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REFINED STABILITY ESTIMATES IN ELECTRICAL IMPEDANCE TOMOGRAPHY WITH MULTI-LAYER STRUCTURE

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Abstract. In this paper we study the inverse problem of determining an electrical inclusion in a multi-layer composite from boundary measurements in 2D. We assume the conductivities in different layers are different and derive a stability estimate for the linearized map with explicit formulae on the conductivity and the thickness of each layer. Intuitively, if an inclusion is surrounded by a highly conductive layer, then, in view of “the principle of the least work”, the current will take a path in the highly conductive layer and disregard the existence of the inclusion. Consequently, a worse stability of identifying the hidden inclusion is expected in this case. Our estimates indeed show that the ill-posedness of the problem increases as long as the conductivity of some layer becomes large. This work is an extension of the previous result by Nagayasu-Uhlmann-Wang[15], where a depth-dependent estimate is derived when an inclusion is deeply hidden in a conductor. Estimates in this work also show the influence of the depth of the inclusion.

1. **Introduction.** Electrical impedance tomography (EIT) arises in medical imaging given that human organs and tissues have quite different conductivities [12]. It has been developed to be an inverse method which consists in determining the electrical properties of a medium by making voltage and current measurements at the boundary of the medium. In the mathematical literature, this is also known as Calderón’s problem [7]. More precisely, let Ω be an open bounded domain with

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smooth boundary in $\mathbb{R}^d$. The conductivity equation can be described by the following second order elliptic equation of divergence form:

$$\nabla \cdot (\gamma(x)\nabla u) = 0, \quad x \in \Omega\,, \tag{1}$$

where $\gamma(x) > 0$.

It is known that for an appropriate function f given on $\partial\Omega$, there exists a unique solution $u(x)$ to the Dirichlet boundary value problem for (1) with $u|_{\partial\Omega} = f$ [10]. Thus, we can define a Dirichlet-to-Neumann operator $\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$, by

$$\Lambda_\gamma(f) = \gamma(x)\frac{\partial u}{\partial \nu}\Big|_{\partial\Omega},$$

where ν is unit exterior normal vector of $\partial\Omega$. Here the boundary information is encoded in the map Λ_γ. The map Λ_γ depends nonlinearly on γ, even though equation (1) is linear. The famous Calderón problem [7] is to determine γ from the knowledge of Λ_γ, meanwhile the EIT problem is notoriously known to be ill-posed. A log-type stability was obtained by Alessandrini[1, 3] for the interior determination and a Lipschitz-type stability was by Alessandrini [2], and Alessandrini-Gaburro [4, 5], Sylvester-Uhlmann[17] for the boundary determination. The log-type stability estimate was proved to be optimal by Mandache [14].

In several practical situations, the conductivity function is of the type $\gamma(x) = \gamma_0(x) + \gamma_1(x)\chi_D$, where $D \subset\subset \Omega$ is an inclusion with abnormal conductivity $\gamma_1(x)$. Assuming that γ_0 is known, we are interested in determining the shape of D by the Dirichlet-to-Neumann map, denoted by Λ_D. Under some appropriate conditions, the uniqueness was proved by Isakov[11]. Numerical results, e.g. [8, 9, 18, 19], showed that the deeper the inclusion, the worse the numerical reconstruction. Nagayasu, Uhlmann and Wang [15], by studying the linearized Dirichlet-to-Neumann map, obtained a quantitative description of this phenomenon in a model case that $\gamma(x) = 1 + (k-1)\chi_D$, $k > 0$, $k \neq 1$, and showed that the ill-posedness increases when the depth of inclusion hidden in the conductor increases.

However, it is known that the crustal structure is multi-layered. Furthermore, in practice, there are many multi-layered composite materials as well. The properties in different layers are always different. Inspired by the result of [15], in this paper, we show that when the conductivity in some middle layer increases, the ill-posedness will increase as well. This result is an extension of Nagayasu-Uhlmann-Wang's work, where the two-layer case was studied. We will follow the method implemented in [15] and use the Fourier series to analytically express the Dirichlet-to-Neumann linearized map for three-layer and n-layer inclusions and then to derive stability estimates. We would like to mention that there are some interesting results related to the linearized map and the stability obtained in [13, 16]. Also, in [6], Alessandrini and Scapin considered the depth dependent resolution in 2D.

To describe precisely the problem we have in mind, we set $d = 2$ and first consider the three-layer case. Let $\Omega := B_R(0)$, $B_i := B_{r_i}(0)$, $0 < r_1 < r_2 < R$, and

$$\gamma(x) = k_1\chi_{B_1} + k_2\chi_{B_2\setminus B_1} + \chi_{\Omega\setminus B_2},$$

where $k_i > 0$ are different with $k_i \neq 1$. Define

$$L_{B_1,B_2}u_0 := \nabla \cdot \Big(\Big(k_1\chi_{B_1} + k_2\chi_{B_2\setminus B_1} + \chi_{\Omega\setminus B_2}\Big)\nabla u_0\Big).$$

Given a smooth function $\psi : \partial B_1 \to \mathbb{R}$, we introduce a perturbation B_1^s of B_1 as in [15], namely, the boundary ∂B_1^s is defined by

$$y = x + s\psi(x)\nu_x \,, \text{ on } \partial B_1,$$

where ν_x is the unit exterior normal vector at $x \in \partial B_1$. For $f \in H^{1/2}(\partial\Omega)$, let u_0 be the solution to the reference problem

$$\begin{cases} L_{B_1,B_2} u_0 = 0, & \text{in } \Omega, \\ u_0 = f, & \text{on } \partial\Omega. \end{cases}$$

Likewise, let u_s be the solution to the perturbed problem

$$\begin{cases} L_{B_1^s,B_2} u_s = 0, & \text{in } \Omega, \\ u_s = f, & \text{on } \partial\Omega. \end{cases}$$

We define the linearized map of the Dirichlet-to-Neumann map at the direction ψ, by

$$d\Lambda_{B_1}(\psi) := \lim_{s\to 0} \frac{1}{s}(\Lambda_{B_1^s,B_2} - \Lambda_{B_1,B_2}), \tag{2}$$

where Λ_{B_1,B_2} and $\Lambda_{B_1^s,B_2}$ are the Dirichlet-to-Neumann maps corresponding to $L_{B_1,B_2}u_0 = 0$ and $L_{B_1^s,B_2}u_s = 0$ in Ω, respectively. Now, we state the stability estimate in the case of three-layer inclusions.

Theorem 1.1. *Let $m > 0$, $M_0 > 0, r_0 > 0$ and $X_0 > 1$ be fixed. Suppose*

$$M \geq M_0, \quad r_1 \leq r_0 \quad \frac{R}{r_1} \geq X_0.$$

Then there exists a positive constant $\bar{k}$ such that for $k_2 > \bar{k}$ and for any $\psi \in H^m(\partial B_1)$ satisfying

$$\|\psi\|_{H^m(\partial B_1)} \leq M \quad \text{and} \quad \|d\Lambda_{B_1}(\psi)\|_{\mathscr{L}} < 1,$$

we have

$$\|\psi\|_{L^2(\partial B_1)} \leq CM(k_2+1)\Big[\ln\Big(\frac{R}{r_1}\Big)\Big]^m \Big|\ln \|d\Lambda_{B_1}(\psi)\|_{\mathscr{L}}\Big|^{-m}, \tag{3}$$

where the positive constant C depends only on m, M_0, r_0, X_0, k_1. Here and after, $\|\cdot\|_{\mathscr{L}}$ denotes the operator norm on the space of bounded linear operators from $H^{1/2}(\partial\Omega)$ into $H^{-1/2}(\partial\Omega)$.

Remark 1. Actually, we can take

$$\bar{k} = \max\Big\{2^{\frac{3m}{2}+\frac{1}{2}}\pi^{-\frac{1}{2}}, \Big((-\ln c)/(\ln X_0) + 16\Big)^m\Big\},$$

for some constant c to be determined in the proof (see (27)). Moreover,

$$C \leq \frac{4 \times 5^{\frac{1}{2}} 10^{\frac{1}{4}} \pi^{\frac{1}{2}} \left(\bar{k} + k_1\right)^2}{\left(\bar{k} - k_1\right)^2}.$$

In particular, if $X_0 \geq e$ and $M_0^2 r_0^{-5} \geq \frac{1}{\sqrt{2}}(\frac{e}{8m})^{-2m}$, then $c = \frac{1}{2}$ and so $16^m \leq \bar{k} \leq 17^m$.

Remark 2. Estimate (3) clearly indicates that the determination of an inclusion by boundary measurements is getting increasingly ill-posed when the inclusion is hidden deeper inside of the conductor, i.e., R/r_1 is large, and also when the conductivity k_2 becomes large. In particular, if $k_2 = \infty$, then u_s will be a constant in $B_2 \setminus B_1^s$. Thus, it is impossible to uniquely determine ∂B_1^s even using the full Dirichlet-to-Neumann map $\Lambda_{B_1^s, B_2}$.

