FINAL EXAM

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1. Final exam (12.28)

Problem 1.1. *Calculate the following limitations.*

(1) $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{k}{n^2}\right)^{1+\frac{k}{n^2}}$;

(2) $\lim_{n\to\infty}\frac{1}{n^k}\int_0^1 \ln^k(1+e^{nx})\,\mathrm{d}x$;

(3) $\lim_{x\to 0}\frac{[1+\ln(1+x)]^{\frac{1}{\tan x}}-e(1-x)}{x^2}$.

Solution. (1) Firstly, we have

$$\left(\frac{k}{n^2}\right)^{1+\frac{k}{n^2}}\le\frac{k}{n^2},\quad k=1,2,\cdots,n,$$

since $\frac{k}{n^2}\le 1$. Then there is

$$\sum_{k=1}^{n}\left(\frac{k}{n^2}\right)^{1+\frac{k}{n^2}}\le\sum_{k=1}^{n}\frac{k}{n^2}=\frac{n(n+1)}{2n^2}=\frac{n+1}{2n}\to\frac{1}{2},\quad \text{as } n\to\infty.$$

On the other hand, we have

$$\left(\frac{k}{n^2}\right)^{1+\frac{k}{n^2}}\ge\left(\frac{k}{n^2}\right)^{1+\frac{1}{n}},\quad k=1,2,\cdots,n.$$

Since $u_n(x)=x^{1+\frac{1}{n}}$ is increasing on $(0,1)$, we know that

$$\begin{aligned}\sum_{k=1}^{n}\left(\frac{k}{n^2}\right)^{1+\frac{1}{n}}&=\frac{1}{\sqrt[n]{n}}\cdot\frac{1}{n}\sum_{k=1}^{n}u_n\left(\frac{k}{n}\right)\\&\ge\frac{1}{\sqrt[n]{n}}\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}}u_n(x)\,\mathrm{d}x\\&=\frac{1}{\sqrt[n]{n}}\int_0^1 u_n(x)\,\mathrm{d}x\end{aligned}$$

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$$= \frac{n^{1-\frac{1}{n}}}{2n+1} \to \frac{1}{2}, \quad \text{as } n \to \infty.$$

Hence

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k}{n^2}\right)^{1+\frac{k}{n^2}} = \frac{1}{2}.$$

(2) By changing of variables and the Stolz theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n^k} \int_0^1 \ln^k(1+e^{nx})\,\mathrm{d}x &\xlongequal{y=nx} \lim_{n\to\infty} \frac{1}{n^{k+1}} \int_0^n \ln^k(1+e^{y})\,\mathrm{d}y \\
&\xlongequal{Stolz} \lim_{n\to\infty} \frac{\int_n^{n+1} \ln^k(1+e^{y})\,\mathrm{d}y}{(n+1)^{k+1} - n^{k+1}} \\
&= \lim_{n\to\infty} \frac{\ln^k(1+e^{\theta_n})}{(k+1)n^k} \\
&\xlongequal{(*)} \frac{1}{k+1},
\end{aligned}$$

where we used $\lim\limits_{n\to\infty} \dfrac{\ln(1+e^{\theta_n})}{n} = 1$ in $(*)$ since

$$\frac{\ln(1+e^{n})}{n} \le \frac{\ln(1+e^{\theta_n})}{n} \le \frac{\ln(1+e^{n+1})}{n}.$$

(3) By Taylor's formula, we have

$$\begin{aligned}
\lim_{x\to 0} \frac{[1+\ln(1+x)]^{\frac{1}{\tan x}} - e(1-x)}{x^2} &= \lim_{x\to 0} \frac{e^{\frac{1}{\tan x}\ln(1+\ln(1+x))} - e(1-x)}{x^2} \\
&= \lim_{x\to 0} \frac{e^{\frac{1}{\tan x}\ln(1+x-\frac{1}{2}x^2+\frac{1}{3}x^3)} - e(1-x)}{x^2} \\
&= \lim_{x\to 0} \frac{e^{\frac{(x-\frac{1}{2}x^2+\frac{1}{3}x^3)-\frac{1}{2}(x-\frac{1}{2}x^2)^2+\frac{1}{3}x^3}{x+\frac{1}{3}x^3}} - e(1-x)}{x^2} \\
&= \lim_{x\to 0} \frac{e^{\frac{x-x^2+\frac{7}{6}x^3}{x+\frac{1}{3}x^3}} - e(1-x)}{x^2} \\
&= \lim_{x\to 0} \frac{e^{1-x+\frac{5}{6}x^2} - e(1-x)}{x^2} \\
&= \lim_{x\to 0} \frac{e(1-x+\frac{5}{6}x^2+\frac{1}{2}x^2) - e(1-x)}{x^2} \\
&= \lim_{x\to 0} \frac{\frac{4e}{3}x^2}{x^2} \\
&= \frac{4e}{3}.
\end{aligned}$$

