LIOUVILLE THEOREMS FOR A CLASS OF DEGENERATE OR SINGULAR MONGE-AMPÈRE EQUATIONS

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Abstract. In this note, we classify solutions to a class of Monge-Ampère equations whose right hand side may be degenerate or singular in the half space. Solutions to these equations are special solutions to a class of fourth order equations, including the affine maximal hypersurface equation, in the half space. Both the Dirichlet boundary value and Neumann boundary value cases are considered.

1. Introduction

The main purpose of this paper is to investigate Liouville theorems for the following class of Monge-Ampère equations

$$\det D^2 u = (a + bx_n)^\alpha, \quad \alpha \in \mathbb{R} \tag{1.1}$$

in the half space $\mathbb{R}^n_+ := \{(x', x_n) \in \mathbb{R}^n : x_n > 0\}$, where $a \geq 0$ and $b > 0$. A motivation to consider (1.1) comes from the study of the following class of fourth order equations

$$U^{ij} w_{ij} = 0, \tag{1.2}$$

where (U^{ij}) denotes the cofactor matrix of (u_{ij}) and $w = (\det D^2 u)^{-\theta}$, $\theta \in \mathbb{R}$ $(\theta \neq 0)$. In particular, when $\theta = \frac{n+1}{n+2}$, it is the *affine mean curvature equation* in affine geometry [Ch] and when $\theta = 1$, it is *Abreu's equation* [Ab]. A first breakthrough for the study of this class of equations is the Chern conjecture, also known as the *affine Bernstein theorem* now, solved by Trudinger-Wang [TW1], which says an entire strictly, uniformly convex solution to (1.2) on $\mathbb{R}^2$ when $\theta = \frac{3}{4}$ must be a quadratic polynomial. Later, it is shown that the Bernstein theorem also holds when $\frac{3}{4} < \theta \leq 1$ [JL, Z] and $\theta < 0$ [TW2]. One of our motivations of the study on (1.1) is for the affine maximal hypersurfaces in the half space. In a subsequent work, we are going to study the Liouville type theorem (1.2) under the boundary condition

$$\begin{cases} u = \dfrac{1}{2}|x'|^2 & \text{on } \partial\mathbb{R}^n_+, \\ w = 1 & \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{1.3}$$

2020 *Mathematics Subject Classification.* 35J96, 35J70, 35B53, 35A09.

Key words and phrases. degenerate Monge-Ampère equation, Liouville theorems, partial Legendre transform, method of moving spheres.

This research is partially supported by NSFC grants 12271008 and National Key R&D Program of China SQ2020YFA0712800.

One can easily find solutions that are not quadratic polynomials. In particular, solutions to (1.1) with $\alpha = -\frac{1}{\theta}$, $a = b = 1$ give a class of special solutions to (1.2), (1.3). The classification of all solutions to (1.1) can help us to study (1.2), (1.3). We also expect this classification will be useful in the blow-up analysis in the investigation of boundary regularity of the affine maximal surface equation as well as the Monge-Ampère equation [S1, TW3, JT2].

When $a = 0$, (1.1) may be degenerate ($\alpha > 0$) or singular ($\alpha < 0$) on $\partial\mathbb{R}^n_+$. When $\alpha \geq 0$, Savin [S1, S2] proved that if the Dirichlet boundary value $u(x', 0) = \frac{1}{2}|x'|^2$ is assigned, any convex continuous solution to (1.1) with the growth condition $u = O(|x|^{3+\alpha-\varepsilon})$ as $|x| \to +\infty$ must be the form of

$$u(Ax) = Bx_n + \frac{1}{2}|x'|^2 + \frac{x_n^{2+\alpha}}{(2+\alpha)(1+\alpha)}$$

for some sliding A along $x_n = 0$, and some constant B. In particular, when $\alpha = 0$, the solution is a quadratic polynomial. This result was later extended to the singular case with $\alpha \in (-1, 0)$ by Savin and Zhang [SZ]. There are examples show that the growth condition at infinity is necessary in general dimensions. When $\alpha = -1$, the local asymptotic behavior of the solution near the boundary in dimension two was studied in [Ru].

In this paper, we concentrate on the two dimensional case. Our first result classifies all solutions to (1.1) with Dirichlet condition in dimension two when $\alpha > -2$.

Theorem 1.1. *Let $u(x, y) \in C^2(\mathbb{R}^2_+) \cap C(\overline{\mathbb{R}^2_+})$ be a convex solution to*

$$\begin{cases} \det D^2 u = (a + by)^\alpha & \text{in } \mathbb{R}^2_+, \\ u(x, 0) = \dfrac{1}{2}x^2 & \text{on } \partial\mathbb{R}^2_+, \end{cases} \tag{1.4}$$

where $a \geq 0$, $b > 0$, and $\alpha > -2$. Then there exist A, B, $C \in \mathbb{R}$ with $A \geq 0$ such that

$$u(x, y) = \begin{cases} \dfrac{(b - aA)(a + by)^{2+\alpha}}{b^3(1+\alpha)(2+\alpha)} + \dfrac{A(a + by)^{3+\alpha}}{b^3(2+\alpha)(3+\alpha)} - By \\ \qquad - \dfrac{(b - aA)a^{2+\alpha}}{b^3(1+\alpha)(2+\alpha)} - \dfrac{Aa^{3+\alpha}}{b^3(2+\alpha)(3+\alpha)} + \dfrac{(x - Cy)^2}{2(1 + Ay)}, & \alpha \neq -1; \\ \dfrac{b - aA}{b^3}(a + by)\ln(a + by) + \dfrac{A}{2b}y^2 - By \\ \qquad - \dfrac{(b - aA)a\ln a}{b^3} + \dfrac{(x - Cy)^2}{2(1 + Ay)}, & \alpha = -1. \end{cases} \tag{1.5}$$

Remark 1.2.

(1) *When $a = 0$, we improve the exponent in the results of* [S1, S2, SZ] *to $\alpha > -2$ in two dimensional case. This exponent is sharp since* (1.4) *admits no solutions continuous up to the boundary for $\alpha \leq -2$ (see details in Remark 3.2). When $\alpha = 0$, Theorem 1.1 can be also found in* [Fi, Page 145-148].

(2) *If we assume* $u = O(|(x,y)|^{3+\alpha-\varepsilon})$ *as* $|(x,y)| \to +\infty$, *then we have that* A *must be* 0 *in* (1.5). *Hence we can recover some of the results in* [S2, SZ] *in dimension two.*

The main idea to prove Theorem 1.1 is as follows. Let $u(x,y)$ be a strictly convex C^2 solution to (1.1). Then its *partial Legendre transform* (see definition in Section 3) in the x-variable is

$$u^\star(\xi,\eta) = xu_x(x,y) - u(x,y), \tag{1.6}$$

where $(\xi,\eta) = (u_x, y)$. It is easy to check that $u^\star$ is a solution to

$$(a+b\eta)^\alpha u^\star_{\xi\xi} + u^\star_{\eta\eta} = 0. \tag{1.7}$$

When $a = 0$, this Grushin type equation was studied in [CS]. By a change of variables $v(x_1,x_2) = u^\star(x_1, f(x_2))$, where

$$\xi = x_1,\ \eta = f(x_2) = b^{\frac{-\alpha}{\alpha+2}} \left(\frac{\alpha+2}{2} x_2\right)^{\frac{2}{\alpha+2}} - \frac{a}{b},$$

we know that v solves the following divergence type equation

$$\operatorname{div}\left(x_2^{\frac{\alpha}{\alpha+2}} \nabla v\right) = 0, \tag{1.8}$$

which may be degenerate or singular. A Liouville theorem for (1.8) on the upper half space has been obtained recently by [WZ]. However, in our case, the domain may shift after the transformations. Hence, we need to extend the result in [WZ] to general upper half spaces. In this situation, we may consider the solution of (1.8) in the weak sense. A result of [YD] will also be used in the proof of our extension.