Next, for n-layer inclusions, we set $\Omega := B_R(0)$, $B_i := B_{r_i}(0)$, $0 < r_1 < r_2 < \cdots < r_n < R$, and

$$L_{B_1,B_2,\cdots,B_n} u_0 := \nabla \cdot \Big(\Big(k_1 \chi_{B_1} + \sum_{i=2}^{n} k_i \chi_{B_i \setminus B_{i-1}} + \chi_{B_R \setminus B_n} \Big) \nabla u_0 \Big) = 0, \tag{4}$$

where $k_i > 0$ and $k_i \neq 1$, $i = 1, 2, \cdots, n$. Let u_0 and u_s be the solutions to the problems

$$\begin{cases} L_{B_1,B_2,\cdots,B_n} u_0 = 0, & \text{in } \Omega, \\ u_0 = f, & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} L_{B_1^s,B_2,\cdots,B_n} u_s = 0, & \text{in } \Omega, \\ u_s = f, & \text{on } \partial\Omega. \end{cases}$$

respectively. The linearized map $d\Lambda_{B_1}(\psi)$ is defined similarly as (2), namely,

$$d\Lambda_{B_1}(\psi) := \lim_{s \to 0} \frac{1}{s} (\Lambda_{B_1^s,B_2,\cdots,B_n} - \Lambda_{B_1,B_2,\cdots,B_n}),$$

where $\Lambda_{B_1,B_2,\cdots,B_n}$ and $\Lambda_{B_1^s,B_2,\cdots,B_n}$ are the Dirichlet-to-Neumann maps corresponding to $L_{B_1,B_2,\cdots,B_n} u_0 = 0$ and $L_{B_1^s,B_2,\cdots,B_n} u_s = 0$ in Ω, respectively. Then

Theorem 1.2. *Let $m > 0$, $M_0 > 0, r_0 > 0$ and $X_0 > 1$ be fixed. Suppose*

$$M \geq M_0, \quad r_1 \leq r_0 \quad \frac{R}{r_1} \geq X_0.$$

Then for any $\psi \in H^m(\partial B_1)$ satisfies

$$\|\psi\|_{H^m(\partial B_1)} \leq M \quad \text{and} \quad \|d\Lambda_{B_1}(\psi)\|_{\mathscr{L}} < 1,$$

the following inequality

$$\|\psi\|_{L^2(\partial B_1)} \leq CM \Big[\ln \Big(\frac{R}{r_1} \Big) \Big]^m \left| \ln \|d\Lambda_{B_1}(\psi)\|_{\mathscr{L}} \right|^{-m}, \tag{5}$$

holds, where C depends only on $m, M_0, r_0, X_0, k_1, k_2, \cdots, k_n$. Moreover, if for some $l \in \{2, 3, \cdots, n\}$, k_l satisfies

$$k_l \geq \max \Big\{ 2^{\frac{3}{2}m + \frac{1}{2}} \pi^{-\frac{1}{2}}, \Big((-\ln c)/(\ln X_0) + 16 \Big)^m \Big\}$$

with some appropriate constant c, then (5) can be written explicitly as

$$\begin{aligned} \|\psi\|_{L^2(\partial B_1)} \leq & \frac{C_n (k_2 + k_1)^2 (k_3 + k_2) \cdots (k_n + 1)}{(k_2 - k_1)^2} \\ & \times M \Big[\ln \Big(\frac{R}{r_1} \Big) \Big]^m \left| \ln \|d\Lambda_{B_1}(\psi)\|_{\mathscr{L}} \right|^{-m}, \end{aligned} \tag{6}$$

where C_n only depends on n.

Remark 3. Estimate (6) shows that the inverse inclusion problem in a multi-layer medium is more unstable whenever the conductivity in any outside layer becomes larger.

The rest of the paper is organized as follows. In Section 2, we consider the problem in the case of three-layer medium. The extension to the n-layer medium is studied in Section 3.

2. **The case of three-layer medium.** In this section, we will derive the solution to the conductivity equation with piecewise constant coefficients and estimate the norm of linearized map $d\Lambda_{B_1}(\psi)$. We then give a proof of Theorem 1.1.

2.1. **Preliminary results for Fourier expression.** In this subsection, we will find the series solution to the Dirichlet problem, preparing for the expression of $d\Lambda(\psi)$. Since we are working in two dimensions, it is convenient to use the polar coordinates: $(x,y) = (\rho\cos\theta, \rho\sin\theta) \in \mathbb{R}^2$ with $\rho > 0$ and $\theta \in [0, 2\pi)$. For any function $f \in L^2(\partial\Omega)$, we denote $\widetilde{f}(\theta) := f(R\cos\theta, R\sin\theta)$. Define the Fourier coefficients and the $H^m(\partial\Omega)$-norm

$$f_l = \int_0^{2\pi} \widetilde{f}(\theta) e^{-\mathrm{i}l\theta} d\theta\,, \qquad \|f\|^2_{H^m(\partial\Omega)} = \frac{R}{2\pi}\sum_{l\in\mathbb{Z}}(1+l^2)^m |f_l|^2.$$

We can also write $\widetilde{f}(\theta) = (2\pi)^{-1}\sum_{l\in\mathbb{Z}} f_l e^{il\theta}$. For functions on other boundaries, they are defined in the same way.

For $f \in H^{1/2}(\partial\Omega)$, suppose that u_0 is the solution of the following Dirichlet problem

$$\begin{cases} L_{B_1,B_2} u_0 = 0, & \text{in } \Omega, \\ u_0 = f, & \text{on } \partial\Omega. \end{cases} \tag{7}$$

The Dirichlet-to-Neumann map $\Lambda_{B_1,B_2} : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ associated with L_{B_1,B_2} is defined by

$$\Lambda_{B_1,B_2}(f) := \Big(k_1\chi_{B_1} + k_2\chi_{B_2\setminus B_1} + \chi_{B_R\setminus B_2}\Big)\frac{\partial u_0}{\partial\nu}\Big|_{\partial\Omega}.$$

Notice that (7) can be rewritten as the following transmission problem

$$\begin{cases} \Delta u_0 = 0, & \text{in } B_1 \cup (B_2\setminus B_1)\cup(\Omega\setminus B_2), \\ u_0|_+ = u_0|_-, & \text{on } \partial B_i,\ i=1,2, \\ k_i\frac{\partial u_0}{\partial\nu}\Big|_+ = k_{i+1}\frac{\partial u_0}{\partial\nu}\Big|_-, & \text{on } \partial B_i,\ i=1,2, \\ u_0 = f, & \text{on } \partial\Omega, \end{cases} \tag{8}$$

where $k_3 = 1$ and $\pm$ stands for taking limit from outside or inside of the inclusion.

Next, we plan to express the solution of (8) in terms of the Fourier series. For $l \in \mathbb{Z}^+$, we denote

$$\begin{aligned} S_l^{-1} :=& -\frac{1}{4l}\begin{vmatrix} -(k_2-k_1)r_1^l & (k_1+k_2)r_1^{-l} & 0 & 0 \\ (k_2+1)r_2^l & -(k_2-1)r_2^{-l} & -2r_2^l & 0 \\ (k_2-1)r_2^l & -(k_2+1)r_2^{-l} & 0 & 2r_2^{-l} \\ 0 & 0 & R^l & R^{-l} \end{vmatrix} \\ =& \frac{1}{2l}\bigg(R^l r_2^{-l}\Big((k_2-k_1)(k_2-1)r_1^l r_2^{-l} - (k_1+k_2)(k_2+1)r_1^{-l}r_2^l\Big) \\ &\quad + R^{-l}r_2^l\Big(-(k_2-k_1)(k_2+1)r_1^l r_2^{-l} + (k_1+k_2)(k_2-1)r_1^{-l}r_2^l\Big)\bigg). \end{aligned} \tag{9}$$

Then

Lemma 2.1. *The series solution of (8) can be written explicitly as*

$$
\begin{aligned}
&u_0(\rho\cos\theta,\rho\sin\theta) \\
(10)\quad &= \begin{cases}
\frac{1}{2\pi}\sum_{l=1}^{\infty}\frac{S_l}{2l}\Big[\Big((k_2-k_1)(k_2-1)r_1^l r_2^{-2l}-(k_1+k_2)(k_2+1)r_1^{-l}\Big)\rho^l \\
\qquad +\Big((k_1+k_2)(k_2-1)r_1^{-l}r_2^{2l}-(k_2-k_1)(k_2+1)r_1^l\Big)\rho^{-l}\Big] \\
\qquad \cdot\Big[f_l e^{\mathrm{i}l\theta}+f_{-l}e^{-\mathrm{i}l\theta}\Big]+f_0, & r_2<\rho<R, \\
\frac{1}{2\pi}\sum_{l=1}^{\infty}\frac{S_l}{l}\Big[-(k_1+k_2)r_1^{-l}\rho^l-(k_2-k_1)r_1^l\rho^{-l}\Big]\cdot\Big[f_l e^{\mathrm{i}l\theta}+f_{-l}e^{-\mathrm{i}l\theta}\Big]+f_0, \\
 & r_1<\rho<r_2, \\
\frac{1}{2\pi}\sum_{l=1}^{\infty}\frac{S_l}{l}(-2k_2)r_1^{-l}\rho^l\Big[f_l e^{\mathrm{i}l\theta}+f_{-l}e^{-\mathrm{i}l\theta}\Big]+f_0, & 0<\rho<r_1.
\end{cases}
\end{aligned}
$$

Recall that

$$f_l=\int_0^{2\pi}\widetilde{f}(\theta)e^{-\mathrm{i}l\theta}\,d\theta.$$

Remark 4. We remark that if $k_2=1$, then ∂B_2 coincides $\partial\Omega$ and

$$S_l^{-1}=\frac{1}{l}\Big((k_1-1)R^{-l}r_1^l-(k_1+1)R^l r_1^{-l}\Big),$$

which is the same formula derived in [15]. In this paper we are mainly concerned about the instability caused by a large k_2.