□

Problem 1.2. *Calculate the following integrals.*

(1) $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt{1+x}+\sqrt{1-x}+2}\,\mathrm{d}x$;

(2) $\displaystyle\int_{1}^{2} \frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}}\,\mathrm{d}x$;

(3) $\displaystyle\int x\sin(\ln x)\,\mathrm{d}x$, *where* $x>0$;

(4) $\displaystyle\int \frac{1}{x+\sqrt{x^2-x+1}}\,\mathrm{d}x$.

Solution. (1) By changing of variables, we have

$$\begin{aligned}
\int_{-1}^{1} \frac{1}{\sqrt{1+x}+\sqrt{1-x}+2}\,\mathrm{d}x &= 2\int_{0}^{1} \frac{1}{\sqrt{1+x}+\sqrt{1-x}+2}\,\mathrm{d}x \\
&\xlongequal{x=\sin 4t} 8\int_{0}^{\frac{\pi}{8}} \frac{\cos 4t}{\sqrt{1+\sin 4t}+\sqrt{1-\sin 4t}+2}\,\mathrm{d}t \\
&= 8\int_{0}^{\frac{\pi}{8}} \frac{\cos 4t}{\sqrt{1+2\sin 2t\cos 2t}+\sqrt{1-2\sin 2t\cos 2t}+2}\,\mathrm{d}t \\
&= 8\int_{0}^{\frac{\pi}{8}} \frac{\cos 4t}{\sin 2t+\cos 2t+\cos 2t-\sin 2t+2}\,\mathrm{d}t \\
&= 4\int_{0}^{\frac{\pi}{8}} \frac{\cos 4t}{\cos 2t+1}\,\mathrm{d}t \\
&= 2\int_{0}^{\frac{\pi}{8}} \frac{\cos 4t}{\cos^2 t}\,\mathrm{d}t \\
&= 2\int_{0}^{\frac{\pi}{8}} \frac{2\cos^2 2t-1}{\cos^2 t}\,\mathrm{d}t \\
&= 2\int_{0}^{\frac{\pi}{8}} \frac{2(2\cos^2 t-1)^2-1}{\cos^2 t}\,\mathrm{d}t \\
&= 2\int_{0}^{\frac{\pi}{8}} \frac{8\cos^4 t-8\cos^2 t+1}{\cos^2 t}\,\mathrm{d}t \\
&= 2\int_{0}^{\frac{\pi}{8}} 8\cos^2 t\,\mathrm{d}t - 2\int_{0}^{\frac{\pi}{8}} 8\,\mathrm{d}t + 2\int_{0}^{\frac{\pi}{8}} \frac{1}{\cos^2 t}\,\mathrm{d}t \\
&= 16\int_{0}^{\frac{\pi}{8}} \frac{1+\cos 2t}{2}\,\mathrm{d}t - 2\pi + 2\tan t\Big|_0^{\frac{\pi}{8}} \\
&= \pi + 2\sqrt{2} - 2\pi + 2\tan\frac{\pi}{8}
\end{aligned}$$

$$= 4\sqrt{2} - 2 - \pi,$$

since

$$1 = \frac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$$

gives us that $\tan\frac{\pi}{8} = \sqrt{2}-1$.