The above approach also works for the case of Neumann problem. Recently, Jhaveri and Savin [JS] obtained a Liouville theorem for the following degenerate Monge-Ampère equation with Neumann boundary value

$$\det D^2 u = \frac{x_n^\alpha}{u_n^\beta} \quad \text{in} \quad \{x_n > 0\} \quad \text{and} \quad u_n = 0 \text{ on } \{x_0 = 0\}$$

when they investigate the regularity of optimal transports, where $\alpha, \beta \geq 0$. They show that that any convex solution (viscosity, Alexandrov, and Brenier solutions are equivalent in this case) must be

$$u(x) = p_0 + p' \cdot x' + P'x' \cdot x' + p_n x_n^{1+\frac{\alpha+1}{\beta+1}}$$

for some $p_0 \in \mathbb{R}$, $p' \in \mathbb{R}^{n-1}$, positive definite matrix P', and constant $p_n > 0$. The case of $\alpha = \beta = 0$ is also included in [JT1]. Next, we partially extend these results in dimension two, but with a more general α, i.e. $\alpha > -1$.

Theorem 1.3. *Let* $u(x,y) \in C^2(\mathbb{R}^2_+) \cap C^1(\overline{\mathbb{R}^2_+})$ *be a convex solution to*

$$\begin{cases} \det D^2 u = y^\alpha & \text{in } \mathbb{R}^2_+, \\ u_y(x,0) = 0 & \text{on } \partial\mathbb{R}^2_+, \end{cases} \tag{1.9}$$

where $\alpha > -1$. *Then there exist a constant* $A > 0$, *and a linear function* $l(x)$ *such that*

$$u(x,y) = \frac{1}{2A}x^2 + \frac{A}{(2+\alpha)(1+\alpha)}y^{2+\alpha} + l(x). \tag{1.10}$$

Remark 1.4.

(1) *When* $\alpha = 0$, *Theorem 1.3 is included in* [JT1, Theorem 1.1]. *In fact, it is proved in* [JT1] *that any convex solution to Neumann problem of Monge-Ampère equations in the half plane must be a quadratic polynomial for two dimensional case, and the conclusion still holds for dimension* $n \geq 3$ *if either the boundary value is zero or the solution restricted on some* $n-2$ *dimensional subspace is bounded from above by a quadratic function. Here we extend this to the degenerate case.*

(2) *The same conclusions in Theorem 1.1 and Theorem 1.3 are also true for Alexandrov or viscosity solutions, it suffices to prove first that those weak solutions should be classical, which use the similar arguments as in* [S2, SZ, JS, JT1, JX], *etc. But to make our ideas clearly, we only consider classical convex solutions in Theorem 1.1 and Theorem 1.3.*

Finally, we turn to the Liouville theorem on the whole space. The celebrated result of Jörgens [Jö], Calabi [Ca] and Pogorelov [Po] states that any entire classical convex solution to the Monge-Ampère equation

$$\det D^2 u = 1 \quad \text{in } \mathbb{R}^n$$

must be a quadratic polynomial. Caffarelli [Caf2] extended this result to viscosity solutions (the proof can be also found in [CL, Theorem 1.1]). For another direction of extension, Jin and Xiong [JX] studied the class of equations

$$\det D^2 u(x,y) = |y|^\alpha \tag{1.11}$$

on the whole plane $\mathbb{R}^2$, and established a Liouville theorem.

Theorem 1.5 ([JX, Theorem 1.1])**.** *Let* $u(x,y)$ *be convex generalized (or Alexandrov) solution to* (1.11) *with* $\alpha > -1$. *Then there exist some constants* $A > 0$, $B \in \mathbb{R}$ *and a linear function* $l(x,y)$ *such that*

$$u(x,y) = \frac{1}{2A}x^2 + \frac{AB^2}{2}y^2 + Bxy + \frac{A}{(2+\alpha)(1+\alpha)}|y|^{2+\alpha} + l(x,y). \tag{1.12}$$

At the end of this paper, we use the approach above to give a new proof of this result in the case of $\alpha \geq 0$. The main idea of Jin and Xiong in [JX] is that using the partial Legendre transform to change (1.11) into a class of linearized Monge-Ampère equations, then applying the Harnack inequality for linearized Monge-Ampère equations derived by Caffarelli and Gutiérrez [CG] and the scaling argument to classify all solutions of the transformed equation. Our new proof is similar to Theorem 1.1 and Theorem 1.3.

The structure of this paper is as follows. In Section 2, we derive the Liouville theorems for a class of linear elliptic equations in divergence form including (1.8). Then we prove Theorem 1.1, Theorem 1.3 and Theorem 1.5 in Section 3.

Acknowledgments. The authors would like to thank the referee for carefully reading the paper, providing constructive comments that help us to improve the exposition of the paper.

2. Liouville theorems for linear elliptic equations in divergence form

In this section, we establish a Liouville theorem for a class of linear elliptic equations in divergence form, which may be degenerate or singular cases, in the half space. This theorem can be viewed as an extension of [WZ, Theorem 1.1]. The proof is very similar to [WZ, Theorem 1.1], where the method of moving sphere will be used. Denote $\mathbb{R}^n_l = \{x = (x', x_n) : x' \in \mathbb{R}^{n-1},\, x_n > l\}$ for $l \geq 0$.

Theorem 2.1. *For $n \geq 2$ and $a \in \mathbb{R}$, let $u \in C^2(\mathbb{R}^n_l) \cap C^0(\overline{\mathbb{R}^n_l})$ be a solution to*

$$\begin{cases} \operatorname{div}(x_n^a \nabla u) = 0, \quad u > -C_0 \text{ in } \mathbb{R}^n_l, \\ u(x', l) = 0, \qquad\quad \text{on } \mathbb{R}^{n-1} \times \{x_n = l\}, \end{cases}$$

where $l \geq 0$ and $C_0 > 0$. Then $u = C_* \left(x_n^{1-a} - l^{1-a}\right)$ for some nonnegative constant C_*. In particular, when $a \geq 1$, $C_* = 0$.

Remark 2.2. *When $l = 0$, Theorem 2.1 is just the Theorem 1.1 of* [WZ].

Proof of Theorem 2.1. We extend u to $\overline{\mathbb{R}^n_+}$ by letting $u(x', x_n) = 0$ in $\mathbb{R}^{n-1} \times [0, l)$, and denote it by $\widetilde{u}$. Hence, we know that $\widetilde{u}(x) \in C\left(\overline{\mathbb{R}^n_+}\right)$.

Firstly, we show that $\widetilde{u}$ is weakly differentiable in $\mathbb{R}^n_+$ and

$$\nabla \widetilde{u} = \begin{cases} \nabla u, & \mathbb{R}^n_l, \\ 0, & \mathbb{R}^{n-1} \times (0, l). \end{cases}$$

Indeed, $\forall\, \varphi \in C_0^\infty(\mathbb{R}^n_+)$, by integration by parts, we have

$$\int_{\mathbb{R}^n_+} \widetilde{u}\, \partial_{x_i} \varphi \, \mathrm{d}x = \int_{\mathbb{R}^{n-1} \times (l, +\infty)} u\, \partial_{x_i} \varphi \, \mathrm{d}x = -\int_{\mathbb{R}^{n-1} \times (l, +\infty)} \partial_{x_i} u\, \varphi \, \mathrm{d}x$$

for $i \leq n - 1$ and

$$\begin{aligned} \int_{\mathbb{R}^n_+} \widetilde{u}\, \partial_{x_n} \varphi \, \mathrm{d}x &= \int_{\mathbb{R}^{n-1} \times (l, +\infty)} u\, \partial_{x_n} \varphi \, \mathrm{d}x \\ &= -\int_{\mathbb{R}^{n-1} \times \{x_n = l\}} u\, \varphi \, \mathrm{d}x' - \int_{\mathbb{R}^{n-1} \times (l, +\infty)} \partial_{x_n} u\, \varphi \, \mathrm{d}x \\ &= -\int_{\mathbb{R}^{n-1} \times (l, +\infty)} \partial_{x_n} u\, \varphi \, \mathrm{d}x, \end{aligned}$$

where we used $u(x', l) = 0$ in the last equality.