Proof. To simplify the notation, we denote $u:=u_0$. Note that

$$u(\rho\cos\theta,\rho\sin\theta)=\sum_{l\in\mathbb{Z}}u_l(\rho)e^{\mathrm{i}l\theta},\quad \theta\in[0,2\pi).$$

By Fourier series, we obtain the following ordinary differential equation for u_l, $l\neq 0$,

$$u_l''(\rho)+\frac{1}{\rho}u_l'(\rho)-\frac{l^2}{\rho^2}u_l(\rho)=0. \tag{11}$$

Clearly, ρ^l is a solution of (11). Thus, its general solution is

$$u_l(\rho)=a\rho^l+b\rho^{-l},$$

for arbitrary constants a and b. Next, we will try to determine the values of a and b in each layer. We set

$$u_l(\rho)=\begin{cases} u_l(r_1)r_1^{-l}\rho^l, & l>0,\\ u_l(r_1)r_1^l\rho^{-l}, & l<0,\end{cases}\qquad \text{if } 0<\rho<r_1,$$

and

$$u_l(\rho)=\begin{cases} a_1\rho^l+b_1\rho^{-l}, & \text{if } r_1<\rho<r_2,\\ a_2\rho^l+b_2\rho^{-l}, & \text{if } r_2<\rho<R.\end{cases}$$

We firstly consider the case where $l>0$. Since $u|_-=u|_+$ and $k_i\frac{\partial u}{\partial\nu}\Big|_-=k_{i+1}\frac{\partial u}{\partial\nu}\Big|_+$ on ∂B_i, it follows that

$$u_l(r_1)=a_1r_1^l+b_1r_1^{-l},\quad \text{on } \partial B_1, \tag{12}$$

and

$$
\begin{cases} k_1 u_l(r_1)\frac{l}{r_1} = k_2\Big(a_1 l {r_1}^{l-1} - b_1 l {r_1}^{-l-1}\Big), & \text{on } \partial B_1, \\ a_1 r_2^l + b_1 r_2^{-l} = a_2 r_2^l + b_2 r_2^{-l}, & \text{on } \partial B_2, \\ k_2\Big(a_1 l {r_2}^{l-1} - b_1 l {r_2}^{-l-1}\Big) = a_2 l {r_2}^{l-1} - b_2 l {r_2}^{-l-1}, & \text{on } \partial B_2. \end{cases} \tag{13}
$$

When $\rho = R$, we have

$$
a_2 R^l + b_2 R^{-l} = u_l(R) = \frac{1}{2\pi}\int_0^{2\pi} \widetilde{f}(\theta) e^{-\mathrm{i}l\theta}\, d\theta = \frac{1}{2\pi} f_l, \quad \text{on } \partial B_R. \tag{14}
$$

Combining (12), (13) with (14) yields the following linear system

$$
\begin{cases} -(k_2-k_1) r_1^l a_1 + (k_1+k_2) r_1^{-l} b_1 = 0, & \cdots\cdots\cdots ① \\ r_2^l a_1 + r_2^{-l} b_1 - r_2^l a_2 - r_2^{-l} b_2 = 0, & \cdots\cdots\cdots ② \\ k_2 r_2^l a_1 - k_2 r_2^{-l} b_1 - r_2^l a_2 + r_2^{-l} b_2 = 0, & \cdots\cdots\cdots ③ \\ R^l a_2 + R^{-l} b_2 = \frac{1}{2\pi} f_l. \end{cases} \tag{15}
$$

Then, ② + ③ and ③ − ② yields

$$
\begin{cases} -(k_2-k_1) r_1^l a_1 + (k_1+k_2) r_1^{-l} b_1 = 0, \\ (k_2+1) r_2^l a_1 - (k_2-1) r_2^{-l} b_1 - 2 r_2^l a_2 = 0, \\ (k_2-1) r_2^l a_1 - (k_2+1) r_2^{-l} b_1 + 2 r_2^{-l} b_2 = 0, \\ R^l a_2 + R^{-l} b_2 = \frac{1}{2\pi} f_l. \end{cases}
$$

Hence, recalling the definition of S_l, (9), and by the Cramer's law, we have

$$
a_1 = -\frac{S_l}{4l}\begin{vmatrix} 0 & (k_1+k_2) r_1^{-l} & 0 & 0 \\ 0 & -(k_2-1) r_2^{-l} & -2r_2^l & 0 \\ 0 & -(k_2+1) r_2^{-l} & 0 & 2r_2^{-l} \\ \frac{1}{2\pi} f_l & 0 & R^l & R^{-l} \end{vmatrix} = -\frac{1}{2\pi} f_l \cdot \frac{S_l}{l}(k_1+k_2) r_1^{-l},
$$

$$
b_1 = -\frac{S_l}{4l}\begin{vmatrix} -(k_2-k_1) r_1^l & 0 & 0 & 0 \\ (k_2+1) r_2^l & 0 & -2r_2^l & 0 \\ (k_2-1) r_2^l & 0 & 0 & 2r_2^{-l} \\ 0 & \frac{1}{2\pi} f_l & R^l & R^{-l} \end{vmatrix} = -\frac{1}{2\pi} f_l \cdot \frac{S_l}{l}(k_2-k_1) r_1^l,
$$

$$
\begin{aligned} a_2 &= -\frac{S_l}{4l}\begin{vmatrix} -(k_2-k_1) r_1^l & (k_1+k_2) r_1^{-l} & 0 & 0 \\ (k_2+1) r_2^l & -(k_2-1) r_2^{-l} & 0 & 0 \\ (k_2-1) r_2^l & -(k_2+1) r_2^{-l} & 0 & 2r_2^{-l} \\ 0 & 0 & \frac{1}{2\pi} f_l & R^{-l} \end{vmatrix} \\ &= \frac{1}{2\pi} f_l \cdot \frac{S_l}{2l}\Big((k_2-k_1)(k_2-1) r_1^l r_2^{-2l} - (k_1+k_2)(k_2+1) r_1^{-l}\Big), \end{aligned}
$$

$$b_2 = -\frac{S_l}{4l}\begin{vmatrix} -(k_2-k_1)r_1^l & (k_1+k_2)r_1^{-l} & 0 & 0 \\ (k_2+1)r_2^l & -(k_2-1)r_2^{-l} & -2r_2^l & 0 \\ (k_2-1)r_2^l & -(k_2+1)r_2^{-l} & 0 & 0 \\ 0 & 0 & R^l & \frac{1}{2\pi}f_l \end{vmatrix}$$
$$= -\frac{1}{2\pi}f_l\cdot\frac{S_l}{2l}\Big((k_2-k_1)(k_2+1)r_1^l-(k_1+k_2)(k_2-1)r_1^{-l}{r_2}^{2l}\Big).$$

Thus, when $l>0$, we have for $r_2<\rho<R$,

$$u_l(\rho)=\frac{1}{2\pi}f_l\cdot\frac{S_l}{2l}\Big[\Big((k_2-k_1)(k_2-1)r_1^lr_2^{-2l}-(k_1+k_2)(k_2+1)r_1^{-l}\Big)\rho^l + \Big(-(k_2-k_1)(k_2+1)r_1^l+(k_1+k_2)(k_2-1)r_1^{-l}{r_2}^{2l}\Big)\rho^{-l}\Big],$$

for $r_1<\rho<r_2$,

$$u_l(\rho)=\frac{1}{2\pi}f_l\cdot\frac{S_l}{l}\Big[-(k_1+k_2)r_1^{-l}\rho^l-(k_2-k_1)r_1^l\rho^{-l}\Big],$$

and for $0<\rho<r_1$,

$$u_l(\rho)=\frac{1}{2\pi}f_l\cdot\frac{S_l}{l}\cdot(-2k_2)\cdot r_1^{-l}\rho^l\,.$$

For $l<0$, we have $u_l(\rho)=u_l(r_1)r_1^l\rho^{-l}$, for $0<\rho<r_1$, which is the only term different from the case of $l>0$. Consequently, we obtain

$$k_1u_l(r_1)\Big(\frac{-l}{r_1}\Big)=k_2\Big(a_1l{r_1}^{l-1}-b_1l{r_1}^{-l-1}\Big)\quad\text{on }\partial B_1\,,$$

so 1 in (15) becomes

$$(k_1+k_2)r_1^la_1-(k_2-k_1)r_1^{-l}b_1=0\,.$$

Thus, it suffices to replace $-(k_2-k_1)$ by (k_1+k_2) in the expression of u for $l>0$. It follows that, for $r_2<\rho<R$,

$$u_l(\rho)=\frac{1}{2\pi}f_l\cdot\frac{S_{-l}}{2l}\Big[\Big(-(k_1+k_2)(k_2-1)r_1^lr_2^{-2l}-(k_2-k_1)(k_2+1)r_1^{-l}\Big)\rho^l + \Big((k_1+k_2)(k_2+1)r_1^l-(k_2-k_1)(k_2-1)r_1^{-l}{r_2}^{2l}\Big)\rho^{-l}\Big],$$

and

$$u_l(\rho)=\frac{1}{2\pi}f_l\cdot\frac{S_{-l}}{l}\Big[(k_2-k_1)r_1^{-l}\rho^l+(k_1+k_2)r_1^l\rho^{-l}\Big],\qquad r_1<\rho<r_2\,,$$

and

$$u_l(\rho)=\frac{1}{2\pi}f_l\cdot\frac{S_{-l}}{l}\cdot(2k_2)\Big(\frac{\rho}{r_1}\Big)^{-l},\qquad 0<\rho<r_1\,.$$