(2) By changing of variables, we have

$$\begin{aligned}\int_1^2 \frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}}\,\mathrm{d}x &\overset{x=\frac{1}{t^2}}{=\!=\!=} \frac{1}{2}\int_{\frac{1}{4}}^1 \frac{1-t}{\sqrt{t^2-2t+2}}\,\mathrm{d}t\\ &= \frac{1}{2}\int_{\frac{1}{4}}^1 \frac{1-t}{\sqrt{(t-1)^2+1}}\,\mathrm{d}t\\ &= -\frac{1}{2}\sqrt{(t-1)^2+1}\Big|_{\frac{1}{4}}^1\\ &= \frac{1}{8}.\end{aligned}$$

(3) By integral by parts, we have

$$\begin{aligned}\int x\sin(\ln x)\,\mathrm{d}x &= \frac{1}{2}x^2\sin(\ln x) - \frac{1}{2}\int x\cos(\ln x)\,\mathrm{d}x\\ &= \frac{1}{2}x^2\sin(\ln x) - \frac{1}{2}\left(\frac{1}{2}x^2\cos(\ln x) + \frac{1}{2}\int x\sin(\ln x)\,\mathrm{d}x\right)\\ &= \frac{1}{2}x^2\sin(\ln x) - \frac{1}{4}x^2\cos(\ln x) - \frac{1}{4}\int x\sin(\ln x)\,\mathrm{d}x,\end{aligned}$$

which yields

$$\int x\sin(\ln x)\,\mathrm{d}x = \frac{2}{5}x^2\sin(\ln x) - \frac{1}{5}x^2\cos(\ln x) + C.$$

(4) By changing of variables, we have

$$\begin{aligned}\int \frac{1}{x+\sqrt{x^2-x+1}}\,\mathrm{d}x &\overset{t=x+\sqrt{x^2-x+1}}{=\!=\!=\!=\!=} 2\int \frac{t^2-t+1}{t(2t-1)^2}\,\mathrm{d}t\\ &= \int \frac{2}{t}\,\mathrm{d}t - \int \frac{3}{2t-1}\,\mathrm{d}t + \int \frac{3}{(2t-1)^2}\,\mathrm{d}t\\ &= \ln t - \frac{3}{2}\ln(2t-1) - \frac{3}{2}\frac{1}{2t-1} + C\\ &= \ln(x+\sqrt{x^2-x+1})\\ &\quad - \frac{3}{2}\ln(2(x+\sqrt{x^2-x+1})-1)\\ &\quad - \frac{3}{2(2(x+\sqrt{x^2-x+1})-1)} + C.\end{aligned}$$

□

Problem 1.3. *Suppose that a curve L can be given by $y = y(x) \in C^4(\mathbb{R})$ in the xy-coordinate system. Rotate the xy-coordinate system against the clockwise $\pi/4$ to get the new coordinate system, say (t, s). Assume that L can be given by $s = s(t) \in C^4(\mathbb{R})$ in the st-coordinate system. If $y'(x) > -1$ and $y''(x) \neq 0$, prove that $s''(t) \neq 0$ and there is*

$$\left[s''(t)^{-\frac{2}{3}}\right]''(t) = \left[y''(x)^{-\frac{2}{3}}\right]''(x),$$

where $(x, y(x))$ and $(t, s(t))$ are the same point in the curve.

Proof. Note that

$$\begin{cases} t = x\cos\dfrac{\pi}{4} + y\sin\dfrac{\pi}{4}, \\ s = -x\sin\dfrac{\pi}{4} + y\cos\dfrac{\pi}{4}. \end{cases}$$

By $y = y(x)$, we know that L can be given by

$$\begin{cases} t = \dfrac{\sqrt{2}}{2}x + \dfrac{\sqrt{2}}{2}y(x), \\ s = -\dfrac{\sqrt{2}}{2}x + \dfrac{\sqrt{2}}{2}y(x). \end{cases}$$

Hence, there is

$$\begin{cases} \mathrm{d}t = \dfrac{\sqrt{2}}{2}(1 + y'(x))\mathrm{d}x, \\ \mathrm{d}s = \dfrac{\sqrt{2}}{2}(-1 + y'(x))\mathrm{d}x. \end{cases}$$

Then we have

$$s'(t) = \frac{-1 + y'(x)}{1 + y'(x)} \quad \text{and} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\sqrt{2}}{1 + y'(x)}.$$

Taking derivative yields

$$s''(t) = \frac{y''(x)\frac{\mathrm{d}x}{\mathrm{d}t}(1 + y'(x)) - y''(x)\frac{\mathrm{d}x}{\mathrm{d}t}(-1 + y'(x))}{(1 + y'(x))^2} = \frac{2\sqrt{2}y''(x)}{(1 + y'(x))^3}.$$

