u \text{ is } b\text{-convex, } u(x) \geq a_0 + b(x, y_0)\},$$

for a constant a_0, and a constant vector $y_0 \in Y$ from the assumption of "null" product.

In a special case when $b(x, y) = x \cdot y$ and the cost $c(y)$ is a quadratic function $\frac{|y|^2}{2}$, it reduces to the famous Rochet-Choné model [17], which corresponds to (1.1) with

$$F(x, z, \mathbf{p}) = \big(|\mathbf{p}|^2/2 - x \cdot \mathbf{p} + z\big)\, \eta_0(x). \tag{1.5}$$

In this case, the C^1 regularity of the minimizer was proved by Carlier and Lachand-Robert [3]. Later, the interior $C^{1,1}$ regularity result was derived by Caffarelli and Lions through a very elegant argument in an unpublished note [1] (see [16, Theorem 6] for a restatement). Very recently, under the assumption of uniformly convexity of the cost function, the C^1 and $C^{1,1}$ regularities for general $b(x, y)$ were extended by Chen [6, 7], and McCann, Rankin and Zhang [16], respectively. The main technique in both results is still from Caffarelli and Lions [1], while the uniform convexity of the cost plays an important role in the proofs.

In this paper, we are concerned with functionals of more general form

$$L(u) := \int_X \big[F^1(x, y_b(x, Du(x))) + F^0(x, u(x))\big]\, \mathrm{d}x, \tag{1.6}$$

with certain conditions on $F^1(x, \mathbf{p})$ and $F^0(x, z)$. One of the main purposes is to relax the uniform convexity assumption of the cost function and to include the Rochet-Choné model with of q-power cost($q > 1$), where

$$F(x, z, \mathbf{p}) = (|\mathbf{p}|^q/q - x \cdot \mathbf{p} + z)\, \eta_0(x). \tag{1.7}$$

See [17, P790]. To make our results more general, we make the following assumptions:

(H1) Fixed $q > 1$, there exists a $\delta > 0$ such that

$$G(x, \mathbf{p}) := F^1(x, y_b(x, \mathbf{p})) - \delta|\mathbf{p}|^q$$

is convex respect to $\mathbf{p}$;

(H2) There exists $M > 0$, such that $|D_z F^0(x, z)| \leq M$ for all $x \in X$ and $z \in \mathbb{R}$;

(H3) There exists $C_0 > 0$, such that

$$|D_{\mathbf{p}} G(x, \mathbf{p})| \leq C_0(|y_b(x, \mathbf{p})|^{q-1} + 1),\ |D_{x_i p_i} G(x, \mathbf{p})| \leq C_0(|y_b(x, \mathbf{p})|^{q-1} + 1)$$

for all $x \in X$, $\mathbf{p} \in \overline{Y}_x$, and for each i.

The main theorem is stated as follows:

Theorem 1.4. *Assume $b(x, y)$ satisfies* **(B0)**-**(B3)**. *Suppose $F^1(x, \mathbf{p})$ and $F^0(x, z)$ satisfy* **(H1)**-**(H3)**. *Let u be a minimizer of the functional* (1.6)*. Then $u \in C^{1,\alpha}_{loc}(X)$, where $\alpha = \frac{1}{q-1}$ for $q > 2$ and $\alpha = 1$ for $1 < q \leq 2$.*

Remark 1.5. *When $q = 2$, we recover the results derived in* [1, 16]. *When $q \geq 2$, examples can show that the regularity is optimal(Section 4).*

The basic idea for the proof of Theorem 1.4 is a perturbation argument used in the unpublished note [1], and there are some extensions in the recent paper [16]. In [16], the authors simplified Caffarelli and Lions's argument and extended it to the principal-agent problem by combining some results and methods from the optimal transport literature. Therefore, we will follow a similar framework as in the proof of [16] to give the proof of Theorem 1.4, using a key lemma(Lemma 2.1).

The main new idea in this paper is that we are more meticulous in the choice of the comparison function with more delicate analysis. Specifically, in [16], the authors perturb a support function of the minimizer to make the geometry of the section controllable, but the magnitude of the perturbation is not a concern for them. However, in our proof we need to introduce a family of perturbations (2.5) to a support function of the minimizer in order to refine the section geometry. The small size of the perturbation is a crucial step in our proof (see the proof of Lemma 2.1 for details). This comparison function allows us to handle more general functionals that may be degenerate. Precisely, we can make assumptions **(H1)**-**(H3)** that cover all q-power cost ($1 < q < \infty$) functions in the Rochet-Choné models, while the assumption of certain types of uniform "convexity" is required in both [1] and [16].

The organization of the paper is organized as follows. First, we will state a crucial technical lemma and give its proof in Section 2. Then, we use this key lemma to prove Theorem 1.4 in Section 3. Finally, an example is provided in Section 4 to demonstrate the optimal regularities of the minimizer of (1.6) when $q \geq 2$.

2. A crucial technical lemma for proving Theorem 1.4

The main technique used to prove the main theorem is the following lemma analogous to [1, 16]. In this section, we show it holds under the conditions in Theorem 1.4.