The proof of Lemma 2.1 is completed. ∎

2.2. **Estimates of the linearized map $d\Lambda_{B_1}(\psi)$.** In this subsection, we will derive the expression of $d\Lambda_{B_1}(\psi)$ and obtain the estimate of $\|d\Lambda_{B_1}(\psi)\|_{\mathscr{L}}$. We now introduce a function

$$U:=\lim_{s\to0}\frac{u_s-u_0}{s},$$

where u_s is the solution of

$$\begin{cases} L_{B_1^s,B_2}u_s=0, & \text{in }\Omega,\\ u_s=f, & \text{on }\partial\Omega.\end{cases}$$

By (2), we have

$$d\Lambda_{B_1}(\psi)(f) = \left.\frac{\partial U}{\partial \nu}\right|_{\partial\Omega}, \quad \forall f \in H^{1/2}(\partial\Omega). \tag{16}$$

By the computation on [15, Page 5], using $y = x + s\psi(x)\nu_x$ on ∂B_1, we have that, as $s \to 0$,

$$\frac{1}{s}\left(u_s(y)|_\pm - u_0(x)|_\pm\right) \to U(x)\Big|_\pm + \psi(x)\frac{\partial u_0}{\partial \nu}(x)\Big|_\pm \quad \text{on } \partial B_1,$$

and

$$\frac{1}{s}\left(\frac{\partial u_s}{\partial \nu_y}(y)\Big|_\pm - \frac{\partial u_0}{\partial \nu_x}(x)\Big|_\pm\right) \to \partial_\rho U(x)\Big|_\pm - r_1^{-2}\widetilde{\psi}'(\theta)\partial_\theta u_0(x)\Big|_\pm + \tilde{\psi}(\theta)\partial_\rho^2 u_0(x)\Big|_\pm$$

on ∂B_1. Notice that

$$\partial_\rho^2 u_0(x)\big|_\pm = -\frac{1}{r_1}\partial_\rho u_0(x)\Big|_\pm - \frac{1}{{r_1}^2}\partial_\theta^2 u_0(x)\Big|_\pm \quad \text{on } \partial B_1.$$

Hence, we can show that U is the solution of

$$\begin{cases} \Delta U = 0 & \text{in } \Omega\setminus\partial B_1, \\ U|_+ - U|_- = \frac{k_2-k_1}{k_1}\psi(x)\,\frac{\partial u_0}{\partial\nu}\big|_+ & \text{on } \partial B_1, \\ k_2\,\frac{\partial U}{\partial \nu}\big|_+ - k_1\,\frac{\partial U}{\partial\nu}\big|_- = r_1^{-2}(k_2-k_1)\partial_\theta\Big(\widetilde{\psi}(\theta)\partial_\theta u_0|_+\Big) & \text{on } \partial B_1, \\ U = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\widetilde{\psi}(\theta) = (2\pi)^{-1}\sum_{l\in\mathbb{Z}}\psi_l e^{il\theta}$.

Noticing that

$$\frac{k_2-k_1}{k_1}\psi\,\frac{\partial u_0}{\partial\nu}\Big|_+ = -\frac{(k_2-k_1)}{2\pi^2 r_1}\sum_{p\in\mathbb{Z}}\sum_{l=1}^{\infty} S_l\Big(\psi_{p-l}f_l + \psi_{p+l}f_{-l}\Big)e^{\mathrm{i}p\theta},$$

and

$$r_1^{-2}(k_2-k_1)\partial_\theta\Big(\widetilde{\psi}(\theta)\partial_\theta u_0|_+\Big) = \frac{(k_2-k_1)k_2}{2\pi^2 r_1^2}\sum_{p\in\mathbb{Z}} p\sum_{l=1}^{\infty} S_l\Big(\psi_{p-l}f_l - \psi_{p+l}f_{-l}\Big)e^{\mathrm{i}p\theta},$$

we rewrite the Dirichlet problem above as

$$\begin{cases} \Delta U = 0, & \text{in } \Omega\setminus\partial B_1, \\ U|_+ - U|_- = -\dfrac{(k_2-k_1)}{2\pi^2 r_1}\sum\limits_{p\in\mathbb{Z}}\sum\limits_{l=1}^{\infty} S_l\Big(\psi_{p-l}f_l + \psi_{p+l}f_{-l}\Big)e^{\mathrm{i}p\theta}, & \text{on } \partial B_1, \\ k_2\frac{\partial U}{\partial\nu}\Big|_+ - k_1\frac{\partial U}{\partial \nu}\Big|_- = \dfrac{(k_2-k_1)k_2}{2\pi^2 r_1^2}\sum\limits_{p\in\mathbb{Z}} p\sum\limits_{l=1}^{\infty} S_l\Big(\psi_{p-l}f_l - \psi_{p+l}f_{-l}\Big)e^{\mathrm{i}p\theta}, & \text{on } \partial B_1, \\ U = 0, & \text{on } \partial\Omega. \end{cases} \tag{17}$$

Similar as the above, we can obtain the series solution of (17) when $r_1 < \rho < R$.

Lemma 2.2. *When $r_1 < \rho < R$, the series solution of (17) is given by*

$$\begin{aligned} U(\rho\cos\theta, \rho\sin\theta) =& \frac{(k_2-k_1)}{2\pi^2 r_1}\sum_{l=1}^{\infty}\frac{T_l}{l}(R^l\rho^{-l} - R^{-l}\rho^l) \\ &\times \sum_{p=1}^{\infty} S_p\Big[(k_1+k_2)\Big(\psi_{-l+p}f_{-p}\,e^{-\mathrm{i}l\theta} + \psi_{l-p}f_p\,e^{\mathrm{i}l\theta}\Big) \\ &\qquad - (k_2-k_1)\Big(\psi_{-l-p}f_p\,e^{-\mathrm{i}l\theta} + \psi_{l+p}f_{-p}\,e^{\mathrm{i}l\theta}\Big)\Big], \end{aligned}$$

where

$$T_l := \frac{-l}{(k_2-k_1)r_1^l R^{-l} + (k_1+k_2)r_1^{-l}R^l}\,.$$

Proof. Let

$$U(\rho\cos\theta, \rho\sin\theta) = \sum_{l\in\mathbb{Z}} U_l(\rho)e^{\mathrm{i}l\theta}, \quad \theta\in[0,2\pi).$$

Similar to Lemma 2.1, we obtain when $l > 0$,

$$\begin{cases} U_l(\rho) = c_1\rho^l + c_2\rho^{-l}, & r_1 < \rho < R, \\ U_l(\rho) = U_l(r_1)\Big(\dfrac{\rho}{r_1}\Big)^l, & 0 < \rho < r_1, \end{cases}$$

where c_1, c_2 are arbitrary constants.

Set

$$A := -\frac{(k_2-k_1)}{2\pi^2 r_1}\sum_{j=1}^{\infty} S_j\Big(\psi_{l-j}f_j + \psi_{l+j}f_{-j}\Big),$$

$$B := \frac{(k_2-k_1)k_2}{2\pi^2 r_1^2}\, l\sum_{j=1}^{\infty} S_j\Big(\psi_{l-j}f_l - \psi_{l+j}f_{-j}\Big).$$

Hence, by the transmission and boundary conditions on ∂B_1 and $\partial\Omega$, we have

$$\begin{cases} c_1 r_1^l + c_2 r_1^{-l} - U_l(r_1) = A, & \text{on } \partial B_1, \\ k_2\left(c_1 l r_1^{l-1} - c_2 l r_1^{-l-1}\right) - k_1 U_l(r_1)\frac{l}{r_1} = B, & \text{on } \partial B_1, \\ c_1 R^l + c_2 R^{-l} = 0, & \text{on } \partial\Omega, \end{cases}$$

that is,

$$\begin{cases} -(k_2-k_1)r_1^1 c_1 + (k_1+k_2)r_1^{-l}c_2 = k_1 A - \frac{r_1}{l}B, & \text{on } \partial B_1, \\ R^l c_1 + R^{-l}c_2 = 0, & \text{on } \partial\Omega, \end{cases}$$

which gives

$$\begin{cases} c_1 = \frac{T_l}{l}R^{-l}\Big(k_1 A - \frac{r_1}{l}B\Big), \\ c_2 = \frac{T_l}{l}\Big(-R^l\Big)\Big(k_1 A - \frac{r_1}{l}B\Big). \end{cases}$$

Hence,

$$U_l(\rho) = \frac{T_l}{l}\Big(k_1 A - \frac{r_1}{l}B\Big)\Big(R^{-l}\rho^l - R^l\rho^{-l}\Big), \quad r_1 < \rho < R.$$

Since

$$k_1 A - \frac{r_1}{l}B = \frac{(k_2-k_1)}{2\pi^2 r_1}\sum_{j=1}^{\infty} S_j\Big(-(k_1+k_2)\psi_{l-j}f_j - (k_1-k_2)\psi_{l+j}f_{-j}\Big),$$

it follows that for $r_1 < \rho < R$,

$$\sum_{l=1}^{\infty} U_l(\rho) e^{\mathrm{i}l\theta}$$
$$= \frac{(k_2-k_1)}{2\pi^2 r_1} \sum_{l=1}^{\infty} \frac{T_l}{l} \Big(R^l \rho^{-l} - R^{-l}\rho^l\Big) \cdot \sum_{p=1}^{\infty} S_p \Big((k_1+k_2)\psi_{l-p} f_p + (k_1-k_2)\psi_{l+p} f_{-p}\Big) e^{\mathrm{i}l\theta}.$$