Next, we show that $\widetilde{u} \in W^{1,2}_{loc}\left(\mathbb{R}^n_+\right) \cap C\left(\overline{\mathbb{R}^n_+}\right)$ is a weak solution to

$$\begin{cases} \operatorname{div}\left(x_n^a \nabla \widetilde{u}\right) = 0, & \widetilde{u} > -C_0 \text{ in } \mathbb{R}^n_+, \\ \widetilde{u} = 0, & \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{2.1}$$

Indeed, for any $\varphi \in C_0^\infty(\mathbb{R}^n_+)$, there is

$$\begin{aligned} \int_{\mathbb{R}^n_+} \operatorname{div}\left(x_n^a \nabla \widetilde{u}\right) \varphi \, \mathrm{d}x &= \int_{\mathbb{R}^{n-1}\times(l,+\infty)} \operatorname{div}\left(x_n^a \nabla \widetilde{u}\right) \varphi \, \mathrm{d}x + \int_{\mathbb{R}^{n-1}\times(0,l)} \operatorname{div}\left(x_n^a \nabla \widetilde{u}\right) \varphi \, \mathrm{d}x \\ &= -\int_{\mathbb{R}^{n-1}\times(l,+\infty)} x_n^a \nabla u \cdot \nabla \varphi \, \mathrm{d}x - \int_{\mathbb{R}^{n-1}\times\{x_n=l\}} \partial_{x_n} u \, \varphi \, \mathrm{d}x' \\ &= \int_{\mathbb{R}^{n-1}\times(l,+\infty)} \operatorname{div}\left(x_n^a \nabla u\right) \varphi \, \mathrm{d}x = 0. \end{aligned}$$

It's clear that $\widetilde{u} > -C_0$ in $\mathbb{R}^n_+$ and $\widetilde{u} = 0$ on $\partial\mathbb{R}^n_+$. Hence, $\widetilde{u} \in W^{1,2}_{loc}\left(\mathbb{R}^n_+\right) \cap C\left(\overline{\mathbb{R}^n_+}\right)$ is a weak solution to (2.1).

For any fixed $x \in \partial\mathbb{R}^n_+$ and $\lambda > 0$, by Kelvin transformation

$$y^{x,\lambda} = x + \frac{\lambda^2 (y-x)}{|y-x|^2}, \quad \forall y \in \overline{\mathbb{R}^n_+},$$

we define

$$\widetilde{u}_{x,\lambda}(y) = \frac{\lambda^{n-2+a}}{|y-x|^{n-2+a}} \widetilde{u}\left(y^{x,\lambda}\right), \quad \forall y \in \overline{\mathbb{R}^n_+}.$$

By [YD, Theorem 2.1], we know that $\widetilde{u}_{x,\lambda}(y) \in W^{1,2}_{loc}\left(\mathbb{R}^n_+\right)$ satisfies $\operatorname{div}\left(y_n^a \nabla \widetilde{u}_{x,\lambda}\right) = 0$ in the weak sense, i.e. $\widetilde{u}_{x,\lambda}$ satisfies the same equation.

For $a > 2-n$, we consider $\bar{u} = \widetilde{u} + C_0$ instead of $\widetilde{u}$. Then $\lim\limits_{|y|\to 0} \bar{u}(x+y) = C_0$ for $x \in \partial\mathbb{R}^n_+$. Let

$$w_{x,\lambda}(y) = \bar{u}(y) - \bar{u}_{x,\lambda}(y), \quad \forall\, y \in \mathbb{R}^n_+.$$

We have

$$\varliminf_{|y|\to+\infty} w_{x,\lambda}(y) \geq 0 - \lim_{|y|\to+\infty} \frac{\lambda^{n-2+a}}{|y-x|^{n-2+a}} \bar{u}\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right) = 0.$$

By the maximum principle, we have $\widetilde{u}_{x,\lambda}(y) \leq \widetilde{u}(y)$, $\forall\, y \in \mathbb{R}^n_+ \backslash B_\lambda(x)$. Hence by Lemma 2.3 below, we know that $\widetilde{u}(y', y_n) = \widetilde{u}(y_n)$. Then solving the corresponding ODE gives us the desired result.

For $a < 2-n$, we consider $\bar{u} = \widetilde{u} - 1$ instead of $\widetilde{u}$. Then $\lim\limits_{|y|\to 0} \bar{u}(x+y) = -1$ for $x \in \partial\mathbb{R}^n_+$. Let

$$w_{x,\lambda}(y) = \bar{u}(y) - \bar{u}_{x,\lambda}(y), \quad \forall\, y \in \mathbb{R}^n_+.$$

We have

$$\begin{aligned}\varliminf_{|y|\to+\infty} w_{x,\lambda}(y) &= \varliminf_{|y|\to+\infty} \bar{u}(y) - \lim_{|y|\to+\infty} \frac{|y-x|^2}{\lambda^2}\bar{u}\left(x+\frac{\lambda^2(y-x)}{|y-x|^2}\right)\\ &\geq -1 - C_0 + \lim_{|y|\to+\infty}\frac{|y-x|^2}{\lambda^2}\\ &= +\infty.\end{aligned}$$

Again by the maximum principle, we have $\widetilde{u}_{x,\lambda}(y) \leq \widetilde{u}(y),\ \forall\, y \in \mathbb{R}^n_+\backslash B_\lambda(x)$. Similarly, by Lemma 2.3, we also have $\widetilde{u}(y', y_n) = \widetilde{u}(y_n)$, then we can obtain the conclusion.

As for $a = 2-n$, we need to modify $\widetilde{u}_{x,\lambda}(y)$ to be

$$\widetilde{u}_{x,\lambda}(y) = \widetilde{u}\left(y^{x,\lambda}\right) + \ln\frac{\lambda}{|y-x|}.$$

Then by similar arguments, we also have $\widetilde{u}_{x,\lambda}(y) \leq \widetilde{u}(y),\ \forall\, y \in \mathbb{R}^n_+\backslash B_\lambda(x)$. The result follows by applying Lemma 2.4. □

In the proof of Theorem 2.1, we used two crucial lemmas of moving spheres [WZ]. For readers' convenience, we include a proof here, which is very similar to the proof of [Li, Lemma 5.7].