Lemma 2.1. *Assume $b(x, y)$ satisfies* **(B0)**-**(B3)**, *$F^1(x, \mathbf{p})$ and $F^0(x, z)$ satisfy* **(H1)**-**(H3)**. *Let $X' \subset\subset X$ and $d = \operatorname{dist}(X', \partial X)$. Then there exist $r_0 > 0$ and constants C_1, $C_2 > 0$ depending only on b, d, q, δ, C_0, and M such that the following property holds: If $u : X \to \mathbb{R}$ is b-convex and $x_0 \in X'$, $y_0 \in y_b(x_0, Du(x_0))$, then for any $r < r_0$ and*

$$h := \sup_{B_r(x_0)} \{u(x) - (b(x, y_0) - b(x_0, y_0) + u(x_0))\} > 0,$$

there is a b-affine function $p_y(x) = b(x, y) + a$ such that

(1) *The section $S := \{x \in X \mid u(x) < p_y(x)\}$ has positive measure.*
(2) *On S, we have*

$$\sup_{x \in S}\{p_y(x) - u(x)\} \leq h. \tag{2.1}$$

(3) *There holds*

$$\frac{1}{|S|}\int_S \left(F^1(x,y) - F^1(x, y_b(x, Du(x)))\right) \mathrm{d}x \leq C_1 h - C_2 \frac{h^q}{r^q}. \tag{2.2}$$

Proof. (1) For simplicity, we assume that $x_0 = 0$, $y_0 = 0$ and $u(0) = 0$. Otherwise, we apply the following transformations as in [16]

$$\begin{aligned} \tilde{u}(\tilde{x}) \;\; &= u(x) - [u(x_0) + b(x, y_0) - b(x_0, y_0)], \\ \tilde{b}(\tilde{x}, \tilde{y}) &= b(x,y) - [b(x_0, y) + b(x, y_0) - b(x_0, y_0)] \end{aligned}$$

for $x = x(\tilde{x})$ and $y = y(\tilde{y})$, where

$$\begin{aligned} \tilde{x}(x) &:= b_y(x, y_0) - b_y(x_0, y_0), \\ \tilde{y}(y) &:= b_x(x_0, y) - b_x(x_0, y_0). \end{aligned}$$

By [16, Lemma 7], we know that $\tilde{u}$ is convex, $\tilde{u}(0) = 0$ and

$$\tilde{b}(\tilde{x}, \tilde{y}) = \tilde{x} \cdot \tilde{y} + a_{ij,kl} \tilde{x}^i \tilde{x}^j \tilde{y}^k \tilde{y}^l \tag{2.3}$$

for smooth functions $a_{ij,kl}$ on $\overline{X} \times \overline{Y}$. We will continue to use the notations x, y, u, and b for the sake of brevity. Furthermore, we can assume that u is convex and b satisfies (2.3).

Without loss of generality, we assume that

$$h = \sup_{B_r} u.$$

It is clear that u attains its maximum over B_r at some point $re_1 \in \partial B_r$, and its tangential derivatives equal 0 at re_1. Then by the convexity of u, we have $Du(re_1) = \kappa e_1$ for some $\kappa \geq h/r$. Here $\kappa \leq \|b\|_{C^1}$. Since the gradient of the b-support of u at re_1 agrees with the gradient of u, we have $y_b(re_1, \kappa e_1) = \kappa e_1$. Note that $y_b(re_1, 0) = 0$. We denote

$$y_\varepsilon := y_b\left(re_1, \frac{\varepsilon h}{r} e_1\right)$$

for some $\varepsilon \in (0,1)$ to be determined later. By the definition of y_b (Definition 1.2), we know that

$$b_x(re_1, y_\varepsilon) = \frac{\varepsilon h}{r} e_1. \tag{2.4}$$

Hence, by (2.3) and **(B1)** we have that

$$\left| y_\varepsilon - \frac{\varepsilon h}{r} e_1 \right| \leq Cr \frac{(\varepsilon h)^2}{r^2}.$$

That is, for sufficiently small h, it holds $|y_\varepsilon| \leq C\frac{\varepsilon h}{r}$. Combining this with (2.3) and $h \leq r\|b\|_{C^1}$, we can show that

$$|b_x(x, y_\varepsilon)| \leq C \frac{\varepsilon h}{r}.$$

Now we choose the b-affine function

$$p_{y_\varepsilon}(x) = b(x, y_\varepsilon) - b(re_1, y_\varepsilon) + u(re_1), \tag{2.5}$$

with

$$|Dp_{y_\varepsilon}(x)| = |b_x(x, y_\varepsilon)| \le C\frac{\varepsilon h}{r}. \tag{2.6}$$

Note that

$$\begin{aligned} p_{y_\varepsilon}(0) - u(0) &\ge -Cre_1 \cdot \frac{\varepsilon h}{r} e_1 + h - Cr^2|y_\varepsilon|^2 \\ &\ge h - C\varepsilon h - C'\varepsilon^2 h^2 > 0 \end{aligned} \tag{2.7}$$

for sufficiently small ε. (2.7) implies that the section $S := \{x \in X \,|\, u(x) < p_{y_\varepsilon}(x)\}$ has positive measure. Hence, (1) is proved.

(2) By Loeper's maximum principle [15, Theorem 3.2], we have

$$b(x, y_\varepsilon) - b(re_1, y_\varepsilon) \le \max\{0, b(x, \kappa e_1) - b(re_1, \kappa e_1)\} \le u(x), \quad x \in S.$$

Hence

$$p_{y_\varepsilon}(x) - u(x) = b(x, y_\varepsilon) - b(re_1, y_\varepsilon) + h - u(x) \le h, \quad x \in S,$$

which yields (2.1).