Similarly, when $l<0$, we have, for $r_1 < \rho < R$,

$$\sum_{l=-1}^{-\infty} U_l(\rho) e^{\mathrm{i}l\theta}$$
$$= \frac{(k_2-k_1)}{2\pi^2 r_1} \sum_{l=-1}^{-\infty} \frac{T_{-l}}{l} (R^l \rho^{-l} - R^{-l}\rho^l) \cdot \sum_{p=1}^{\infty} S_p \Big((k_1+k_2)\psi_{l+p} f_{-p} - (k_2-k_1)\psi_{l-p} f_p\Big) e^{\mathrm{i}l\theta}$$
$$= \frac{(k_2-k_1)}{2\pi^2 r_1} \sum_{l=1}^{\infty} \frac{T_l}{l} (R^l \rho^{-l} - R^{-l}\rho^l) \cdot \sum_{p=1}^{\infty} S_p \Big((k_1+k_2)\psi_{-l+p} f_{-p} - (k_2-k_1)\psi_{-l-p} f_p\Big) e^{-\mathrm{i}l\theta}.$$

Thus, we obtain Lemma 2.2. □

Therefore, we have the series expression of $d\Lambda_{B_1}(\psi)$ thanks to Lemma 2.2, which is

$$d\Lambda_{B_1}(\psi)(f)(R\cos\theta, R\sin\theta) = \frac{\partial U}{\partial \rho}\Big|_{\partial\Omega} = \sum_{l\in\mathbb{Z}} \lambda_l e^{\mathrm{i}l\theta}, \quad \theta \in [0, 2\pi), \tag{18}$$

where $\lambda_0 := 0$ and for $l > 0$

$$\lambda_{-l} := \frac{k_1-k_2}{\pi^2} (Rr_1)^{-1} T_l \sum_{p=1}^{\infty} S_p \Big[(k_1+k_2)\psi_{-l+p} f_{-p} - (k_2-k_1)\psi_{-l-p} f_p\Big],$$
$$\lambda_l := \frac{k_1-k_2}{\pi^2} (Rr_1)^{-1} T_l \sum_{p=1}^{\infty} S_p \Big[(k_1+k_2)\psi_{l-p} f_p - (k_2-k_1)\psi_{l+p} f_{-p}\Big].$$

Then, with the help of the expression of $d\Lambda_{B_1}(\psi)$, we can estimate its norm. The method used in next lemma is similar to that of [15]. Lemma 2.3 and Corollary 1 below will not be used in the proof of Theorem 1.1. But it is interesting to see how the norm of the linearized Dirichlet-to-Neumann map depends on conductivities and radii.

Lemma 2.3. *The bounded operator $d\Lambda_{B_1}(\psi) : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ satisfies the following estimate:*

(19)
$$\|d\Lambda_{B_1}(\psi)(f)\|_{H^{-1/2}(\partial\Omega)} \le \frac{2^{5/2} C|k_1-k_2|}{(k_1+k_2)(k_2+1)\pi} \frac{(Rr_1)^{-1}}{(1-(r_1/R)^2)}$$
$$\times \left[\sum_{l=1}^{\infty} l \left(\frac{r_1}{R}\right)^{2l} \sum_{p=1}^{\infty} p \left(\frac{r_1}{R}\right)^{2p} (|\psi_{-l-p}|^2 + |\psi_{-l+p}|^2 + |\psi_{l-p}|^2 + |\psi_{l+p}|^2)\right]^{1/2}$$
$$\times \|f\|_{H^{1/2}(\partial(\Omega))},$$

where C is explicitly given in (20). Note that C remains bounded as $k_2 \to \infty$.

Proof. First by (9) we have

$$
\begin{aligned}
2l|S_l^{-1}| = &\Big| R^l r_2^{-l}\Big((k_2-k_1)(k_2-1)r_1^l r_2^{-l} - (k_1+k_2)(k_2+1)r_1^{-l}r_2^l\Big) \\
&+ R^{-l}r_2^l\Big(-(k_2-k_1)(k_2+1)r_1^l r_2^{-l} + (k_1+k_2)(k_2-1)r_1^{-l}r_2^l\Big)\Big| \\
= &(k_1+k_2)(k_2+1)\left(\frac{R}{r_1}\right)^l \\
&\times\left|1 - \frac{k_1-k_2}{k_1+k_2}\left(\frac{r_1}{R}\right)^{2l} + \frac{k_1-k_2}{k_1+k_2}\frac{k_2-1}{k_2+1}\left(\frac{r_1}{r_2}\right)^{2l} - \frac{k_2-1}{k_2+1}\left(\frac{r_2}{R}\right)^{2l}\right|.
\end{aligned}
$$

Noticed that

$$
\lim_{l\to+\infty}\left|1 - \frac{k_1-k_2}{k_1+k_2}\left(\frac{r_1}{R}\right)^{2l} + \frac{k_1-k_2}{k_1+k_2}\frac{k_2-1}{k_2+1}\left(\frac{r_1}{r_2}\right)^{2l} - \frac{k_2-1}{k_2+1}\left(\frac{r_2}{R}\right)^{2l}\right| = 1\,,
$$

which means that there exists a sufficiently large $l_0\in\mathbb{Z}^+$, such that for $l>l_0$,

$$
\left|1 - \frac{k_1-k_2}{k_1+k_2}\left(\frac{r_1}{R}\right)^{2l} + \frac{k_1-k_2}{k_1+k_2}\frac{k_2-1}{k_2+1}\left(\frac{r_1}{r_2}\right)^{2l} - \frac{k_2-1}{k_2+1}\left(\frac{r_2}{R}\right)^{2l}\right| > \frac{1}{2}\,.
$$

Let

$$
c_0 := \min_{1\le l\le l_0}\left|1 - \frac{k_1-k_2}{k_1+k_2}\left(\frac{r_1}{R}\right)^{2l} + \frac{k_1-k_2}{k_1+k_2}\frac{k_2-1}{k_2+1}\left(\frac{r_1}{r_2}\right)^{2l} - \frac{k_2-1}{k_2+1}\left(\frac{r_2}{R}\right)^{2l}\right|\,,
$$

and for $S_l^{-1}\neq 0$ for all $l>0$, we know $c_0>0$. Setting

$$
C := (\min\{1/2\,,\,c_0\})^{-1}, \tag{20}
$$

we obtain

$$
|S_l| \le \frac{2lC}{(k_1+k_2)(k_2+1)}\left(\frac{r_1}{R}\right)^l\,.
$$

Similarly, we can prove

$$
\begin{aligned}
|T_l| &= \frac{l}{k_1+k_2}\left(\frac{r_1}{R}\right)^l\frac{1}{1-((k_1-k_2)/(k_1+k_2))(r_1/R)^{2l}} \\
&\le \frac{l}{k_1+k_2}\frac{1}{1-(r_1/R)^2}\left(\frac{r_1}{R}\right)^l\,.
\end{aligned}
$$

So, it follows from (18) that

$$
|\lambda_{\pm l}| \le \frac{2C|k_1-k_2|}{(k_1+k_2)(k_2+1)\pi^2}\frac{(Rr_1)^{-1}l(r_1/R)^l}{(1-(r_1/R)^2)}\sum_{p=1}^{\infty}p\left(\frac{r_1}{R}\right)^p(|\psi_{\pm l-p}||f_p| + |\psi_{\pm l+p}||f_{-p}|),
$$

for any positive integer l. Hence, we have

$$
\begin{aligned}
\|d\Lambda_{B_1}(\psi)(f)\|^2_{H^{-1/2}(\partial\Omega)} &= 2\pi R\sum_{l\in\mathbb{Z}}(1+l^2)^{-1/2}|\lambda_l|^2 \le 2\pi R\sum_{l=1}^{\infty}l^{-1}(|\lambda_1| + |\lambda_{-l}|)^2 \\
&\le \frac{8C^2|k_1-k_2|^2}{(k_1+k_2)^2(k_2+1)^2\pi^3}\frac{R^{-1}r_1^{-2}}{(1-(r_1/R)^2)^2}\sum_{l=1}^{\infty}l\left(\frac{r_1}{R}\right)^{2l} \\
&\quad\times\left[\sum_{p=1}^{\infty}p\left(\frac{r_1}{R}\right)^p(|\psi_{-l-p}||f_p| + |\psi_{-l+p}||f_{-p}| + |\psi_{l-p}||f_p| + |\psi_{l+p}||f_{-p}|)\right]^2.
\end{aligned}
$$

By the Cauchy-Schwarz inequality, we then derive

$$\begin{aligned}
&\left[\sum_{p=1}^{\infty} p\left(\frac{r_1}{R}\right)^{p}(|\psi_{-l-p}||f_p|+|\psi_{-l+p}||f_{-p}|+|\psi_{l-p}||f_p|+|\psi_{l+p}||f_{-p}|)\right]^2\\
&\le \left[\sum_{p=1}^{\infty} p\left(\frac{r_1}{R}\right)^{2p}(|\psi_{-l-p}|^2+|\psi_{-l+p}|^2+|\psi_{l-p}|^2+|\psi_{l+p}|^2)\right]\\
&\quad\times\left[\sum_{p=1}^{\infty} p(|f_p|^2+|f_{-p}|^2+|f_p|^2+|f_{-p}|^2)\right]\\
&= \left[\sum_{p=1}^{\infty} p\left(\frac{r_1}{R}\right)^{2p}(|\psi_{-l-p}|^2+|\psi_{-l+p}|^2+|\psi_{l-p}|^2+|\psi_{l+p}|^2)\right] 2\sum_{p\in\mathbb{Z}}|p||f_p|^2\\
&\le \left[\sum_{p=1}^{\infty} p\left(\frac{r_1}{R}\right)^{2p}(|\psi_{-l-p}|^2+|\psi_{-l+p}|^2+|\psi_{l-p}|^2+|\psi_{l+p}|^2)\right]\frac{4\pi}{R}\|f\|^2_{H^{1/2}(\partial\Omega)}\,,
\end{aligned}$$

which concludes the proof of this lemma. □

Corollary 1. *We have the estimate*

$$\|d\Lambda_{B_1}(\psi)\|_{\mathscr{L}} \le \frac{16C|k_1-k_2|}{\pi^{1/2}(k_1+k_2)(k_2+1)}\frac{1}{(1-(r_1/R)^2)^3}r_1^{1/2}R^{-3}\|\psi\|_{L^2(\partial B_1)}\,,$$

where C is the constant obtained in Lemma *2.3.*

Proof. First note that

$$|\psi_l|^2 \le \sum_{l\in\mathbb{Z}}|\psi_l|^2 = \frac{2\pi}{r_1}\|\psi\|^2_{L^2(\partial B_1)} \quad \text{and} \quad \sum_{j=1}^{\infty} jt^j = \frac{t}{(1-t)^2}\,,\ \forall\ |t|<1\,.$$