Since $y''(x) \neq 0$, it is clear that $s''(t) \neq 0$. What's more, since

$$s''(t)^{-\frac{2}{3}} = y''(x)^{-\frac{2}{3}}\frac{(1 + y'(x))^2}{2},$$

we have

$$\begin{aligned} \left[s''(t)^{-\frac{2}{3}}\right]'(t) &= \left[y''(x)^{-\frac{2}{3}}\right]'\frac{\mathrm{d}x}{\mathrm{d}t}\frac{(1 + y'(x))^2}{2} + y''(x)^{-\frac{2}{3}}(1 + y'(x))y''(x)\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= \left[y''(x)^{-\frac{2}{3}}\right]'\frac{1 + y'(x)}{\sqrt{2}} + \sqrt{2}y''(x)^{\frac{1}{3}}. \end{aligned}$$

Then

$$\left[s''(t)^{-\frac{2}{3}}\right]''(t) = \left[y''(x)^{-\frac{2}{3}}\right]''(x) + \left[y''(x)^{-\frac{2}{3}}\right]'\frac{y''(x)}{\sqrt{2}}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\sqrt{2}}{3}y''(x)^{-\frac{2}{3}}y'''(x)\frac{\mathrm{d}x}{\mathrm{d}t}$$

$$
\begin{aligned}
&= \left[y''(x)^{-\frac{2}{3}}\right]''(x) + \left[y''(x)^{-\frac{2}{3}}\right]' y''(x)\frac{1}{1+y'(x)} + \frac{2}{3}y''(x)^{-\frac{2}{3}}y'''(x)\frac{1}{1+y'(x)} \\
&= \left[y''(x)^{-\frac{2}{3}}\right]''(x) - \frac{2}{3}y''(x)^{-\frac{2}{3}}y'''(x)\frac{1}{1+y'(x)} + \frac{2}{3}y''(x)^{-\frac{2}{3}}y'''(x)\frac{1}{1+y'(x)} \\
&= \left[y''(x)^{-\frac{2}{3}}\right]''(x).
\end{aligned}
$$

□

Problem 1.4. *Suppose that $f \in C^\infty(\mathbb{R})$ and for any $k \in \mathbb{N}$, there is*

$$\sup_{x\in\mathbb{R}} \big||x|^k|f(x)| + |f^{(k)}(x)|\big| < +\infty.$$

Prove that for any $k, l \in \mathbb{N}$, there is

$$\sup_{x\in\mathbb{R}} \big||x|^k|f^{(l)}(x)|\big| < +\infty.$$

Proof. We prove the conclusion by induction. For $l = 0$, it's clear that $\sup_{x\in\mathbb{R}} \big||x|^k|f(x)|\big| < +\infty$ for any $k \in \mathbb{N}$. Assume that for any $0 \le l \le n$ and $k \in \mathbb{N}$, there is

$$\sup_{x\in\mathbb{R}} \big||x|^k|f^{(l)}(x)|\big| < +\infty.$$

We will show that $\sup_{x\in\mathbb{R}} \big||x|^k|f^{(n+1)}(x)|\big| < +\infty$ for any $k \in \mathbb{N}$. Indeed, by Taylor's formula, we have for any $x > 0$ that

$$f(x+h) = f(x) + f'(x)h + \cdots + \frac{f^{(n+1)}(x)}{(n+1)!}h^{n+1} + \frac{f^{(n+2)}(\xi)}{(n+2)!}h^{n+2}.$$

Taking $h = |x|^{-k}$, we have

$$
\begin{aligned}
\big||x|^k|f^{(n+1)}(x)|\big| \le (n+1)!\Bigg(&|x|^{(n+2)k}|f(x+|x|^{-k})| + |x|^{(n+2)k}|f(x)| \\
&+ |x|^{(n+1)k}|f'(x)| + \cdots + \frac{1}{n!}|x|^{2k}|f^{(n)}(x)| + \frac{|f^{(n+2)}(\xi)|}{(n+2)!}\Bigg).
\end{aligned}
$$