(3) First, by a further transformation

$$\tilde{x}(x) := b_y(x, y_\varepsilon), \quad \overline{u}(\tilde{x}) := u(x), \quad \overline{p}_{y_\varepsilon}(\tilde{x}) := p_{y_\varepsilon}(x),$$

we can assume that S is convex.

Next, we show that

$$S \subset \{x \in \mathbb{R}^n \,|\, -\bar{C}\varepsilon^{-1} r \le x_1 \le \bar{C} r\}. \tag{2.8}$$

Indeed, by the proof of Lemma 9 in [16, P12-P13], we already have $S \subset \{x \,|\, x_1 \le \bar{C}r\}$. Then it suffices to show $S \subset \{x \,|\, x_1 \ge -\bar{C}\varepsilon^{-1}r\}$. Using (2.3) and (2.4), we know that

$$\frac{\varepsilon h}{r} = b_x(re_1, y_\varepsilon) \cdot e_1 \le y_\varepsilon \cdot e_1 + Cr|y_\varepsilon|^2,$$

i.e.

$$y_\varepsilon \cdot e_1 \ge \frac{\varepsilon h}{r} - Cr|y_\varepsilon|^2. \tag{2.9}$$

Combining (2.3) and (2.9) then gives

$$\begin{aligned} p_{y_\varepsilon}((1 - 2\varepsilon^{-1})re_1) &= b((1 - 2\varepsilon^{-1})re_1, y_\varepsilon) - b(re_1, y_\varepsilon) + u(re_1) \\ &\le (1 - 2\varepsilon^{-1})re_1 \cdot y_\varepsilon - re_1 \cdot y_\varepsilon + Cr^2|y_\varepsilon|^2 + u(re_1) \\ &= -2\varepsilon^{-1}re_1 \cdot y_\varepsilon + Cr^2|y_\varepsilon|^2 + u(re_1) \\ &\le -2h + C\varepsilon^{-1}r^2|y_\varepsilon|^2 + Cr^2|y_\varepsilon|^2 + h \end{aligned}$$

$$\le -2h + h + Ch^2 < 0$$

for sufficiently small h, which implies that $\{x \in \mathbb{R}^n \,|\, p_{y_\varepsilon}(x) \ge 0\}$ has a boundary point te_1 for some $t \in ((1-2\varepsilon^{-1})r, 0)$. Note that by (2.4),

$$\begin{aligned} D_x p_{y_\varepsilon}(te_1) = b_x(te_1, y_\varepsilon) &= b_x(te_1, y_\varepsilon) - b_x(re_1, y_\varepsilon) + b_x(re_1, y_\varepsilon) \\ &= b_{xx}(\xi, y_\varepsilon) \cdot (te_1 - re_1) + \frac{\varepsilon h}{r} e_1 \\ &= O(\varepsilon^{-1} r) + \frac{\varepsilon h}{r} e_1. \end{aligned}$$

So we know that the outer normal $D_x p_{y_\varepsilon}(te_1)$ makes an angle with the negative axis e_1, say θ, which satisfies $\sin\theta \le C\varepsilon^{-1} r$. This means that

$$\begin{aligned} \{x \in \mathbb{R}^n \,|\, 0 \le p_{y_\varepsilon}(x)\} &\subset \{x \in \mathbb{R}^n \,|\, x_1 \ge (1-2\varepsilon^{-1})r - C\varepsilon^{-1} r \cdot \mathrm{diam}(X)\} \\ &\subset \{x \in \mathbb{R}^n \,|\, x_1 \ge -\bar{C}\varepsilon^{-1} r\}. \end{aligned}$$

Therefore, we have

$$S \subset \{x \in \mathbb{R}^n \,|\, 0 \le p_{y_\varepsilon}(x)\} \subset \{x \in \mathbb{R}^n \,|\, x_1 \ge -\bar{C}\varepsilon^{-1} r\}.$$

Now we are ready to prove (2.2). Let $G(x, \mathbf{p}) = F^1(x, y_b(x, \mathbf{p})) - \delta|\mathbf{p}|^q$. Then we have

$$\begin{aligned} &\int_S \big(F^1(x, y_b(x, Dp_{y_\varepsilon})) - F^1(x, y_b(x, Du))\big)\, \mathrm{d}x \\ &= \delta \int_S \left(|Dp_{y_\varepsilon}|^q - |Du|^q\right) \mathrm{d}x + \int_S (G(x, Dp_{y_\varepsilon}) - G(x, Du))\, \mathrm{d}x. \end{aligned} \tag{2.10}$$

By **(H1)**, we know that $G(x, \mathbf{p})$ is convex respect to $\mathbf{p}$. Hence, we have

$$G(x, Du) \ge G(x, Dp_{y_\varepsilon}) + D_{\mathbf{p}} G(x, Dp_{y_\varepsilon}) \cdot (Du - Dp_{y_\varepsilon}).$$

By an elementary inequality[1], we obtain

$$|Du|^q \ge \frac{1}{2^{q-1}} |Du - Dp_{y_\varepsilon}|^q - |Dp_{y_\varepsilon}|^q.$$

Combining these inequalities with (2.10), we conclude that

$$\begin{aligned} &\int_S \big(F^1(x, y_b(x, Dp_{y_\varepsilon})) - F^1(x, y_b(x, Du))\big)\, \mathrm{d}x \\ &\quad \le \delta \int_S \left(2|Dp_{y_\varepsilon}|^q - \frac{1}{2^{q-1}} |Du - Dp_{y_\varepsilon}|^q\right) \mathrm{d}x + \int_S D_{\mathbf{p}} G(x, Dp_{y_\varepsilon}) \cdot (Dp_{y_\varepsilon} - Du)\, \mathrm{d}x. \end{aligned}$$