Hence, it follows from (19) that

$$\begin{aligned}
\|d\Lambda_{B_1}(\psi)\|_{\mathscr{L}} &\le \frac{2^{5/2}C|k_1-k_2|}{(k_1+k_2)(k_2+1)\pi}\frac{(Rr_1)^{-1}}{(1-(r_1/R)^2)}2\sqrt{\frac{2\pi}{r_1}}\|\psi\|_{L^2(\partial B_1)}\sum_{j=1}^{\infty} j\left(\frac{r_1}{R}\right)^{2j}\\
&= \frac{16C|k_1-k_2|}{(k_1+k_2)(k_2+1)\sqrt{\pi}}\frac{(Rr_1)^{-1}r_1^{-1/2}}{(1-(r_1/R)^2)}\frac{r_1^2/R^2}{(1-(r_1/R)^2)^2}\|\psi\|_{L^2(\partial B_1)}\\
&= \frac{16C|k_1-k_2|}{(k_1+k_2)(k_2+1)\sqrt{\pi}}\frac{1}{(1-(r_1/R)^2)^3}r_1^{1/2}R^{-3}\|\psi\|_{L^2(\partial B_1)}\,,
\end{aligned}$$

and the corollary is established. □

2.3. **Proof of Theorem 1.1.** In this subsection, we will give a detailed proof of Theorem 1.1 based on the method in [15]. Firstly, we need the following two useful lemmas.

Lemma 2.4. *For any $j\in\mathbb{Z}$, we define $g_j := e^{\mathrm{i}j\theta}$ on $\partial\Omega$. Then*

$$\int_{\partial\Omega} d\Lambda_{B_1}(\psi)(g_{\pm l})g_{\pm p}dS = 4(k_2-k_1)^2 r_1^{-1}S_lT_p\psi_{\mp(l+p)}, \tag{21}$$

$$\int_{\partial\Omega} d\Lambda_{B_1}(\psi)(g_{\pm l})g_{\mp p}dS = -4(k_2-k_1)(k_2+k_1) r_1^{-1}S_lT_p\psi_{\mp(l-p)}, \tag{22}$$

where l,p are arbitrary integers.

Proof. We prove this lemma by straightforward calculations. We begin with the case of $f = g_l$, $g = g_p$. By (18), it yields

$$\int_{\partial\Omega} d\Lambda_{B_1}(\psi)(g_l)g_p dS = R\int_0^{2\pi} d\Lambda_{B_1}(\psi)(g_l)g_p d\theta = 2\pi R\lambda_{-p}.$$

Then, in view of the formula of λ_l, we have

$$\lambda_{-p} = \frac{2}{\pi}(k_2-k_1)^2(Rr_1)^{-1}S_lT_p\psi_{-(l+p)},$$

since here $f_l = \int_0^{2\pi} e^{\mathrm{i}l\theta}e^{-\mathrm{i}l\theta}d\theta = 2\pi$. In other words, we have derived

$$\int_{\partial\Omega} d\Lambda_{B_1}(\psi)(g_l)g_p dS = 2\pi R\lambda_{-p} = 4(k_2-k_1)^2 r_1^{-1}S_lT_p\psi_{-(l+p)}.$$

Next we consider the case where $f = g_{-l}, g = g_{-p}$. Similarly, by (18), we have that

$$\int_{\partial\Omega} d\Lambda_{B_1}(\psi)(g_{-l})g_{-p} dS = R\int_0^{2\pi} d\Lambda_{B_1}(\psi)(g_{-l})g_{-p} d\theta = 2\pi R\lambda_p,$$

and

$$\lambda_p = \frac{2}{\pi}(k_2-k_1)^2(Rr_1)^{-1}S_lT_p\psi_{l+p},$$

which immediately implies

$$\int_{\partial\Omega} d\Lambda_{B_1}(\psi)(g_{-l})g_{-p} dS = 2\pi R\lambda_p = 4(k_2-k_1)^2 r_1^{-1}S_lT_p\psi_{l+p}.$$

In conclusion, we have established (21). The relation (22) can be derived similarly. □

Now we turn to the estimates of ψ_j's.

Lemma 2.5. *We can show that*

$$|\psi_0| \le C(k_1,k_2)\frac{R^3}{r_1}\|d\Lambda_{B_1}(\psi)\|_{\mathscr{L}}, \quad |\psi_{\pm1}| \le C(k_1,k_2)\frac{R^4}{r_1^2}\|d\Lambda_{B_1}(\psi)\|_{\mathscr{L}}$$

and

$$|\psi_{\pm l}| \le \frac{C(k_1,k_2)}{l}\frac{R^{l+1}}{r_1^{l-1}}\|d\Lambda_{B_1}(\psi)\|_{\mathscr{L}}\,, \quad \forall\, l\ge 2\,,$$

where

$$C(k_1,k_2) = \frac{4\times 10^{\frac14}\pi(k_2+k_1)^2(k_2+1)}{(k_2-k_1)^2}. \tag{23}$$

Proof. For any integer j, we define $g_j := e^{\mathrm{i}j\theta}$ on $\partial\Omega$. Note that

$$\|g_{\pm l}\|_{H^{1/2}(\partial\Omega)} = (1+l^2)^{\frac14}(2\pi R)^{\frac12}, \quad \forall\ l\ge 1.$$

By the duality property, we have

$$\begin{aligned}\left|\int_{\partial\Omega} d\Lambda_{B_1}(\psi)(g_j)g_{j'}dS\right| &\le \|d\Lambda_{B_1}(\psi)(g_j)\|_{H^{-1/2}(\partial\Omega)}\|g_{j'}\|_{H^{1/2}(\partial\Omega)}\\ &\le \|d\Lambda_{B_1}(\psi)\|_{\mathscr{L}}\|g_j\|_{H^{1/2}(\partial\Omega)}\|g_{j'}\|_{H^{1/2}(\partial\Omega)}\\ &= 2\pi(1+j^2)^{\frac14}(1+j'^2)^{\frac14}R\|d\Lambda_{B_1}(\psi)\|_{\mathscr{L}},\end{aligned}$$

where $j, j' \neq 0$ are arbitrary integers. Moreover, we can estimate

$$\begin{aligned}\frac{2}{|S_l|} \leq \frac{(k_1+k_2)(k_2+1)}{l}\left(\frac{R}{r_1}\right)^l \Big(&\frac{|k_1-k_2|}{k_1+k_2}\frac{|k_2-1|}{k_2+1}\Big(\frac{r_1}{r_2}\Big)^{2l}+1 \\ &+\frac{|k_1-k_2|}{k_1+k_2}\Big(\frac{r_1}{R}\Big)^{2l}+\frac{|k_2-1|}{k_2+1}\Big(\frac{r_2}{R}\Big)^{2l}\Big) \\ \leq \frac{4(k_1+k_2)(k_2+1)}{l}\left(\frac{R}{r_1}\right)^l,&\end{aligned}$$

which gives us

$$\frac{1}{|S_l|} \leq \frac{2(k_1+k_2)(k_2+1)}{l}\left(\frac{R}{r_1}\right)^l, \tag{24}$$

and

$$\frac{1}{|T_l|} = \frac{k_1+k_2}{l}\left(\frac{R}{r_1}\right)^l\left[1-\frac{k_1-k_2}{k_1+k_2}\Big(\frac{r_1}{R}\Big)^{2l}\right] \leq \frac{2(k_1+k_2)}{l}\left(\frac{R}{r_1}\right)^l. \tag{25}$$

Then, in (22), we take $l = p = 1$ and obtain

$$\begin{aligned}|\psi_0| &= \frac{r_1}{4|k_2-k_1|(k_2+k_1)}\frac{1}{S_1T_1}\left|\int_{\partial\Omega} d\Lambda_{B_1}(\psi)(g_1)g_{-1}dS\right| \\ &\leq \frac{2\sqrt{2}\pi(k_1+k_2)(k_2+1)}{|k_1-k_2|}\frac{R^3}{r_1}\|d\Lambda_{B_1}(\psi)\|_{\mathscr{L}}.\end{aligned}$$