By $\sup_{x\in\mathbb{R}} \big||x|^k|f(x)| + |f^{(k)}(x)|\big| < +\infty$, $\sup_{x>0} \frac{x}{x+|x|^{-k}} < +\infty$ and the assumption, we know that

$$\sup_{x>0} \big||x|^k f^{(n+1)}(x)|\big| < +\infty \quad \text{for any } k \in \mathbb{N}.$$

For any $x < 0$, we just need to take $h = -|x|^{-k}$. For $x = 0$, it's clear. Hence we know that for any $k, l \in \mathbb{N}$, there is

$$\sup_{x\in\mathbb{R}} \big||x|^k f^{(l)}(x)|\big| < +\infty.$$

□

Problem 1.5. *Suppose that $f(x)$ is twice differentiable on $[-2,2]$, $|f(x)| \le 1$ and $[f(0)]^2 + [f'(0)]^2 = 4$. Prove that there exists $\xi \in (-2,2)$ such that $f''(\xi) + f(\xi) = 0$.*

Proof. Let

$$F(x) = f(x)^2 + f'(x)^2, \quad \forall\, x \in [-2,2].$$

Then $F(0) = 4$. By the Lagrange Mean Value Theorem, we know that there exists $x_1 \in (-2,0)$ such that

$$f'(x_1) = \frac{f(0) - f(-2)}{2}.$$

Since $|f(x)| \le 1$, we have that $|f'(x_1)| \le 1$. Similarly, we know that there exists $x_2 \in (0,2)$ such that $|f'(x_2)| \le 1$. Then $F(x_1) \le 2$ and $F(x_2) \le 2$. Note that $x_1 < 0 < x_2$ and $F(0) = 4 > 2$, we know that there must be at least a maximum point in (x_1, x_2). Hence, there exists $\xi \in (-2,2)$ such that $F'(\xi) = 0$, i.e. $f(\xi) + f''(\xi) = 0$ since $f'(\xi) \ne 0$. To prove $f'(\xi) \ne 0$, it suffices to note that $F(\xi) \ge 4$ and $f(\xi)^2 \le 1$. Then we are done. □

Problem 1.6. *Suppose that $f(x)$ is nonnegative convex function on $[-1,1]$, satisfying $f(0) = 0$ and $f(-1) = f(1) = 1$. Define $S(h) = \{x | f(x) \le h\}$, $\forall\, h \in [0,1]$.*

(1) *If there exists $\varepsilon > 0$ such that $\forall\, x \in [-1,1]$, there is $f\left(\frac{x}{2}\right) \le \frac{1-\varepsilon}{2} f(x)$. Prove that there exist $\alpha > 0$ and $C > 0$ such that $f(x) \le C|x|^{1+\alpha}$, $\forall\, x \in [-1,1]$.*

(2) *If there exists $\varepsilon \in (0, 1/2)$ such that $\forall\, h \in [0,1]$, there is $l\left(\frac{h}{2}\right) \le (1-\varepsilon) l(h)$, where $l(h)$ is the length of $S(h)$. Prove that there exist $\beta > 0$ and $C > 0$ such that $f(x) \ge C|x|^{1+\beta}$, $\forall\, x \in [-1,1]$.*

Proof. (1) By $f\left(\frac{x}{2}\right) \le \frac{1-\varepsilon}{2} f(x)$, we have

$$f\left(\frac{x}{2^k}\right) \le \left(\frac{1-\varepsilon}{2}\right)^k f(x), \quad \forall\, x \in [-1,1],\, k \ge 0.$$

Since $f(x)$ is convex and $f(-1) = f(1) = 1$, we know that $f(x) \le 1$, $\forall\, x \in [-1,1]$. Choosing $\alpha > 0$ such that $2^{-\alpha} = 1 - \varepsilon$, i.e. $\alpha = -\frac{\ln(1-\varepsilon)}{\ln 2}$. Then there is

$$f\left(\frac{x}{2^k}\right) \le \left(\frac{1-\varepsilon}{2}\right)^k = \left(\frac{1}{2^k}\right)^{1+\alpha}, \quad \forall\, x \in [-1,1],\, k \ge 0.$$