Hence, to prove (2.2), it suffices to estimate

$$\int_S |Du - Dp_{y_\varepsilon}|^q\, \mathrm{d}x \tag{2.11}$$

[1] $|\xi + \eta|^q \ge \frac{1}{2^{q-1}} |\xi|^q - |\eta|^q$ for $\xi, \eta \in \mathbb{R}^n$.

and

$$\int_S D_{\mathbf{p}}G(x, Dp_{y_\varepsilon}) \cdot (Dp_{y_\varepsilon} - Du) \, \mathrm{d}x. \tag{2.12}$$

We first estimate the second term (2.12). By the divergence theorem, we have

$$\int_S D_{\mathbf{p}}G(x, Dp_{y_\varepsilon}) \cdot (Dp_{y_\varepsilon} - Du) \, \mathrm{d}x = \int_S D_{\mathbf{p}}G(x, Dp_{y_\varepsilon}) \cdot D_x(p_{y_\varepsilon} - u) \, \mathrm{d}x$$

$$= \int_{\partial S \cap \partial\Omega} (p_{y_\varepsilon} - u) D_{\mathbf{p}}G(x, Dp_{y_\varepsilon}) \cdot \mathbf{n} \, \mathrm{d}S - \int_S (p_{y_\varepsilon} - u) \mathrm{div}_x \left(D_{\mathbf{p}}G(x, Dp_{y_\varepsilon}) \right) \, \mathrm{d}x, \tag{2.13}$$

where $\mathbf{n}$ is the unit outer normal vector. By **(H2)** and **(H3)**, we know

$$\begin{aligned} |D_{\mathbf{p}}G(x, Dp_{y_\varepsilon})| &\le C_0 \, |y_b(x, Dp_{y_\varepsilon})|^{q-1} + C_0 \\ &= C_0 |y_\varepsilon|^{q-1} + C_0, \end{aligned}$$

and

$$|\mathrm{div}_x (D_{\mathbf{p}}G(x, y_\varepsilon))| \le C_0 \, |y_b(x, Dp_{y_\varepsilon})|^{q-1} + C_0 = C_0 |y_\varepsilon|^{q-1} + C_0,$$

where we have used the fact $y_b(x, Dp_{y_\varepsilon}) = y_\varepsilon$. Indeed, by the definition of y_b and $Dp_{y_\varepsilon}(x) = b_x(x, y_\varepsilon)$, we have

$$b_x(x, y_b(x, Dp_{y_\varepsilon})) = Dp_{y_\varepsilon} = b_x(x, y_\varepsilon).$$

Then by **(B1)**, we obtain $y_b(x, Dp_{y_\varepsilon}) = y_\varepsilon$. By $|y_\varepsilon| \le C\frac{\varepsilon h}{r}$, (2.1) and (2.13), we have

$$\int_S D_{\mathbf{p}}G(x, Dp_{y_\varepsilon}) \cdot (Dp_{y_\varepsilon} - Du) \, \mathrm{d}x \le Ch|S| + C\frac{h^q}{r^{q-1}}|S|, \tag{2.14}$$

where we used the estimate $|\partial S \cap \partial\Omega| \le C|S|$, which was proved by Carlier and Lachand-Robert [3], and Chen [6, P82].

Next, we estimate (2.11). For $x = (x_1, x')$, we let $P(x) := (0, x')$ be its projection onto $\{x \in \mathbb{R}^n \,|\, x_1 = 0\}$. Denote $\frac{1}{K}S$ as the dilation of S by a factor $\frac{1}{K}$ with respect to the origin. Choose $K = 2\,\mathrm{diam}(\Omega)/d$, where $d = \mathrm{dist}(\Omega', \partial\Omega)$. Hence, $P\left(\frac{1}{K}S\right) + \frac{d}{2}e_1 \subset \mathrm{int}\,\Omega$. Choose r_0 sufficiently small depending on d such that $S \subset \left\{x : x_1 \le \frac{d}{2}\right\}$. For $(0, x') \in P\left(\frac{1}{K}S\right)$, we let $l_{x'}$ be the line segment with greater x_1 component of the set $(P^{-1}\left(\frac{1}{K}S\right) \cap S) \backslash \left(\frac{1}{K}S\right)$ and write

$$l_{x'} = [a_{x'}, b_{x'}] \times \{x'\}, \text{ where } b_{x'} > a_{x'}.$$

Then the point $(b_{x'}, x')$ satisfies $b_{x'} \le \frac{d}{2}$. Hence, $(b_{x'}, x') \in \partial S \cap \mathrm{int}\,\Omega$.