Similarly, in (22), choosing $l = 2, p = 1$ gives

$$\begin{aligned}|\psi_{\pm1}| &= \frac{r_1}{4|k_2-k_1|(k_2+k_1)}\frac{1}{T_1S_2}\left|\int_{\partial\Omega} d\Lambda_{B_1}(\psi)(g_{\pm2})g_{\mp1}dS\right| \\ &\leq \frac{10^{\frac{1}{4}}\pi(k_1+k_2)(k_2+1)}{|k_1-k_2|}\frac{R^4}{r_1^2}\|d\Lambda_{B_1}(\psi)\|_{\mathscr{L}}.\end{aligned}$$

Next, letting $l = p \geq 1$ in (21), we have

$$\begin{aligned}|\psi_{\mp2l}| &= \frac{r_1}{4(k_2-k_1)^2}\frac{1}{S_lT_l}\left|\int_{\partial\Omega} d\Lambda_{B_1}(\psi)(g_{\pm l})g_{\pm l}dS\right| \\ &\leq \frac{2\pi(k_2+k_1)^2(k_2+1)}{(k_2-k_1)^2}\frac{(1+l^2)^{\frac{1}{2}}}{l^2}\frac{R^{2l+1}}{r_1^{2l-1}}\|d\Lambda_{B_1}(\psi)\|_{\mathscr{L}} \\ &\leq \frac{4\sqrt{2}\pi(k_2+k_1)^2(k_2+1)}{(k_2-k_1)^2}\frac{1}{2l}\frac{R^{2l+1}}{r_1^{2l-1}}\|d\Lambda_{B_1}(\psi)\|_{\mathscr{L}}.\end{aligned}$$

Likewise, taking $l \geq 1, p = l+1$ in (21) yields

$$\begin{aligned}|\psi_{\mp(2l+1)}| &= \frac{r_1}{4(k_2-k_1)^2}\frac{1}{S_lT_{l+1}}\left|\int_{\partial\Omega} d\Lambda_{B_1}(\psi)(g_{\pm l})g_{\pm(l+1)}dS\right| \\ &\leq \frac{2\pi(k_2+k_1)^2(k_2+1)}{(k_2-k_1)^2}\frac{(1+l^2)^{\frac{1}{4}}(1+(l+1)^2)^{\frac{1}{4}}}{l(l+1)}\frac{R^{(2l+1)+1}}{r_1^{(2l+1)-1}}\|d\Lambda_{B_1}(\psi)\|_{\mathscr{L}} \\ &\leq \frac{4\times10^{\frac{1}{4}}\pi(k_2+k_1)^2(k_2+1)}{(k_2-k_1)^2}\frac{1}{2(l+1)}\frac{R^{(2l+1)+1}}{r_1^{(2l+1)-1}}\|d\Lambda_{B_1}(\psi)\|_{\mathscr{L}} \\ &\leq \frac{4\times10^{\frac{1}{4}}\pi(k_2+k_1)^2(k_2+1)}{(k_2-k_1)^2}\frac{1}{2l+1}\frac{R^{(2l+1)+1}}{r_1^{(2l+1)-1}}\|d\Lambda_{B_1}(\psi)\|_{\mathscr{L}}.\end{aligned}$$

Hence, we can choose

$$
\begin{aligned}
C(k_1,k_2) = \max\Bigg\{ & \frac{2\sqrt{2}\pi(k_1+k_2)(k_2+1)}{|k_1-k_2|}, \frac{10^{\frac{1}{4}}\pi(k_1+k_2)(k_2+1)}{|k_1-k_2|}, \\
& \frac{4\sqrt{2}\pi(k_2+k_1)^2(k_2+1)}{(k_2-k_1)^2}, \frac{4\times 10^{\frac{1}{4}}\pi(k_2+k_1)^2(k_2+1)}{(k_2-k_1)^2} \Bigg\} \\
= & \frac{4\times 10^{\frac{1}{4}}\pi(k_2+k_1)^2(k_2+1)}{(k_2-k_1)^2},
\end{aligned}
$$

where $\sqrt{2}=4^{\frac{1}{4}}<10^{\frac{1}{4}}$ and $|k_2^2-k_1^2|<(k_2+k_1)^2$ used. The proof is completed. □

By lemmas established above, we are ready to prove Theorem 1.1 similar to [15].

Proof of Theorem 1.1. Repeating the argument used in the proof of [15, Theorem 1.1], we can prove this theorem. For readers' convenience, we give a sketch of the argument here. We first consider the case that $A := \|d\Lambda_{B_1}(\psi)\|_{\mathscr{L}}^2$ is sufficiently small. In this situation, combining Lemma 2.5, estimate $\sum_{l\in\mathbb{Z}}\left(1+l^2\right)^m|\psi_l|^2 \le \frac{2\pi}{r_1}M^2$, and following the computation on [15, page 10], we can derive

$$
\sum_{l\in\mathbb{Z}}|\psi_l|^2 \le F(t) := 5C(k_1,k_2)^2 r_1^4 (R/r_1)^{2\left[\left(2\pi M^2/r_1\right)^{1/2m} t^{-1/2m}+1\right]} A + t, \tag{26}
$$

where $0<t<2\times 3^{-2m}\pi M^2 r_1^{-1}$. Furthermore, one can show that there exists t_0 satisfying $0<t_0<2\times 3^{-2m}\pi M^2 r_1^{-1}$ such that

$$
F(t_0) \le C_2 M^2 r_1^{-1}(\log(R/r_1))^{2m}(-\log A)^{-2m},
$$

where C_2 is explicitly given by

$$
C_2 = 5C(k_1,k_2)^2 + 2^{6m+1}\pi
$$

(see [15, (3.6)]). Next, it follows immediately from (26) that

$$
\begin{aligned}
\|\psi\|_{L^2(\partial B_1)} = \left(\frac{r_1}{2\pi}\sum_{l\in\mathbb{Z}}|\psi_l|^2\right)^{1/2} & \le \left(\frac{r_1}{2\pi}F(t_0)\right)^{1/2} \\
& \le \left(\frac{C_2}{2\pi}\right)^{1/2} M \left(\ln\frac{R}{r_1}\right)^m (-\ln A)^{-m}
\end{aligned}
$$

for $0<A<\min\{A_0,1\}$, where A_0 is the same constant obtained in [15, page 11]. On the other hand, in the case $A_0\le A<1$, we can use the formula of A_0 (see [15, page 11]) to derive that

$$
\begin{aligned}
\|\psi\|_{L^2(\partial B_1)} & \le \|\psi\|_{H^m(\partial B_1)} \le M \\
& \le (-\ln A_0)^m M(-\ln A)^{-m} \le C_3^m M\left(\ln\frac{R}{r_1}\right)^m(-\ln A)^{-m},
\end{aligned}
$$

where $C_3=(-\ln c)/(\ln X_0)+16$. Recall that X_0 is the constant satisfying $X_0\le R/r_1$. We thus obtain the estimate (3).

Now we would like to pay attention to how the constant C in (3) is estimated. Note that

$$
C = \max\left\{(C_2/2\pi)^{1/2}, {C_3}^m\right\}
$$

with

$$c := \min\left\{ \left[\left(\frac{e}{8m}\right)^{2m} M_0^2 r_0^{-5} (\ln X_0)^{2m} \right]^2, \frac{1}{2} \right\}, \tag{27}$$

and $C(k_1, k_2)$ is given in (23). From (23), we see that

$$C_2 = \frac{80 \times 10^{1/2} \pi^2 (k_2 + k_1)^4 (k_2 + 1)^2}{(k_2 - k_1)^4} + 2^{6m+1}\pi.$$

Now, if

$$k_2 \geq \max\left\{ 2^{3m+\frac{1}{2}} \pi^{-\frac{1}{2}}, \Big((-\ln c)/(\ln X_0) + 16 \Big)^m \right\} =: \bar{k},$$

then

$$C_3{}^m \leq (k_2 + 1) < (C_2/2\pi)^{1/2} \quad \text{and} \quad C_2 \leq 2\pi^2 \cdot \frac{80 \times 10^{1/2} (k_2 + k_1)^4}{(k_2 - k_1)^4} (k_2 + 1)^2.$$

Hence, we can obtain

$$C \leq \frac{4 \times 5^{\frac{1}{2}} 10^{\frac{1}{4}} \pi^{\frac{1}{2}} (k_2 + k_1)^2}{(k_2 - k_1)^2} (k_2 + 1).$$

Since $\frac{(k_2+k_1)^2}{(k_2-k_1)^2}$ is monotonically decreasing in k_2, we immediately conclude that

$$C \leq \frac{4 \times 5^{\frac{1}{2}} 10^{\frac{1}{4}} \pi^{\frac{1}{2}} (\bar{k} + k_1)^2}{(\bar{k} - k_1)^2} (k_2 + 1) \quad \text{for} \;\; k_2 > \bar{k}\,.$$

$\square$

Remark 5. If there is no lower bound assumption for k_2 in the proof of Theorem 1.1, we can let $k_2 = 1$ and, therefore, $r_2 = R$. We observe that the solution in Lemma 2.1 is the same as in the two-layer case [15]. Estimate (24) becomes

$$\frac{1}{|S_l|} \leq \frac{2(k_1 + 1)}{l} \left(\frac{R}{r_1}\right)^l,$$

which implies that $C(k_1, k_2)$ is same as the estimate in [15]. In other words, in this situation we obtain the same estimates as those in [15].