Lemma 2.3 ([WZ, Lemma 3.3])**.** *Assume $f(y) \in C^0\left(\overline{\mathbb{R}^n_+}\right)$, $n \geq 2$, and $\tau \in \mathbb{R}$. Suppose*

$$\left(\frac{\lambda}{|y-x|}\right)^\tau f\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right) \leq f(y) \tag{2.2}$$

for $\lambda > 0$, $x \in \partial\mathbb{R}^n_+$, $y \in \mathbb{R}^n_+$ satisfying $|y-x| \geq \lambda$. Then

$$f(y) = f\left(y', y_n\right) = f\left(0', y_n\right), \quad \forall y = (y', y_n) \in \mathbb{R}^n_+.$$

Proof. For any fixed $y', z' \in \mathbb{R}^{n-1}$ with $y' \neq z'$ and $y_n > 0$, we denote $y = (y', y_n)$ and $z = (z', z_n)$, where $z_n = \frac{b-1}{b} y_n$ for $b > 1$. Then we have

$$x = y + b(z-y) \in \partial\mathbb{R}^n_+$$

and

$$z = x + \frac{\lambda^2(y-x)}{|y-x|^2},$$

where $\lambda = \sqrt{|z-x|\cdot|y-x|}$. By (2.2), we have

$$\left(\frac{\lambda}{|y-x|}\right)^\tau f(z) \leq f(y). \tag{2.3}$$

Since

$$\lim_{b\to+\infty}\frac{\lambda}{|y-x|} = \lim_{|x|\to\infty}\sqrt{\frac{|z-x|}{|y-x|}} = 1, \quad \lim_{b\to+\infty} z_n = \lim_{b\to+\infty}\frac{b-1}{b}y_n = y_n.$$

and f is continuous, we have $f(z', y_n) \leq f(y', y_n)$. By the arbitrariness of $y' \neq z'$, the proof is completed. □

Lemma 2.4. *Suppose that $f \in C^0\left(\overline{\mathbb{R}^n_+}\right)$ satisfies that for all $x \in \partial\mathbb{R}^n_+$ and $\lambda > 0$,*

$$f(y) \geq f\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right) + \ln \frac{\lambda}{|y-x|}, \quad \forall y \in \mathbb{R}^n_+ \backslash B_\lambda(x).$$

Then

$$f(y) = f\left(y', y_n\right) = f\left(0', y_n\right), \quad \forall y = (y', y_n) \in \mathbb{R}^n_+.$$

Proof. The proof is the same as Lemma 2.3. It suffices to replace (2.3) by

$$\ln \frac{\lambda}{|y-x|} + f(z) \leq f(y).$$

□

A Liouville theorem for the Neumman boundary value is also derived in [WZ].

Theorem 2.5 ([WZ, Theorem 1.2])**.** *Assume $n \geq 2$ and $\max\{-1, 2-n\} < a < 1$. Suppose $u(x) \in C^2\left(\mathbb{R}^n_+\right) \cap C^1\left(\overline{\mathbb{R}^n_+}\right)$ satisfies*

$$\begin{cases} \operatorname{div}\left(x_n^a \nabla u\right) = 0, \quad u > 0, & \text{in } \mathbb{R}^n_+, \\ x_n^a \frac{\partial u}{\partial x_n} = 0 & \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{2.4}$$

Then $u = C$ for some positive constant C. The boundary condition in (2.4) *holds in the following sense:*

$$\lim_{x_n \to 0^+} x_n^a \frac{\partial u}{\partial x_n} = 0.$$

Remark 2.6. *From the proof of Theorem 1.2 in* [WZ]*, we know that the codition on u can be weaken to $\frac{\partial u}{\partial x_i} \in C\left(\overline{\mathbb{R}^n_+}\right)$, $i = 1, 2, \cdots, n-1$ and $x_n^a \frac{\partial u}{\partial x_n} \in C\left(\overline{\mathbb{R}^n_+}\right)$.*

3. Proof of main theorems

In this section, we first derive the new equation under the partial Legendre transform. Let $\Omega \subset \mathbb{R}^2$ and $u(x, y)$ be a convex function on Ω. The partial Legendre transform in the x-variable is

$$u^\star(\xi, \eta) = \sup\{x\xi - u(x, \eta)\}, \tag{3.1}$$

where the supremum is taken respect to x on the slice η is the fixed constant, namely for all x such that $(x, y) \in \Omega$. This definition is from [Liu]. Hence, when $u \in C^2(\Omega)$ is a strictly convex function, we will have a injective mapping $\mathcal{P}$ satisfying

$$(\xi, \eta) = \mathcal{P}(x, y) := (u_x, y) \in \mathcal{P}(\Omega) := \Omega^\star. \tag{3.2}$$

In this situation, we know that

$$u^\star(\xi, \eta) = xu_x(x, y) - u(x, y).$$

Indeed, it just needs u to be strictly convex respect to x-variable [GP]. Then a direct calculation yields

$$\frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{pmatrix} u_{xx} & u_{xy} \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \frac{\partial(x,y)}{\partial(\xi,\eta)} = \begin{pmatrix} \frac{1}{u_{xx}} & -\frac{u_{xy}}{u_{xx}} \\ 0 & 1 \end{pmatrix}.$$

Hence,

$$u^\star_\xi = x, \quad u^\star_\eta = -u_y, \tag{3.3}$$

$$u^\star_{\xi\xi} = \frac{1}{u_{xx}}, \quad u^\star_{\eta\eta} = -\frac{\det D^2 u}{u_{xx}}, \quad u^\star_{\xi\eta} = -\frac{u_{xy}}{u_{xx}}. \tag{3.4}$$

Then if $u \in C^2(\Omega)$ is a strictly convex solution to

$$\det D^2 u = (a+bx)^\alpha,$$

we know that $u^\star \in C^2(\Omega^\star)$ is a solution to

$$(a+b\eta)^\alpha u^\star_{\xi\xi} + u^\star_{\eta\eta} = 0.$$

We will apply the results in Section 2 related to this equation to prove the main theorems.

3.1. **The case of Dirichlet boundary value.** We use Theorem 2.1 to prove Theorem 1.1. We consider the the partial Legendre transform $u^\star$ of u on $\mathbb{R}^2_+$. First, by [Fi, Theorem 2.19] we know that any solution to (1.4) is strictly convex in $\mathbb{R}^2_+$. Note that $\xi = u_x$, $\eta = y$ and $u_x = x$ on $\{y=0\}$ by (1.4), which gives us that $\mathcal{P}(\{y=0\}) = \{\eta = 0\}$. Hence, we have $\mathcal{P}\left(\mathbb{R}^2_+\right) = \mathbb{R}^2_+$. Then if $u \in C^2(\mathbb{R}^2_+) \cap C(\overline{\mathbb{R}^2_+})$ is a strictly convex solution to (1.4), we have that $u^\star \in C^2(\mathbb{R}^2_+) \cap C(\overline{\mathbb{R}^2_+})$ is a solution to

$$\begin{cases} (a+b\eta)^\alpha u^\star_{\xi\xi} + u^\star_{\eta\eta} = 0 & \text{in } \mathbb{R}\times(0,+\infty), \\ u^\star(\xi,0) = \dfrac{\xi^2}{2} & \text{on } \mathbb{R}\times\{0\}, \end{cases} \tag{3.5}$$

where we used the fact that the Legendre transform of $x \mapsto \frac{1}{2}x^2$ is $\xi \mapsto \frac{1}{2}\xi^2$. To use Theorem 2.1, we first need to differentiate (3.5) twice respect to ξ. Hence, we first show the following lemma:

Lemma 3.1. $u^\star_{\xi\xi} \in C^2(\mathbb{R}^2_+) \cap C(\overline{\mathbb{R}^2_+})$.

Proof. It is easy to see that $u^\star_{\xi\xi} \in C^2(\mathbb{R}^2_+)$. Indeed, for any point $(\xi,\eta) \in \mathbb{R}\times(0,+\infty)$, we know that there exists a sufficiently small neighborhood of (ξ,η) such that (3.5) becomes a uniformly elliptic equation with smooth coffecients, then the classical theory of uniformly elliptic equations gives us that $u^\star$ is C^4 at (ξ,η) (even C^∞) [GT], and thus $u^\star_{\xi\xi}$ is C^2 at (ξ,η).