Hence, for $\forall\, x' \in [-1,1]$, we know that there exists $k = k(x')$ such that

$$\frac{1}{2^{k+1}} < |x'| \le \frac{1}{2^k}.$$

Then taking $x = 2^k x' \in [-1,1]$, we have

$$f(x') = f\left(\frac{x}{2^k}\right) \le \left(\frac{1}{2^k}\right)^{1+\alpha} = \left(\frac{1}{2^{k+1}}\right)^{1+\alpha} 2^{1+\alpha} \le 2^{1+\alpha}|x'|^{1+\alpha}, \quad \forall\, x' \in [-1,1].$$

i.e. there exist $\alpha = -\dfrac{\ln(1-\varepsilon)}{\ln 2}$ and $C = 2^{1+\alpha}$ such that

$$f(x) \le C|x|^{1+\alpha}, \quad \forall\, x \in [-1,1].$$

(2) Similar to (1), we have

$$l(h) \le 2^{1+\alpha} h^{\alpha}, \quad \forall\, h \in [0,1],$$

where $\alpha = -\dfrac{\ln(1-\varepsilon)}{\ln 2}$. Next, we prove that $\forall\, x \in [-1,1]$, there is

$$f(x) \ge 2^{-\left(\frac{2}{\alpha}+1\right)}|x|^{\frac{1}{\alpha}}.$$

We prove the claim by contradiction. Assume that there exists $x_0 \in [-1,1]$ such that

$$f(x_0) < 2^{-\left(\frac{2}{\alpha}+1\right)}|x_0|^{\frac{1}{\alpha}}.$$

Without loss of generality, we may assume that $x_0 > 0$, and it's similar for $x_0 < 0$. Since $f(x)$ is a convex function, we know that $\forall\, x \in [0, x_0]$, there is

$$f(x) \le \lambda f(x_0) + (1-\lambda) f(0) \le f(x_0) < 2^{-\left(\frac{2}{\alpha}+1\right)}|x_0|^{\frac{1}{\alpha}}.$$

Hence $[0, x_0] \subset S(h_0)$, where $h_0 = 2^{-\left(\frac{2}{\alpha}+1\right)}|x_0|^{\frac{1}{\alpha}} < 1$. Then

$$|x_0| \le l(h_0) \le 2^{1+\alpha}\left(2^{-\left(\frac{2}{\alpha}+1\right)}|x_0|^{\frac{1}{\alpha}}\right)^{\alpha} = 2^{1+\alpha} \cdot 2^{-(2-\alpha)}|x_0| = \frac{1}{2}|x_0|,$$

contradiction. Hence, $\forall\, x \in [-1,1]$, there is

$$f(x) \ge 2^{-\left(\frac{2}{\alpha}+1\right)}|x|^{\frac{1}{\alpha}}.$$

Therefore, we can take $C = 2^{-\left(\frac{2}{\alpha}+1\right)}$ and $\beta = \dfrac{1}{\alpha} - 1$. Since $\varepsilon \in (0, 1/2)$ and $\alpha = -\dfrac{\ln(1-\varepsilon)}{\ln 2}$, we know that $\alpha \in (0,1)$, then $\beta > 0$. □

Problem 1.7. *Suppose that* $f(x) \in C^1(\mathbb{R})$ *satisfying* $\sup\limits_{x\in\mathbb{R}} |f(x)| \le A \in (0, +\infty)$ *and* $\sup\limits_{x\in\mathbb{R}, y>x} \left|\dfrac{f'(y)-f'(x)}{y-x}\right| \le B \in (0,+\infty)$. *Prove that* $\forall\, x \in \mathbb{R}$, *there is* $|f'(x)| \le \sqrt{2AB}$.

Proof. By the Newton-Leibniz formula, we have

$$f(x+h) = f(x) + f'(x)h + \int_x^{x+h} \left(f'(t) - f'(x)\right)\, \mathrm{d}t,$$
$$f(x-h) = f(x) - f'(x)h + \int_{x-h}^{x} \left(f'(t) - f'(x)\right)\, \mathrm{d}t.$$

Hence there are

$$|f(x+h) - f(x) - f'(x)h| \le B\int_x^{x+h} (t-x)\, \mathrm{d}t = \frac{B}{2}h^2,$$

$$|f(x-h) - f(x) + f'(x)h| \le B\int_{x-h}^{x} (x-t)\ \mathrm{d}t = \frac{B}{2}h^2.$$

Then

$$|2hf'(x) + f(x-h) - f(x+h)| \le Bh^2,$$

which yields

$$|f'(x)| \le \frac{1}{2h}\left(Bh^2 + |f(x+h) - f(x-h)|\right) \le \frac{A}{h} + \frac{Bh}{2}.$$

Choosing $h = \sqrt{\dfrac{2A}{B}}$, we have

$$|f'(x)| \le \sqrt{2AB}.$$

□

Problem 1.8. *Suppose $f(x) \in C[0,1]$ is positive, and* $\int_0^1 f(x)\,\mathrm{d}x = A$, $\int_0^1 f^2(x)\,\mathrm{d}x = B$.