By (2.7) and $u(Ka_{x'}, Kx') - p_{y_\varepsilon}(Ka_{x'}, Kx') \le 0$, we use the convexity of $u - p_{y_\varepsilon}$ to obtain

$$u(a_{x'}, x') - p_{y_\varepsilon}(a_{x'}, x') \le -\frac{K-1}{K}\left(\frac{3}{4} - C\varepsilon\right) h.$$

Since $(b_{x'}, x') \in \partial S \cap \mathrm{int}\,\Omega$, it is clear that

$$u(b_{x'}, x') - p_{y_\varepsilon}(b_{x'}, x') = 0.$$

Then by Jensen's inequality, we have

$$\begin{aligned}
&\int_{a_{x'}}^{b_{x'}} |D_{x_1}u(t,x') - D_{x_1}p_{y_\varepsilon}(t,x')|^q \, \mathrm{d}t \\
&\qquad \geq \frac{1}{{d_{x'}}^{q-1}} \left(\int_{a_{x'}}^{b_{x'}} D_{x_1}u(t,x') - D_{x_1}p_{y_\varepsilon}(t,x') \, \mathrm{d}t \right)^q \\
&\qquad \geq \frac{1}{{d_{x'}}^{q-1}} \left[\frac{K-1}{K} \left(\frac{3}{4} - C\varepsilon \right) h \right]^q \\
&\qquad \geq \left[\frac{K-1}{K} \left(\frac{3}{4} - C\varepsilon \right) \right]^q \frac{h^q}{(Cr)^{q-1}},
\end{aligned}$$

where $d_{x'} = b_{x'} - a_{x'} \leq Cr$. Hence, it holds

$$\begin{aligned}
\int_S |Du - Dp_{y_\varepsilon}|^q \, \mathrm{d}x \geq & \int_{\frac{1}{K}S} |Du - Dp_{y_\varepsilon}|^q \, \mathrm{d}x \\
= & \int_{P(\frac{1}{K}S)} \int_{a_{x'}}^{b_{x'}} |D_{x_1}u(t,x') - D_{x_1}p_{y_\varepsilon}(t,x')|^q \, \mathrm{d}t \, \mathrm{d}x' \\
\geq & \int_{P(\frac{1}{K}S)} \left[\frac{K-1}{K} \left(\frac{3}{4} - C\varepsilon \right) \right]^q \frac{h^q}{(Cr)^{q-1}} \, \mathrm{d}x' \\
= & \left[\frac{K-1}{K} \left(\frac{3}{4} - C\varepsilon \right) \right]^q \frac{h^q}{(Cr)^{q-1}} \left| P\left(\frac{1}{K} S \right) \right| \\
= & \frac{(K-1)^q \left(\frac{3}{4} - C\varepsilon \right)^q}{K^{q+n-1}} \frac{h^q}{(Cr)^{q-1}} |P(S)|.
\end{aligned}$$

By (2.8), we have

$$\bar{C}(1 + \varepsilon^{-1}) r |P(S)| \geq |S|.$$

Therefore, we obtain

$$\int_S |Du - Dp_{y_\varepsilon}|^q \, \mathrm{d}x \geq \frac{(K-1)^q \left(\frac{3}{4} - C\varepsilon \right)^q}{C^{q-1} K^{q+n-1} \bar{C}(\varepsilon + 1)} \frac{\varepsilon h^q}{r^q} |S|. \tag{2.15}$$

Substituting (2.6), (2.14) and (2.15) into (2.10), we have

$$\begin{aligned}
&\frac{1}{|S|} \int_S \left(F^1(x, y_b(x, Dp_{y_\varepsilon})) - F^1(x, y_b(x, Du)) \right) \mathrm{d}x \\
&\quad = \frac{\delta}{|S|} \int_S (|Dp_{y_\varepsilon}|^q - |Du|^q) \, \mathrm{d}x + \frac{1}{|S|} \int_S (G(x, Dp_{y_\varepsilon}) - G(x, Du)) \, \mathrm{d}x \\
&\quad \leq \frac{\delta}{|S|} \int_S \left(2|Dp_{y_\varepsilon}|^q - \frac{1}{2^{q-1}} |Du - Dp_{y_\varepsilon}|^q \right) \, \mathrm{d}x + \frac{1}{|S|} \int_S (G(x, Dp_{y_\varepsilon}) - G(x, Du)) \, \mathrm{d}x
\end{aligned}$$

$$\leq 2C\delta\left(\frac{\varepsilon h}{r}\right)^q - \frac{\delta(K-1)^q\left(\frac{3}{4}-C\varepsilon\right)^q}{(2C)^{q-1}K^{q+n-1}\bar{C}(\varepsilon+1)}\frac{\varepsilon h^q}{r^q} + Ch + C\frac{h^q}{r^{q-1}}$$
$$= \left(2C\varepsilon^{q-1} - \frac{(K-1)^q\left(\frac{3}{4}-C\varepsilon\right)^q}{(2C)^{q-1}K^{q+n-1}\bar{C}(\varepsilon+1)}\right)\delta\varepsilon\,\frac{h^q}{r^q} + Ch + C\frac{h^q}{r^{q-1}}.$$

We choose $\varepsilon > 0$ sufficiently small such that

$$2C\varepsilon^{q-1} - \frac{(K-1)^q\left(\frac{3}{4}-C\varepsilon\right)^q}{(2C)^{q-1}K^{q+n-1}\bar{C}(\varepsilon+1)} \leq -\widetilde{C}$$

for some constant $\widetilde{C} > 0$ depending only on b, d and q. Then we choose r sufficiently small such that $Cr \leq \frac{1}{2}\widetilde{C}\delta\varepsilon$. In conclusion, there exist $C_1, C_2 > 0$ depending on b, d, q, δ, C_0 and M such that

$$\frac{1}{|S|}\int_S \left(F^1(x, Dp_{y_\varepsilon}) - F^1(x, Du(x))\right) \mathrm{d}x \leq C_1 h - C_2\frac{h^q}{r^q},$$

i.e. (2.2) holds. □

Remark 2.2. *From the proof of Lemma 2.1, we can see that C_1 is identical 0 if $G(x,\mathbf{p}) \equiv 0$, i.e., $F^1(x, y_b(x,\mathbf{p})) = \delta|\mathbf{p}|^q$. Hence,* (2.2) *becomes*

$$\frac{1}{|S|}\int_S (|y|^q - |Du(x)|^q)\,\mathrm{d}x \leq -C_2\frac{h^q}{r^q}. \tag{2.16}$$