Next, we show $u^\star_{\xi\xi} \in C(\overline{\mathbb{R}^2_+})$. The argument is very similar as Lemm 2.4 in [DS] and Proposition 3.2 in [JX]. It suffices to prove $u^\star_{\xi\xi}$ is continuous at $(0,0)$, other points on $\partial\mathbb{R}^2_+$ are similar. When $a>0$, we can see that (3.5) becomes a uniformly elliptic equation with smooth coffecients in the neighborhood of $(0,0)$ in $\overline{\mathbb{R}^2_+}$, hence $u^\star_{\xi\xi}$ is continuous at $(0,0)$. Then it only needs to consider the case for $a=0$. For simplicity, we may assume

$b=1$. First we prove that $u^\star_\xi$ is bounded in $B_1^+ := B_1((0,0))\cap\mathbb{R}^2_+$. Since $u^\star_\xi\in C^2(B_1^+)$ (we already show $u^\star_{\xi\xi}\in C^2(\mathbb{R}^2_+)$ in the above), $u^\star_\xi$ satisfies $Lu^\star_\xi := \eta^\alpha(u^\star_\xi)_{\xi\xi}+(u^\star_\xi)_{\eta\eta}=0$. We only show $u^\star_\xi$ is bounded. The same argument applied inductively would imply that the derivatives of $u^\star$ with respect to ξ of any order are bounded in the neighborhod of $(0,0)$, which will imply $u^\star_{\xi\xi}$ is continuous at $(0,0)$.

To establish the bound on $u^\star_\xi$, we consider $z:=\lambda(u^\star)^2+\varphi^2(u^\star{}_\xi)^2$, where $\lambda>0$ is to be determined later, φ is a cutoff function in B_2^+ satisfying $\varphi=1$ in B_1^+, $\varphi=0$ in $B_2^+\backslash B_{3/2}^+$, $\varphi_\eta=0$ for $0\le\eta\le\frac14$, and $|\varphi_\eta|\le C\eta^{\frac{\alpha}{2}}$. First we show that $Lz\ge 0$ in B_2^+. Indeed, a direct computation shows that

$$L(u^{\star 2})=2[\eta^\alpha(u^\star_\xi)^2+(u^\star_\eta)^2],$$

and

$$\begin{aligned}L(\varphi^2(u^\star_\xi)^2)) &= L(\varphi^2)\cdot(u^\star_\xi)^2+\varphi^2\cdot L((u^\star_\xi)^2)+2(\varphi^2)_\eta\cdot((u^\star{}_\xi)^2)_\eta+2\eta^\alpha(\varphi^2)_\xi\cdot((u^\star{}_\xi)^2)_\xi\\ &= L(\varphi^2)(u^\star_\xi)^2+2\varphi^2[\eta^\alpha(u^\star_{\xi\xi})^2+(u^\star_{\xi\eta})^2]+8(\varphi_\eta u^\star_\xi)\cdot(\varphi u^\star_{\xi\eta})\\ &\quad+8\eta^\alpha(\varphi_\xi u^\star_\xi)\cdot(\varphi u^\star_{\xi\xi}),\end{aligned}$$

hence

$$\begin{aligned}Lz &\ge 2(\lambda\eta^\alpha+\varphi^2L(\varphi^2))\cdot(u^\star_\xi)^2+2\varphi^2[\eta^\alpha(u^\star_{\xi\xi})^2+(u^\star_{\xi\eta})^2]\\ &\quad+8(\varphi_\eta u^\star_\xi)\cdot(\varphi u^\star_{\xi\eta})+8\eta^\alpha(\varphi_\xi u^\star_\xi)\cdot(\varphi u^\star_{\xi\xi}).\end{aligned}$$

Note that

$$L(\varphi^2)\ge -C_1\eta^\alpha,\quad |\varphi_\eta|\le C_1\eta^{\frac{\alpha}{2}}.$$

By Cauchy's inequality, we obtain $Lz\ge 0$ if λ is suffficiently large. Note that $u^\star$ is continous in B_2^+ and $u^\star(\xi,0)=\frac{\xi^2}{2}$ on $B_2^+\cap\partial\mathbb{R}^2_+$. Therefore by the maximum principle, $u^\star_\xi$ is bounded in B_1^+. Similarly, we have $u^\star_{\xi\xi}$ is bounded in $B_{1/2}^+$, and $u^\star_{\xi\xi\xi}$ is bounded in $B_{1/4}^+$ and so on, which implies $u^\star_{\xi\xi}$ is continuous at $(0,0)$. Using the same argument for other points on $\partial\mathbb{R}^2_+$, we have $u^\star_{\xi\xi}\in C(\overline{\mathbb{R}^2_+})$. Hence, $u^\star_{\xi\xi}\in C^2(\mathbb{R}^2_+)\cap C(\overline{\mathbb{R}^2_+})$. □

Proof of Theorem 1.1. Since Legendre transform does not change the convexity, we have that $u^\star_{\xi\xi}\ge 0$. Denote $v:=u^\star_{\xi\xi}-1$. By Lemma 3.1, differentiating (3.5) twice respect to ξ, we have that $v\in C^2(\mathbb{R}^2_+)\cap C(\overline{\mathbb{R}^2_+})$ and $v\ge -1$ solves

$$\begin{cases}(a+b\eta)^\alpha v_{\xi\xi}+v_{\eta\eta}=0 & \text{in } \mathbb{R}\times(0,+\infty),\\ v(\xi,0)=0 & \text{on } \mathbb{R}\times\{\eta=0\}.\end{cases}\tag{3.6}$$

Let $\xi=x_1$, $\eta=f(x_2)=b^{\frac{-\alpha}{\alpha+2}}\left(\frac{\alpha+2}{2}x_2\right)^{\frac{2}{\alpha+2}}-\frac{a}{b}$ and

$$\widetilde{v}(x_1,x_2)=v\left(x_1,f(x_2)\right).$$

A direct calculation yields

$$\begin{aligned}\widetilde{v}_{11}&=v_{\xi\xi},\\ \widetilde{v}_2&=b^{\frac{-\alpha}{\alpha+2}}\left(\frac{\alpha+2}{2}x_2\right)^{\frac{-\alpha}{\alpha+2}}v_\eta,\end{aligned}$$

$$\widetilde{v}_{22} = -\frac{\alpha}{\alpha+2} x_2^{-1}\widetilde{v}_2 + (a+b\eta)^{-\alpha} v_{\eta\eta}.$$

Note that $\eta = 0$ gives us that $x_2 = \frac{2}{b(\alpha+2)} a^{\frac{\alpha+2}{2}}$. Denote $l = \frac{2}{b(\alpha+2)} a^{\frac{\alpha+2}{2}}$. Hence by (3.6), we know that $\widetilde{v} \in C^2(\mathbb{R}^2_l) \cap C(\overline{\mathbb{R}^2_l})$ and $\widetilde{v} \geq -1$ solves

$$\begin{cases} \widetilde{v}_{11} + \widetilde{v}_{22} + \dfrac{\alpha}{\alpha+2} x_2^{-1}\widetilde{v}_2 = 0 & \text{in } \mathbb{R}\times(l,+\infty), \\ \widetilde{v}(x_1,0) = 0 & \text{on } \mathbb{R}\times\{x_2 = l\}, \end{cases}$$

i.e.,

$$\begin{cases} \operatorname{div}\left(x_2^{\frac{\alpha}{\alpha+2}} \nabla \widetilde{v}\right) = 0 & \text{in } \mathbb{R}\times(l,+\infty), \\ \widetilde{v}(x_1,0) = 0 & \text{on } \mathbb{R}\times\{x_2 = l\}. \end{cases}$$