(1) *Prove that for any $n \in \mathbb{N}_+$, there exists a partition $\Delta : 0 = x_0 < \cdots < x_n = 1$ such that* $\int_{x_{k-1}}^{x_k} f(x)\,\mathrm{d}x = \dfrac{A}{n}$, $k = 1, 2, \cdots, n$.

(2) *Find* $\lim_{n\to\infty} \dfrac{1}{n}\sum_{k=1}^{n} f(x_k)$.

Proof. (1) Since $f(x)$ is continuous and positive, we know that $\int_0^x f(t)\,\mathrm{d}t$ is continuous and increasing. By the intermediate value theorem, we have that there exist $0 = x_0 < \cdots < x_n = 1$ such that $\int_0^{x_k} f(t)\,\mathrm{d}t = \dfrac{kA}{n}$, hence $\int_{x_{k-1}}^{x_k} f(x)\,\mathrm{d}x = \dfrac{A}{n}$, $k = 1, 2, \cdots, n$.

(2) By the mean value theorems for definite integrals, we know that there exists $\xi_k \in (x_{k-1}, x_k)$, such that

$$\int_{x_{k-1}}^{x_k} f(x)\,\mathrm{d}x = f(\xi_k)(x_k - x_{k-1}) = f(\xi_k)\Delta x_k = \frac{A}{n}, \quad k = 1, 2, \cdots, n.$$

Since $f(x)$ is continuous on $[0,1]$, we know that $f(x)$ is uniformly continuous. Then for $\forall\,\varepsilon > 0$, there exits $\delta > 0$ such that $\forall\, x, x' : |x - x'| < \delta$, there is $|f(x) - f(x')| < \varepsilon$. Hence for n large enough, we have $\Delta x_k < \delta$, which gives us that $|f(x_k) - f(\xi_k)| < \varepsilon$. Then

$$\begin{aligned}
\left|\frac{1}{n}\sum_{k=1}^{n} f(x_k) - \frac{B}{A}\right| &= \left|\frac{1}{n}\sum_{k=1}^{n} f(x_k) - \frac{1}{A}\int_0^1 f^2(x)\,\mathrm{d}x\right| \\
&\le \left|\frac{1}{n}\sum_{k=1}^{n} f(x_k) - \frac{1}{A}\sum_{k=1}^{n} f^2(x_k)\Delta x_k\right| \\
&\quad + \left|\frac{1}{A}\sum_{k=1}^{n} f^2(x_k)\Delta x_k - \frac{1}{A}\int_0^1 f^2(x)\,\mathrm{d}x\right|
\end{aligned}$$

$$
\begin{aligned}
&\xlongequal{\frac{1}{n}=\frac{1}{A}f(\xi_k)\Delta x_k} \left|\frac{1}{A}\sum_{k=1}^{n} f(x_k)f(\xi_k)\Delta x_k - \frac{1}{A}\sum_{k=1}^{n} f^2(x_k)\Delta x_k\right| \\
&\qquad + \left|\frac{1}{A}\sum_{k=1}^{n} f^2(x_k)\Delta x_k - \frac{1}{A}\int_0^1 f^2(x)\,\mathrm{d}x\right| \\
&\le \varepsilon \left|\frac{1}{A}\sum_{k=1}^{n} f(x_k)\Delta x_k\right| + \left|\frac{1}{A}\sum_{k=1}^{n} f^2(x_k)\Delta x_k - \frac{1}{A}\int_0^1 f^2(x)\,\mathrm{d}x\right| \\
&\to \varepsilon, \quad \text{as } n \to \infty.
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary, we have

$$
\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f(x_k) = \frac{B}{A}.
$$

□

Problem 1.9. *Prove that for any* $n \in \mathbb{N}_+$, *there is* $\left|\int_1^2 \sin\left(nx - \frac{1}{x}\right)\,\mathrm{d}x\right| < \frac{2}{n}$.