In the rest of this section we assume that u is convex in the classical sense, i.e. $b(x,y) = x\cdot y$. From Remark 2.2, we can obtain an interesting corollary of Lemma 2.1, which is also mentioned in [1] for the case $q = 2$.

Corollary 2.3. *Let $F^1(x,\mathbf{p}) = |\mathbf{p}|^q$ and $F^0(x,z) = f(x)z$ in* (1.6)*. Let u be a convex minimizer of* (1.6)*. Then a non-trivial section of u cannot be contained in the region where $f \leq 0$. In particular, u is a ruled surface in the region where $f \leq 0$.*

Before presenting the proof, we first review the definition of extreme points [10, 18]. Let Ω be a bounded convex domain in $\mathbb{R}^n$, $n \geq 2$. A boundary point $z \in \partial\Omega$ is an *extreme point* of Ω if there exists a hyperplane L such that $\{z\} = L \cap \partial\Omega$, namely z is the unique point in $L \cap \partial\Omega$. It is known that any interior point of Ω can be expressed as a linear combination of extreme points of Ω.

Proof of Corollary 2.3. We prove this corollary by contradiction. Suppose that there exists a non-trivial section of u is contained in $\{x \in X : f(x) \leq 0\}$. Then there exists a point x' such that the contact set

$$\mathcal{T}_{x'} := \{x \in X : u(x) = u(x') + Du(x')\cdot(x - x')\}$$

contains at least one extreme point x_0 lying in the interior of $\{x \in X : f(x) \leq 0\}$. Thus, for sufficiently small $r > 0$, we have

$$h := \sup_{B_r(x_0)} (u - l_{x_0}) > 0$$

and

$$\{x \in X : u(x) < l_{x_0}(x) + h\} \subset \{x \in X : f(x) \leq 0\},$$

where

$$l_{x_0}(x) = u(x_0) + Du(x_0) \cdot (x - x_0).$$

Then, by Lemma 2.1 we choose an affine function $p_y(x)$ with the associated section $S = \{x : \ u(x) < p_y(x)\}$ and set

$$u_h := \max\{u, p_y\}.$$

It is clear that u_h is convex, then there is $L(u_h) \geq L(u)$ since u is a minimizer of (1.6). Note that u_h is different from u only on S. By (2.1), (2.16) in Remark 2.2 and

$$S \subset \{x \in X : u(x) < l_{x_0}(x) + h\} \subset \{x \in X : f(x) \leq 0\},$$

we have

$$\begin{aligned} 0 \leq L(u_h) - L(u) &= \int_S [(|y|^q + f(x)p_y(x)) - (|Du(x)|^q + f(x)u(x))] \,\mathrm{d}x \\ &= \int_S (|y|^q - |Du(x)|^q) \,\mathrm{d}x + \int_S (p_y(x) - u(x))f(x) \,\mathrm{d}x \\ &\leq -C_2 \frac{h^q}{r^q}|S| \\ &< 0, \end{aligned}$$

which makes a contradiction. Hence, any non-trivial section of u can not be contained in the region where $f \leq 0$. This implies that the contact sets of u have no extreme points inside $\{x \in X : f(x) \leq 0\}$, which gives us that u should be a ruled surface in $\{x \in X : f(x) \leq 0\}$. □

3. Proof of Theorem 1.4

In this section, we will use Lemma 2.1 to prove Theorem 1.4.

Proof of Theorem 1.4. Fix $X' \subset\subset X$, $x_0 \in X'$ and $y_0 \in y_b(x_0, Du(x_0))$. First, we show that for any r less than a given r_0 (independent of u) there exists $C > 0$, such that

$$\sup_{B_r(x_0)} |u(x) - u(x_0) - b(x, y_0) + b(x_0, y_0)| \leq C r^{1+\frac{1}{q-1}}, \tag{3.1}$$

where $p_0(x) := u(x_0) + b(x, y_0) - b(x_0, y_0)$ is a b-support function of u at x_0. Indeed, let

$$h = \sup_{B_r(x_0)} (u - p_0).$$

We assume $h > 0$. Otherwise, the proof is finished. Then we choose a b-affine function p_y with the associated section $S = \{x : \ u(x) < p_y(x)\}$ by Lemma 2.1 and set

$$u_h := \max\{u, p_y\}.$$

It is clear that u_h is b-convex, then $L(u_h) \geq L(u)$ since u is a minimizer of (1.6). Note that u_h differs from u only on S. Since p_y is a b-affine function, we have $y_b(x, Dp_y(x)) = y$. Hence, we can deduce from (**H2**), (2.1), and (2.2) that

$$\begin{aligned}
0 &\leq L(u_h) - L(u) \\
&= \int_S \left[(F^1(x,y) + F^0(x,p_y)) - (F^1(x, y_b(x,Du)) + F^0(x,u)) \right] \mathrm{d}x \\
&= \int_S (F^1(x,y) - F^1(x, y_b(x,Du)))\, \mathrm{d}x + \int_S (F^0(x,p_y) - F^0(x,u))\, \mathrm{d}x \\
&\leq \left(C_1 h - C_2 \frac{h^q}{r^q} + Mh \right) |S|,
\end{aligned}$$

which gives us that

$$h \leq C r^{\frac{q}{q-1}},$$

i.e.