Applying Theorem 2.1 with $n = 2$ and $a = \frac{\alpha}{\alpha+2} < 1$, we know that $\widetilde{v}(x_1,x_2) = C_*\left(x_2^{\frac{2}{\alpha+2}} - l^{\frac{2}{\alpha+2}}\right)$ for some nonnegative constant C_*. Transforming back to (ξ,η), we have $v(\xi,\eta) = A\eta$ for some $A \geq 0$, i.e., $u^\star_{\xi\xi}(\xi,\eta) = 1 + A\eta$. Then

$$u^\star(\xi,\eta) = h_1(\eta) + \xi h_2(\eta) + \frac{\xi^2}{2}(1 + A\eta)$$

for some functions $h_1, h_2 : [0,+\infty) \to \mathbb{R}$. Recalling (3.5), we have $h_1(0) = h_2(0) = 0$ and

$$h_1''(\eta) + \xi h_2''(\eta) + (1 + A\eta)(a+b\eta)^\alpha = 0$$

on $\mathbb{R}\times(0,+\infty)$. This implies that $h_1''(\eta) + (1+A\eta)(a+b\eta)^\alpha = 0$ and $h_2''(\eta) = 0$. By solving the ODEs, we obtain

$$u^\star(\xi,\eta) = \begin{cases} B\eta - \dfrac{(b-aA)(a+b\eta)^{2+\alpha}}{b^3(1+\alpha)(2+\alpha)} - \dfrac{A(a+b\eta)^{3+\alpha}}{b^3(2+\alpha)(3+\alpha)} + C\xi\eta \\ \qquad + \dfrac{(b-aA)a^{2+\alpha}}{b^3(1+\alpha)(2+\alpha)} + \dfrac{Aa^{3+\alpha}}{b^3(2+\alpha)(3+\alpha)} + \dfrac{\xi^2}{2}(1+A\eta), & \alpha \neq -1; \\ B\eta - \dfrac{b-aA}{b^3}(a+b\eta)\ln(a+b\eta) - \dfrac{A}{2}\eta^2 + C\xi\eta \\ \qquad + \dfrac{(b-aA)a\ln a}{b^3} + \dfrac{\xi^2}{2}(1+A\eta), & \alpha = -1, \end{cases}$$

for some constants $B, C \in \mathbb{R}$. Recalling that the Legendre transform is an involution on convex functions, we recover u by taking the partial Legendre transform of $u^\star$:

$$u(x,y)=\begin{cases}\dfrac{(b-aA)(a+by)^{2+\alpha}}{b^3(1+\alpha)(2+\alpha)}+\dfrac{A(a+by)^{3+\alpha}}{b^3(2+\alpha)(3+\alpha)}-By\\ \qquad -\dfrac{(b-aA)a^{2+\alpha}}{b^3(1+\alpha)(2+\alpha)}-\dfrac{Aa^{3+\alpha}}{b^3(2+\alpha)(3+\alpha)}+\dfrac{(x-Cy)^2}{2(1+Ay)}, & \alpha\neq -1;\\ \dfrac{b-aA}{b^3}(a+by)\ln(a+by)+\dfrac{A}{2b}y^2-By\\ \qquad\qquad -\dfrac{(b-aA)a\ln a}{b^3}+\dfrac{(x-Cy)^2}{2(1+Ay)}, & \alpha=-1.\end{cases}$$

This gives us a complete classification of all solutions to (1.4). ☐

Remark 3.2. *$\alpha > -2$ in Theorem 1.1 is sharp since* (1.4) *has no convex solutions continuous up to boundary in $\mathbb{R}^2_+$ when $\alpha \leq -2$. Indeed, if there exists a convex function $u \in C^2(\mathbb{R}^2_+) \cap C(\overline{\mathbb{R}^2_+})$ solves* (1.4)*, by* [S2, Theorem 5.1]*, we will have a Pogorelov type estimate*

$$(1-u)u_{xx} \leq C(\max|u_x|)$$

in S_1, where $S_h = \{x \in \mathbb{R}^2_+ : u(x) < u(0) + \nabla u(0)\cdot x + h\}$ for $h > 0$. Since $u(x,0) = \frac{1}{2}x^2$ on $\partial\mathbb{R}^2_+$, we know that $|u_x|$ is bounded in S_1 (depends on $\|u\|_{L^\infty(S_2)}$). Then there exists a small $c_0 > 0$ such that $u_{xx} \leq C(\|u\|_{L^\infty(S_2)})$ in $B^+_{c_0}$. Hence, we have

$$Cu_{yy} \geq u_{xx}u_{yy} \geq u_{xx}u_{yy} - u_{xy}^2 = y^\alpha \ \text{ in } B^+_{c_0},$$

i.e. in $B^+_{c_0}$, it holds

$$u(x,y)\geq\begin{cases}\dfrac{1}{C(1+\alpha)(2+\alpha)}y^{2+\alpha}+D(x)y+E(x), & \alpha<-2,\\ -\dfrac{1}{C}\ln y+D(x)y+E(x), & \alpha=-2,\end{cases}$$

which means that $\lim\limits_{y\to 0+} u(x,y) = +\infty$. This contradicts with $u \in C(\overline{\mathbb{R}^2_+})$.

3.2. **The case of Neumann boundary value.** We prove Theorem 1.3 in this section.

Proof of Theorem 1.3. First, we know that u is strictly convex in $\{y \geq 0\}$ as in [JS, Section 4] when $\alpha > 0$ and by [Fi, Theorem 2.19] when $-1 < \alpha \leq 0$, and also by the classical regularity theory for the Monge-Ampère equation [Fi], we have $u \in C^\infty(\mathbb{R}^2_+)$. Similarly as in the last section, we know that if $u \in C^\infty(\mathbb{R}^2_+) \cap C^1(\overline{\mathbb{R}^2_+})$ is a solution to (1.9), $u^\star \in C^\infty(\mathbb{R}^2_+) \cap C^1(\overline{\mathbb{R}^2_+})$ is a solution to

$$\begin{cases}\eta^\alpha u^\star_{\xi\xi}+u^\star_{\eta\eta}=0 & \text{in } \mathbb{R}\times(0,+\infty),\\ u^\star_\eta(\xi,0)=0 & \text{on } \mathbb{R}\times\{0\}.\end{cases} \tag{3.7}$$

Denote $v := u^\star_\eta$, then $v \in C^\infty(\mathbb{R}^2_+) \cap C(\overline{\mathbb{R}^2_+})$. Note that $v = u^\star_\eta = -u_y$ by (3.3) and $u_y(x,0) = 0$, we have by the strict convexity that $v < 0$ in $\mathbb{R}^2_+$. Differentiating (3.7) once respect to η and noting that $u^\star_{\xi\xi} = -\eta^\alpha v_\eta$, we have that $v < 0$ solves

$$
\begin{cases} \eta^\alpha v_{\xi\xi} + v_{\eta\eta} - \dfrac{\alpha}{\eta} v_\eta = 0 & \text{in } \mathbb{R} \times (0, +\infty), \\ v(\xi, 0) = 0 & \text{on } \mathbb{R} \times \{\eta = 0\}. \end{cases} \tag{3.8}
$$

Let $\xi = x_1$, $\eta = \left(\frac{\alpha+2}{2}\right)^{\frac{2}{\alpha+2}} x_2^{\frac{2}{\alpha+2}}$ and

$$
\widetilde{v}(x_1, x_2) = v\left(x_1, \left(\frac{\alpha+2}{2}\right)^{\frac{2}{\alpha+2}} x_2^{\frac{2}{\alpha+2}}\right).
$$

Then (3.8) gives us that $\widetilde{v} \in C^\infty(\mathbb{R}^2_+) \cap C(\overline{\mathbb{R}^2_+})$ and $\widetilde{v} > 0$ solves

$$
\begin{cases} \widetilde{v}_{11} + \widetilde{v}_{22} - \dfrac{\alpha}{\alpha+2} x_2^{-1} \widetilde{v}_2 = 0 & \text{in } \mathbb{R} \times (0, +\infty), \\ \widetilde{v}(x_1, 0) = 0 & \text{on } \mathbb{R} \times \{x_2 = 0\}, \end{cases}
$$

i.e.,

$$
\begin{cases} \operatorname{div}\left(x_2^{-\frac{\alpha}{\alpha+2}} \nabla \widetilde{v}\right) = 0 & \text{in } \mathbb{R} \times (0, +\infty), \\ \widetilde{v}(x_1, 0) = 0 & \text{on } \mathbb{R} \times \{x_2 = 0\}. \end{cases}
$$