3. **The case of n-layer inclusions.** In this section, we set $\Omega := B_R(0)$, $B_i := B_{r_i}(0)$, $0 < r_1 < r_2 < \cdots < r_n < R$,

$$L_{B_1, B_2, \cdots, B_n} u_0 := \nabla \cdot ((k_1 \chi_{B_1} + k_2 \chi_{B_2 \setminus B_1} + \cdots + k_n \chi_{B_n \setminus B_{n-1}} + \chi_{B_R \setminus B_n}) \nabla u_0),$$

where $k_i > 0$, and $k_i \neq 1$, $i = 1, 2, \cdots, n$. Now the Dirichlet boundary problem is

$$\begin{cases} L_{B_1, B_2, \cdots, B_n} u_0 = 0 & \text{in } \Omega, \\ u_0 = f & \text{on } \partial\Omega. \end{cases} \tag{28}$$

From the method used in Section 2, we know that the key step is to evaluate the determinant of the matrix for the linear system derived from the method of Fourier series, i.e.

$$
\left\{
\begin{array}{rcl}
R^l x_1 + R^{-l} x_2 & = & \frac{1}{2\pi} f_l, \\
r_n^l x_1 + r_n^{-l} x_2 - r_n^l x_3 - r_n^{-l} x_4 & = & 0, \\
k_{n+1} r_n^l x_1 - k_{n+1} r_n^{-l} x_2 - k_n r_n^l x_3 + k_n r_n^{-l} x_4 & = & 0, \\
r_{n-1}^l x_3 + r_{n-1}^{-l} x_4 - r_{n-1}^l x_5 - r_{n-1}^{-l} x_6 & = & 0, \\
k_n r_{n-1}^l x_3 - k_n r_{n-1}^{-l} x_4 - k_{n-1} r_{n-1}^l x_5 + k_{n-1} r_{n-1}^{-l} x_6 & = & 0, \\
& \vdots & \\
-(k_2 - k_1) r_1^l x_{2n-1} + (k_1 + k_2) r_1^{-l} x_{2n} & = & 0,
\end{array}
\right.
\tag{29}
$$

where we choose $k_{n+1} = 1$. Denoting the determinant in (29) by $D_n(R^l, R^{-l})$, that is,

$$
D_n(R^l, R^{-l}) :=
\begin{vmatrix}
R^l & R^{-l} & 0 & 0 & \cdots & 0 & 0 \\
r_n^l & r_n^{-l} & -r_n{}^l & -r_n^{-l} & \cdots & 0 & 0 \\
k_{n+1} r_n^l & -k_{n+1} r_n^{-l} & -k_n r_n^l & k_n r_n^{-l} & \cdots & 0 & 0 \\
0 & 0 & r_{n-1}^l & r_{n-1}^{-l} & \cdots & 0 & 0 \\
0 & 0 & k_n r_{n-1}^l & -k_n r_{n-1}^{-l} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & -r_2^l & -r_2^{-l} \\
0 & 0 & 0 & 0 & \cdots & -k_2 r_2^l & k_2 r_2^{-l} \\
0 & 0 & 0 & 0 & \cdots & -(k_2 - k_1) r_1^l & (k_1 + k_2) r_1^{-l}
\end{vmatrix}.
$$

Then we can show that

Theorem 3.1. *The determinant $D_n(R^l, R^{-l})$ satisfies the following bound:*

$$
|D_n(R^l, R^{-l})| \le 2^n (k_1 + k_2)(k_2 + k_3) \cdots (k_n + k_{n+1}) X^l,
\tag{30}
$$

where $X := \frac{R}{r_1}$.

Proof. We prove (30) by induction. Firstly, calculating D_1, D_2, D_3 directly, we have

$$
D_1(R^l, R^{-l}) = (k_1 + k_2) R^l r_1^{-l} (k_2 - k_1) R^{-l} r_1^l,
$$

$$
\begin{aligned}
& D_2(R^l, R^{-l}) \\
&= R^l r_2^{-l} \Big(-(k_2 + k_3) r_2^l (k_1 + k_2) r_1^{-l} - (k_2 - k_3) r_2^{-l} - (k_2 - k_1) r_1^l \Big) \\
&- R^{-l} r_2^l \Big(-(k_2 - k_3) r_2^l (k_1 + k_2) r_1^{-l} - (k_2 + k_3) r_2^{-l} - (k_2 - k_1) r_1^l \Big) \\
&= R^l r_2^{-l} D_1(-(k_2 + k_3) r_2^l, (k_2 - k_3) r_2^{-l}) - R^{-l} r_2^l D_1(-(k_2 - k_3) r_2^l, (k_2 + k_3) r_2^{-l}),
\end{aligned}
$$

$$
\begin{aligned}
&D_3(R^l, R^{-l})\\
&= R^l r_3^{-l}\Big(- (k_3+k_4) r_3^l r_2^{-l}\Big(- (k_2+k_3)(k_1+k_4) r_2^l r_1^{-l} - (k_2-k_3) - (k_2-k_1) r_2^{-l} r_1^l\Big)\\
&\qquad\qquad - (k_3-k_4) r_3^{-l} r_2^l \Big(- (k_2-k_3)(k_1+k_2) r_2^l r_1^{-l} - (k_2+k_3) - (k_2-k_1) r_2^{-l} r_1^l\Big)\Big)\\
&- R^{-l} r_3^l\Big(- (k_3-k_4) r_3^l r_2^{-l}\Big(- (k_2+k_3)(k_1+k_2) r_2^l r_1^{-l} - (k_2-k_3) - (k_2-k_1) r_2^{-l} r_1^l\Big)\\
&\qquad\qquad - (k_3+k_4) r_3^{-l} r_2^l \Big(- (k_2-k_3)(k_1+k_2) r_2^l r_1^{-l} - (k_2+k_3) - (k_2-k_1) r_2^{-l} r_1^l\Big)\Big)\\
&= R^l r_3^{-l} D_2(-(k_3+k_4) r_3^l, (k_3-k_4) r_3^{-l}) - R^{-l} r_3^l D_2(-(k_3-k_4) r_3^l, (k_3+k_4) r_3^{-l}).
\end{aligned}
$$

In general, we can show the recurrence formula:

$$
\begin{aligned}
D_n(R^l, R^{-l}) =& R^l r_n^{-l} D_{n-1}(-(k_n+k_{n+1}) r_n^l, (k_n-k_{n+1}) r_n^{-l})\\
&- R^{-l} {r_n}^l D_{n-1}(-(k_n-k_{n+1}) r_n^l, (k_n+k_{n+1}) r_n^{-l}).
\end{aligned} \tag{31}
$$

To establish (30), it is easy to see that

$$
|D_1| = (k_1+k_2) X^l \Big[1 - \frac{k_1-k_2}{k_1+k_2}\Big(\frac{1}{X^{2l}}\Big)\Big] \le 2(k_1+k_2) X^l.
$$

For D_2, we have

$$
\begin{aligned}
|D_2| &= \Big| R^l r_2^{-l} D_1(-(k_2+k_3) r_2^l, (k_2-k_3) r_2^{-l}) - R^{-l} r_2^l D_1(-(k_2-k_3) r_2^l, (k_2+k_3) r_2^{-l})\Big|\\
&\le \Big| R^l r_2^{-l} D_1(-(k_2+k_3) r_2^l, (k_2-k_3) r_2^{-l})\Big| + \Big| R^{-l} r_2^l D_1(-(k_2-k_3) r_2^l, (k_2+k_3) r_2^{-l})\Big|\\
&\le R^l r_2^{-l} \cdot 2(k_1+k_2) \frac{(k_2+k_3) r_2^l}{r_1^l} + R^{-l} r_2^l \cdot 2(k_1+k_2) \frac{(k_2+k_3) r_2^{-l}}{r_1^l}\\
&\le 4(k_1+k_2)(k_2+k_3) X^l.
\end{aligned}
$$

By induction and using (31), we obtain

$$
|D_n| \le 2^n (k_1+k_2)(k_2+k_3)\cdots(k_n+k_{n+1}) X^l,
$$

and hence the proof. □

Proof of Theorem 1.2. Combining (30) with $k_{n+1} = 1$ and the definition of S_l in (9) gives

$$
\frac{1}{|S_l|} = \frac{|D_n|}{4l} \le \frac{2^{n-2}(k_1+k_2)(k_2+k_3)\cdots(k_n+1) X^l}{l},
$$

which is used to replace (24). Estimate (25) remains unchanged. Following the proof of Lemma 2.5, we can derive similar estimates as in Lemma 2.5 except that $C(k_1, k_2)$ is replaced by

$$
C(k_1, k_2, \cdots, k_n) = \frac{C_n (k_2+k_1)^2 (k_3+k_2)\cdots(k_n+1)}{(k_2-k_1)^2},
$$

where C_n is a constant number depends only on n. The rest of the proof of Theorem 1.2 is similar to that of Theorem 1.1. The only modification is that C_2 now is given by $C_2 = 5C(k_1, k_2, \cdots, k_n)^2 + 2^{6m+1}\pi$. Observe that in the proof of Theorem 1.1, we estimate C_2, C_3 in terms of k_2+1. Here, k_2+1 is replaced by $(k_3+k_2)\cdots(k_n+1)$. □

4. **Conclusion.** In this paper, we establish the stability estimates of the linearized problems for the inverse three-layer and n-layer inclusions problems. Our first result is the stability estimate for the case of three-layer inclusions. The estimate demonstrates that the stability deteriorates when the unknown inclusion is hidden deeply inside the domain or the conductivity outside of the inclusion is large. Following the same strategy, we also prove a similar estimate for the case of n-layer inclusions in which the explicit dependence of constant on conductivities k_j's is derived. The estimate shows the phenomenon of deteriorating stability when the conductivity of any layer outside of the unknown inclusion increases. Moreover, in our estimates, like the main conclusion of [15], the influence of the depth of the unknown inclusion in the stability estimate is also observed.

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