Proof. Let

$$
t = x - \frac{1}{nx}.
$$

It's clear that

$$
\frac{\mathrm{d}t}{\mathrm{d}x} = 1 + \frac{1}{nx^2} > 0.
$$

Hence we know that there exists inverse function of $t = t(x)$, i.e. $x = x(t)$. What's more, we have

$$
\frac{\mathrm{d}x}{\mathrm{d}t} = \left(1 + \frac{1}{nx^2}\right)^{-1}.
$$

By changing of variables, we have

$$
\int_1^2 \sin\left(nx - \frac{1}{x}\right)\,\mathrm{d}x = \int_{1-\frac{1}{n}}^{2-\frac{1}{2n}} \sin(nt)x'(t)\,\mathrm{d}t.
$$

Note that

$$
\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\left(1 + \frac{1}{nx^2}\right)^{-2} \frac{-2}{nx^3}\frac{\mathrm{d}x}{\mathrm{d}t} = \left(1 + \frac{1}{nx^2}\right)^{-3} \frac{2}{nx^3} > 0,
$$

which gives us that $x'(t)$ is monotonic increasing. Then by the second mean value theorem for definite integrals, we know that there exists ξ such that

$$
\begin{aligned}
\left|\int_1^2 \sin\left(nx - \frac{1}{x}\right)\,\mathrm{d}x\right| &= \left|\int_{1-\frac{1}{n}}^{2-\frac{1}{2n}} \sin(nt)x'(t)\,\mathrm{d}t\right| \\
&= \left|x'\left(2 - \frac{1}{2n}\right)\int_{\xi}^{2-\frac{1}{2n}} \sin(nt)\,\mathrm{d}t\right|
\end{aligned}
$$

$$
\begin{aligned}
&= \left(1+\frac{1}{4n}\right)^{-1}\frac{1}{n}\left|\cos\left(2-\frac{1}{2n}\right)-\cos\xi\right| \\
&\le \left(1+\frac{1}{4n}\right)^{-1}\frac{2}{n} \\
&< \frac{2}{n}.
\end{aligned}
$$

□

Problem 1.10. *Suppose that $f(x)$ is a nonnegative monotonic increasing function on $[0, \frac{\pi}{2}]$. Prove that when $x \in [0, \frac{\pi}{2}]$, there is $(1-\cos x)\int_0^x f(t)\,\mathrm{d}t \le x\int_0^x f(t)\sin t\,\mathrm{d}t$.*

Proof. Let

$$g(x) = \frac{1-\cos x}{x},$$

and

$$h(x) = \int_0^x f(t)\sin t\,\mathrm{d}t - g(x)\int_0^x f(t)\,\mathrm{d}t.$$

Then

$$
\begin{aligned}
h'(x) &= f(x)\sin x - g(x)f(x) - g'(x)\int_0^x f(t)\,\mathrm{d}t \\
&= f(x)\sin x - f(x)\frac{1-\cos x}{x} - \frac{x\sin x - 1 + \cos x}{x^2}\int_0^x f(t)\,\mathrm{d}t \\
&= \frac{x\sin x - 1 + \cos x}{x^2}\left(xf(x) - \int_0^x f(t)\,\mathrm{d}t\right).
\end{aligned}
$$

It's easy to see that $x\sin x - 1 + \cos x \ge 0$ on $[0, \frac{\pi}{2}]$ (Leave to the reader). Since $f(x)$ is nonnegative and monotonic increasing, we have

$$\int_0^x f(t)\,\mathrm{d}t \le xf(x),$$

which implies

$$h'(x) \ge 0$$

on $[0, \frac{\pi}{2}]$. Note that $h(0) = 0$, we have $h(x) \ge h(0) = 0$, $\forall\, x \in [0, \frac{\pi}{2}]$. Hence

$$(1-\cos x)\int_0^x f(t)\,\mathrm{d}t \le x\int_0^x f(t)\sin t\,\mathrm{d}t, \quad \forall\, x \in \left[0, \frac{\pi}{2}\right].$$

□

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