Applying Theorem 2.1 with $n = 2$ and $a = -\frac{\alpha}{\alpha+2} \in (-1, 1)$, we know that $\widetilde{v} = C x_2^{1+\frac{\alpha}{\alpha+2}}$ for some negative constant C. Transforming back to (ξ, η), we have $v(\xi, \eta) = -C_1 \eta^{1+\alpha}$ for some $C_1 > 0$, i.e., $u^\star_\eta(\xi, \eta) = -C_1 \eta^{1+\alpha}$ for some $C_1 > 0$. Then

$$
u^\star(\xi, \eta) = h(\xi) - \frac{C_1}{2+\alpha} \eta^{2+\alpha}
$$

for some functions $h : \mathbb{R} \to \mathbb{R}$. Recalling (3.7), we have

$$
\eta^\alpha h''(\xi) - C_1(1+\alpha)\eta^\alpha = 0 \quad \text{in } \mathbb{R} \times (0, +\infty).
$$

This implies that $h''(\xi) = C_1(1+\alpha)$. By solving the ODE, we obtain

$$
u^\star(\xi, \eta) = \frac{A}{2}\xi^2 + B\xi - \frac{A\eta^{2+\alpha}}{(1+\alpha)(2+\alpha)} + C
$$

for some constants $A = C_1(1+\alpha) > 0$, and B, $C \in \mathbb{R}$. Recalling that $u = (u^\star)^\star$, we have

$$
u(x, y) = \frac{1}{2A}(x - B)^2 + \frac{Ay^{2+\alpha}}{(1+\alpha)(2+\alpha)} - C,
$$

which yields (1.10). □

3.3. **The entire space case.** Before proving Theorem 1.5, we first recall a definition and two theorems for (1.11) in [JX].

Definition 3.3. [JX, Definition 3.1] *Let Ω be a bounded C^2-domain in $\mathbb{R}^2$ with $\Omega \cap \{(x,y) \in \mathbb{R}^2 \mid y = 0\} \neq \emptyset$. A function u is the strong solution to the Grushin type equation*

$$|y|^\alpha u_{xx} + u_{yy} = 0$$

in Ω if $u \in C^1(\Omega) \cap C^2(\Omega \backslash \{y = 0\})$ and satisfies $|y|^\alpha u_{xx} + u_{yy} = 0$ in $\Omega \backslash \{y = 0\}$.

Theorem 3.4 ([JX, Theorem 4.1])**.** *Let Ω be an open bounded convex set in $\mathbb{R}^2$, and u be the Alexandrov solution of*

$$\det D^2 u(x,y) = |y|^\alpha \quad \text{in } \Omega,$$

with $u = 0$ on $\partial\Omega$. Then u is strictly convex in Ω, and $u \in C^{1,\delta}_{loc}(\Omega)$ for some $\delta > 0$ depending only on α. Furthermore, the partial Legendre transform $u^\star$ of u is a strong solution of

$$|\eta|^\alpha u^\star_{\xi\xi} + u^\star_{\eta\eta} = 0 \quad \text{in } \mathcal{P}(\Omega),$$

where the map $\mathcal{P}$ is given in (3.2).

Remark 3.5. *From the proof of the above theorem, we can see that $u^\star$ is in fact in $C^2(\Omega^* \setminus \{\eta = 0\}) \cap C^1(\Omega^*)$, and hence is a strong solution of $|\eta|^\alpha u^\star_{\xi\xi} + u^\star_{\eta\eta} = 0$ in $\mathcal{P}(\Omega)$.*

Theorem 3.6 ([JX, Theorem 4.2])**.** *Let u be a Alexandrov solution of*

$$\det D^2 u(x,y) = |y|^\alpha \quad \text{in } \mathbb{R}^2.$$

Then u is strictly convex.

Hence Theorem 3.4 and Theorem 3.6 give us that u is strictly convex, and by the classical regularity theory for the Monge-Ampère equation, we know u is smooth away from $\{y = 0\}$ [Caf1, Fi]. Furthermore, we know that $u \in C^{1,\delta}_{\text{loc}}(\mathbb{R}^2)$ and the partial Legendre transform $u^\star$ of u is a strong solution of (3.10) in the sense of Remark 3.5.

Next, we need a Liouville theorem for degenerate elliptic equations in divergence form. This theorem is a partial extension of [WZ, Corollary 1.4], where they assumed stronger conditions.

Theorem 3.7. *Assume that $n = 2$ and $a \geq 0$. Let u be a positive function satisfying $u \in C^2(\mathbb{R}^2 \backslash \{x_2 = 0\})$, $\frac{\partial u}{\partial x_1} \in C(\mathbb{R}^2)$ and $|x_2|^a \frac{\partial u}{\partial x_2} \in C(\mathbb{R}^2)$ with $\lim_{x_2 \to 0} |x_2|^a \dfrac{\partial u}{\partial x_2} = 0$. If u is a strong solution to*

$$\operatorname{div}(|x_2|^a \nabla u) = 0 \quad \text{in } \mathbb{R}^2 \backslash \{x_2 = 0\}. \tag{3.9}$$

Then u is a constant function.

Proof. This theorem is a corollary of Theorem 2.5 if we noticed Remark 2.6. Hence we can repeat the same process as in the proof of Theorem 1.2 in [WZ]. Due to its similarity, we omit the details here. ∎

Now, we are ready to give the proof of Theorem 1.5.

Proof of Theorem 1.5. Our proof only works for the case $\alpha \geq 0$. We consider the partial Legendre transform $u^\star$ of u. By Theorem 3.6, we know $u^\star$ is a strong solution to

$$|\eta|^\alpha u^\star_{\xi\xi} + u^\star_{\eta\eta} = 0 \quad \text{in } \mathbb{R}^2. \tag{3.10}$$

Since (3.10) becomes uniformly elliptic with smooth coefficients in any bounded domain away from $\{\eta = 0\}$, we know $u^\star \in C^\infty(\mathbb{R}^2 \backslash \{\eta = 0\})$ by the classical theory of elliptic equations [GT]. By [DS, Lemma 2.4] or [JX, Position 3.2] we know that $u^\star$ is smooth respect to ξ and $u^\star \in C^{2,\beta}_{loc}(\mathbb{R}^2)$ when $\alpha \geq 0$ for some $\beta > 0$ depending only on α. Hence, we have $u^\star_\xi \in C^2(\mathbb{R}^2 \backslash \{\eta = 0\}) \cap C^1(\mathbb{R}^2)$ is a strong solution to (3.10). Similarly, we know that $u^\star_\xi \in C^{2,\beta}_{loc}(\mathbb{R}^2)$, which means that $u^\star_{\xi\xi} \in C^2(\mathbb{R}^2 \backslash \{\eta = 0\}) \cap C^1(\mathbb{R}^2)$.