$$\sup_{B_r(x_0)} |u - p_0| \leq C r^{1+\frac{1}{q-1}}.$$

Next, we show that for any r less than a given r_0 there exists

$$\sup_{B_r(x_0)} |u(x) - u(x_0) - Du(x_0)\cdot(x - x_0)| \leq C r^{1+\alpha}$$

for $\alpha = 1/(q-1)$ when $q > 2$ and $\alpha = 1$ when $1 < q \leq 2$. Indeed, by Definition 1.2 we have $Du(x_0) = b_x(x_0, y_0)$. Then by (3.1) and Lagrange's Mean Value Theorem, for any x, there exists ξ, such that

$$\begin{aligned}
&|u(x) - u(x_0) - Du(x_0)\cdot(x - x_0)| \\
&\leq |u(x) - u(x_0) - (b(x,y_0) - b(x_0,y_0))| + |(b(x,y_0) - b(x_0,y_0)) - Du(x_0)\cdot(x-x_0)| \\
&= |u(x) - u(x_0) - (b(x,y_0) - b(x_0,y_0))| + |b_x(\xi,y_0)\cdot(x-x_0) - b_x(x_0,y_0)\cdot(x-x_0)| \\
&\leq C|x - x_0|^{1+\frac{1}{q-1}} + \|b_{xx}\|_{L^\infty(\overline{X}\times\overline{Y})}|x - x_0|^2 \\
&\leq C|x - x_0|^{1+\alpha}
\end{aligned}$$

for $\alpha = 1/(q-1)$ when $q > 2$ and $\alpha = 1$ when $1 < q \leq 2$. Then the proof is completed by noting that a b-convex function is semi-convex and applying Lemma 3.1. □

In the above proof, we used a criterion for $C^{1,\alpha}$ regularity of convex functions, which states that if a convex function separates its supporting planes in a $C^{1,\alpha}$ fashion pointwisely, then it is indeed of class $C^{1,\alpha}$. This lemma can be found in many references, see, for example, in Figalli's book [8, Lemma A.32]. For readers' convenience, we include it here.

Lemma 3.1 ([8, Lemma A.32]). *Let Z be an open convex set satisfying*

$$B_r(\bar{x}) \subset Z \subset B_R(\bar{x})$$

for some $0 < r \le R$ *and* $\bar{x} \in \mathbb{R}^n$. *Let* $u : Z \to \mathbb{R}$ *be a convex function, and assume that there exist constants* K, C, $\varrho > 0$ *and* $\alpha \in (0, 1]$ *such that the following holds:* u *is* K*-Lipschitz in* Z, *and for every* $x \in Z$ *there exists* $p_x \in \partial u(x)$ *satisfying*

$$u(z) - u(x) - p_x \cdot (z - x) \le C|z - x|^{1+\alpha}, \quad \forall\, z \in Z \cap B_\varrho(x).$$

Then $u \in C^{1,\alpha}(Z)$ *with*

$$\|Du\|_{C^\alpha(Z)} \le \bar{C} = \bar{C}\Big(r, R, K, C, \varrho\Big).$$

4. Optimal regularities of minimizers

In this section, we will provide examples to demonstrate that the regularity in Theorem 1.4 is optimal for $q \ge 2$. When $q = 2$, there is an example constructed in [16, Remark 5] to show that Theorem 1.4 is optimal.

As mentioned in the introduction, we do not have an explicit Euler-Lagrange equation for the minimizers of (1.1) with convexity constraints. In [14], Lions has shown that the Euler-Lagrange equation for the minimizers of (1.1) with convexity constraints has the following form:

$$\sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} \mu_{ij} = \frac{\partial F}{\partial z}(x, u(x), Du(x)) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\frac{\partial F}{\partial p_i}(x, u(x), Du(x)) \right), \tag{4.1}$$

where $\mu = (\mu_{ij})$ is a matrix-valued Radon measure and (4.1) holds in the sense of distribution. See [2] for a different proof and some extensions. Since very little is known about the measure μ, the regularity of the minimizers of (1.1) with convexity constraints via (4.1) is still inaccessible. However, in the one-dimensional case it was shown in [2] by using (4.1) that the minimizers of (1.1) with certain conditions on F must belong to the class of C^1.

Theorem 4.1 ([2, Theorem 3])**.** *Assume that* $n = 1$ *and suppose that* $F(t, x, v)$ *satisfies*

(1) F *is of class* C^1 *over* $(a, b) \times \mathbb{R} \times \mathbb{R}$,

(2) *there exists* $\beta > 0$, $\alpha \in L^{p'}((a, b))$ *and* $\gamma \in L^1((a, b))$ *such that for all* $(t, x, v) \in [a, b] \times \mathbb{R} \times \mathbb{R}$,

$$\left|\frac{\partial F}{\partial v}(t, x, v)\right| \le \alpha(t) + \beta\left(1 + |v|^{p-1}\right),$$

$$\left|\frac{\partial F}{\partial x}(t, x, v)\right| \le \gamma(t) + \beta\left(1 + |v|^{p}\right)$$

where we assume $p > 1$, *and* $\frac{1}{p} + \frac{1}{p'} = 1$,

(3) F *is strictly convex with respect to* v.