Let $v := u^\star_{\xi\xi} \geq 0$. Differentiating (3.10) twice respect to ξ, we have that $v \geq 0$ is a strong solution to

$$|\eta|^\alpha v_{\xi\xi} + v_{\eta\eta} = 0 \quad \text{in } \mathbb{R}^2. \tag{3.11}$$

Note by [DS, Lemma 2.4] that there must be $v_\eta(\xi, 0) = 0$ since v is C^1 near the point $(\xi, 0)$ and $v \geq 0$. By a change of variables, we let

$$\widetilde{v}(x_1, x_2) = \begin{cases} v\left(x_1, \left(\dfrac{\alpha+2}{2}\right)^{\frac{2}{\alpha+2}} x_2^{\frac{2}{\alpha+2}}\right), & \eta \geq 0, \\ v\left(x_1, -\left(\dfrac{\alpha+2}{2}\right)^{\frac{2}{\alpha+2}} (-x_2)^{\frac{2}{\alpha+2}}\right), & \eta < 0. \end{cases}$$

A direct calculation yields

$$\begin{aligned} \widetilde{v}_{11} &= v_{\xi\xi}, \\ \widetilde{v}_2 &= \left(\frac{\alpha+2}{2}\right)^{\frac{-\alpha}{\alpha+2}} |x_2|^{\frac{-\alpha}{\alpha+2}} v_\eta \\ \widetilde{v}_{22} &= -\frac{\alpha}{\alpha+2} x_2^{-1} \widetilde{v}_2 + |\eta|^{-\alpha} v_{\eta\eta}. \end{aligned}$$

By (3.11), we know that $\widetilde{v} \geq 0$ satisfying $\widetilde{v} \in C^2(\mathbb{R}^2 \backslash \{x_2 = 0\})$, $\widetilde{v}_1 \in C(\mathbb{R}^2)$ and $|x_2|^{\frac{\alpha}{\alpha+2}} \widetilde{v}_2 \in C(\mathbb{R}^2)$ with $\lim_{x_2 \to 0} |x_2|^{\frac{\alpha}{\alpha+2}} \widetilde{v}_2 = 0$ solves

$$\widetilde{v}_{11} + \widetilde{v}_{22} + \frac{\alpha}{\alpha+2} x_2^{-1} \widetilde{v}_2 = 0 \quad \text{in } \mathbb{R}^2,$$

i.e.,

$$\operatorname{div}\left(|x_2|^{\frac{\alpha}{\alpha+2}} \nabla \widetilde{v}\right) = 0 \quad \text{in } \mathbb{R}^2.$$

Hence, by Theorem 3.7 with $a = \frac{\alpha}{\alpha+2} \geq 0$ in (3.9), we obtain $\widetilde{v} \equiv$ constant. Thus $u^\star_{\xi\xi} \equiv A$, where A is a constant. Similar to the proofs of Theorem 1.1 and Theorem 1.3, by solving these ODEs, we have

$$u^\star(\xi,\eta)=\frac{A}{2}\xi^2-\frac{A}{(1+\alpha)(2+\alpha)}|\eta|^{2+\alpha}+B\xi\eta+l(\xi,\eta).$$

Again by $u=(u^\star)^\star$, we have (1.12). □

References

[Ab] Abreu, M., Kähler geometry of toric varieties and extremal metrics. *Int. J. Math.* **9** (1998), no. 6, 641-651.

[Caf1] Caffarelli, L. A., Interior $W^{2,p}$ estimates for solutions of the Monge-Ampère equation. *Ann. of Math. (2)* **131** (1990), no. 1, 135-150.

[Caf2] Caffarelli, L. A., *Topics in PDEs: The Monge-Ampère Equation, Graduate course*. Courant Institute, New York University, 1995.

[CG] Caffarelli, L. A., Gutiérrez, C. E., Properties of solutions of the linearized Monge-Ampère equation. *Amer. J. Math.* **119** (1997), no. 2, 423-465.

[CL] Caffarelli, L. A., Li, Y. Y., An extension to a theorem of Jörgens, Calabi, and Pogorelov. *Comm. Pure Appl. Math.* **56** (2003), no. 5, 549-583.

[CS] Caffarelli, L. A., Silvestre, L., An extension problem related to the fractional Laplacian. *Comm. Part. Diff. Eqns.* **32** (2007), 1245-1260.

[Ca] Calabi, E., Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens. *Mich. Math. J.* **5** (1958), 105-126.

[Ch] Chern, S. S., Affine minimal hypersurfaces. *Minimal submanifolds and geodesics* (Proc. Japan-United States Sem., Tokyo, 1977), pp. 17-30, North-Holland, Amsterdam-New York, 1979.

[DS] Daskalopoulos, P., Savin, O., On Monge-Ampère equations with homogeneous right-hand sides. *Comm. Pure Appl. Math.* **62** (2009), no. 5, 639–676.

[Fi] Figalli, A., *The Monge-Ampère equation and its applications*. Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2017.

[GT] Gilbarg, D., Trudinger, N. S., *Elliptic partial differential equations of second order*. Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001.

[GP] Guan, P., Phong, D. H., Partial Legendre transforms of non-linear equations. *Proc. Amer. Math. Soc.* **140** (2012), no. 11, 3831-3842.

[JS] Jhaveri, Y., Savin, O., On the regularity of optimal transports between degenerate densities. *Arch. Ration. Mech. Anal.* **245** (2022), no. 2, 819-861.

[JL] Jia, F., Li, A. M., A Bernstein property of some fourth order partial differential equations. *Results Math.* **56** (2009), 109-139.

[JT1] Jian, H. Y., Tu, X. S., A Liouville theorem for the Neumann problem of Monge-Ampère equations. *J. Funct. Anal.* **284** (2023), no. 6, Paper No. 109817.

[JT2] Jian, H. Y., Tu, X. S., The Oblique boundary value problem for Monge-Ampère equations. *arXiv:2111.09765.*

[JX] Jin, T. L., Xiong, J. G., A Liouville theorem for solutions of degenerate Monge-Ampère equations. *Comm. Part. Diff. Eqns.* **39** (2014), no. 2, 306-320.

[Jö] Jörgens, K., Über die Lösungen der Differentialgleichung $rt-s^2=1$. *Math. Ann.* **127** (1954), 130-134.

[Li] Li, Y. Y., Remark on some conformally invariant integral equations: the method of moving spheres. *J. Eur. Math. Soc. (JEMS)* **6** (2004), no. 2, 153-180.

[Liu] Liu, J. K., Interior C^2 estimate for Monge-Ampère equations in dimension two. *Proc. Amer. Math. Soc.* **149** (2021), no. 6, 2479-2486.

[Po] Pogorelov, A.V., On the improper affine hyperspheres. *Geom. Dedic.* **1** (1972), no. 1, 33-46.

[Ru] Rubin, D., The Monge-Ampère equation with Guillemin boundary conditions. *Cal. Var. Part. Diff. Eqns.* **54** (2015), 951-965.

[S1] Savin, O., Pointwise $C^{2,\alpha}$ estimates at the boundary for the Monge-Ampère equation. *J. Amer. Math. Soc.* **26** (2013), no. 1, 63-99.

[S2] Savin, O., A localization theorem and boundary regularity for a class of degenerate Monge-Ampère equations. *J. Diff. Eqns.* **256** (2014), no. 2, 327-388.

[SZ] Savin, O., Zhang, Q., Boundary regularity for Monge-Ampère equations with unbounded right hand side. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **20** (2020), no. 4, 1581-1619.

[TW1] Trudinger, N. S., Wang, X. -J., The Bernstein problem for affine maximal hypersurfaces. *Invent. Math.* **140** (2000), no. 2, 399-422.

[TW2] Trudinger, N. S., Wang, X. -J., Bernstein-Jörgens theorem for a fourth order partial differential equation. *J. Part. Diff. Eqns.* **15** (2002), no. 1, 78-88.

[TW3] Trudinger, N. S., Wang, X. -J., Boundary regularity for the Monge-Ampère and affine maximal surface equations. *Ann. of Math. (2)* **167** (2008), no. 3, 993-1028.

[WZ] Wang, L., Zhu, M. J., Liouville theorems on the upper half space. *Disc. Cont. Dyn. Syst.* **40** (2020), no. 9, 5373-5381.

[YD] Yao, J. G., Dou, J. B., Classification of positive solutions to a divergent equation on the upper half space. *Acta Math. Sin. (Engl. Ser.)* **38** (2022), no. 3, 499-509.

[Z] Zhou, B., The Bernstein theorem for a class of fourth order equations. *Calc. Var. Part. Diff. Eqns.* **43** (2012), no. 1-2, 25-44.

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