Then the minimizers of (1.1) *with convexity constraints belongs to* $C^1(a, b)$.

The assumptions outlined in **(H1)**-**(H3)** are partially inspired by Theorem 4.1 and have a certain naturalness in their formulation.

Let $\Omega = [-1, 1]$, $q > 2$, we consider the functional

$$L[u] := \int_{-1}^{1} \frac{1}{q}|u'(x)|^q + u(x)\ \mathrm{d}x \tag{4.2}$$

over the set

$$\{u : [-1, 1] \to \mathbb{R} \,|\, u \text{ is convex and } u(1) = u(-1) = 0\}\,.$$

By Theorem 4.1, we know that the minimizer of (4.2) with convexity constraints is already C^1. What's more, the minimizer of (4.2) is $C^{1,1/(q-1)}$ for $q > 2$ according to Theorem 1.4. Now, we show that when $q > 2$, the minimizer is at most $C^{1,1/(q-1)}$.

It is easy to see that the Euler-Lagrange equation of (4.2) without convexity constraint is

$$(|u'|^{q-2}u')' = 1. \tag{4.3}$$

Solving (4.3) with boundary conditions $u(-1) = u(1) = 0$ yields

$$u(x) = \frac{q-1}{q}\left(|x|^{1+\frac{1}{q-1}} - 1\right). \tag{4.4}$$

It is clear that u is a convex function on $[-1, 1]$, which implies that the minimizers of $L(u)$ with or without a convexity constraint coincide. Then we know that the regularity of $u(x)$ is at most $C^{1,1/(q-1)}$ when $q > 2$.

When $1 < q < 2$, we can observe that $u(x)$ in (4.4) has higher regularity than $C^{1,1}$. This implies the optimal regularity of u remains undetermined in the $1 < q < 2$ case. Consequently, at the end of this section, we present two questions for future consideration. Firstly, what is the optimal regularity for the minimizer of (1.6) when $1 < q < 2$? Secondly, can the regularity established in Theorem 1.4 be extended to the boundary under certain boundary conditions?

References

[1] Caffarelli, L. A.; Lions, P. -L.: Unpublished and handwritten notes.

[2] Carlier, G.: Calculus of variations with convexity constraint. *J. Nonlinear Convex Anal.* **3** (2002), no. 2, 125-143.

[3] Carlier, G.; Lachand-Robert, T.: Regularity of solutions for some variational problems subject to a convexity constraint. *Comm. Pure Appl. Math.* **54** (2001), no. 5, 583-594.

[4] Carlier, G.; Lachand-Robert, T.: Representation of the polar cone of convex functions and applications. *J. Convex Anal.* **15** (2008), no. 3, 535-546.

[5] Carlier, G.; Radice, T.: Approximation of variational problems with a convexity constraint by PDEs of Abreu type. *Calc. Var. Part. Diff. Eqns.* **58** (2019), no. 5, Paper No. 170, 13 pp.

[6] Chen, S. B.: *Convex solutions to the power-of-mean curvature flow, conformally invariant inequalities and regularity results in some applications of optimal transportation.* Thesis (Ph.D.)–University of Toronto.

[7] Chen, S. B.: Regularity of the solution to the principal-agent problem. *Geometric and Functional Inequalities and Recent Topics in Nonlinear PDEs*, 41-47.

[8] Figalli, A.: *The Monge-Ampère equation and its applications.* Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2017.
[9] Figalli, A.; Kim, Y. -H.; McCann, R. J.: When is multidimensional screening a convex program? *J. Econom. Theory* **146** (2011), no. 2, 454-478.
[10] Gutiérrez, C.: *The Monge-Ampère equation*, Progress in Nonlinear Differential Equations and Their Applications **44** (2001), Birkhäuser, Massachusetts.
[11] Le, N. Q.: Singular Abreu equations and minimizers of convex functionals with a convexity constraint. *Comm. Pure Appl. Math.* **73** (2020), no. 10, 2248-2283.
[12] Le, N. Q.: Twisted Harnack inequality and approximation of variational problems with a convexity constraint by singular Abreu equations. *Adv. Math.* **434** (2023), Paper No. 109325.
[13] Le, N. Q.; Zhou, B.: Solvability of a class of singular fourth order equations of Monge-Ampère type. *Ann. PDE* **7** (2021), no. 2, Paper No. 13, 32 pp.
[14] Lions, P. -L.: Identification du cône dual des fonctions convexes et applications. *C. R. Acad. Sci. Paris Sér. I Math.* **326** (1998), no. 12, 1385-1390.
[15] Loeper, G.: On the regularity of solutions of optimal transportation problems. *Acta Math.* **202** (2009), no. 2, 241-283.
[16] McCann, R. J.; Rankin, C.; Zhang, S. J.: $C^{1,1}$ regularity for principal-agent problems. *arXiv:2303.04937v3.*
[17] Rochet, J. -C.; Choné, P.: Ironing, Sweeping, and Multidimensional Screening. *Econometrica* **66** (1998), no. 4, 783-826.
[18] Trudinger, N. S.; Wang, X. -J.: *The Monge-Ampère equations and its geometric applications*, Handbook of geometric analysis, International Press, 2008, pp. 467-524.

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