A HIGHER-DIMENSIONAL PARTIAL LEGENDRE TRANSFORM

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ABSTRACT. In this short note, we outline the application of a higher-dimensional partial Legendre transform in the study of Monge-Ampère equations. The main focus includes presenting interior regularity results for certain degenerate Monge-Ampère equations established by Rios, Sawyer, and Wheeden [RSW1], the global smoothness of the eigenfunctions of the Monge-Ampère equation obtained by Le and Savin [LS2], and a Bernstein-type theorem for some singular Monge-Ampère equations due to Huang, Tang, and Wang [HTW]. Additionally, we discuss possible extensions of this approach to fourth-order equations at the end of the note.

1. INTRODUCTION

In this short note, we introduce a higher-dimensional partial Legendre transform associated with a convex solution u. This transform is a natural generalization of the classical two-dimensional partial Legendre transform and serves as a powerful tool for deriving interior estimates for certain degenerate Monge-Ampère equations.

The two-dimensional partial Legendre transform has been widely applied in the study of Monge-Ampère equations, including Monge-Ampère type fourth-order equations and linearized Monge-Ampère equations [WZ1, Wa]. For a comprehensive overview of these applications, we refer to the survey by the author and Zhou [WZ2]. However, the study of the higher-dimensional partial Legendre transform remains relatively sparse. One possible reason is that, unlike the two-dimensional case, the transformed equations in higher dimensions often retain some nonlinearity, making their analysis still more challenging. Nevertheless, by imposing certain nondegeneracy conditions on part of the Hessian matrix of the convex function u, some progress has been made using the higher-dimensional partial Legendre transform; see [RSW1, RSW2, LS2, HTW]. The main purpose of this note it to outline the role of the higher-dimensional partial Legendre transform in these works. Before presenting the main results, we first define the higher-dimensional partial Legendre transform and derive some useful identities that will aid our analysis.

We consider a convex solution u to the Monge-Ampère equation

$$\det D^2 u = f(x, u, Du), \quad x \in \Omega, \tag{1.1}$$

where f is smooth and nonnegative in $\Omega \times \mathbb{R} \times \mathbb{R}^n$, and Ω is a convex domain in $\mathbb{R}^n$.

The partial Legendre transform in the x'-variable is given by

$$u^\star(y', y_n) = \sup \left(x' \cdot y' - u(x', y_n) \right),$$

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where the supremum is taken with respect to x' for fixed y_n, i.e., for all x' such that $(x', y_n) \in \Omega$. This definition is generalized two-dimensional partial Legendre transform. When $u \in C^2(\Omega)$ is strictly convex in the x'-variable, the mapping

$$\mathcal{P} : (x', x_n) \mapsto (D_{x'}u(x), x_n)$$

is injective, and we denote the image of Ω under $\mathcal{P}$ as $\Omega^\star$. In this case, we have

$$u^\star(y) = x' \cdot D_{x'}u - u(x) \quad \text{in} \quad \Omega^\star.$$

This follows directly from the strict convexity of u with respect to the x'-variable [GP].

By direct computations, we have

$$\begin{aligned} &\frac{\partial y_n}{\partial x'} = 0, \quad \frac{\partial y_n}{\partial x_n} = 1, \\ &\frac{\partial y'}{\partial x'} = D^2_{x'}u, \quad \frac{\partial y'}{\partial x_n} = D^2_{x'x_n}u, \end{aligned}$$

and

$$\begin{aligned} &\frac{\partial x_n}{\partial y'} = 0, \quad \frac{\partial x_n}{\partial y_n} = 1, \\ &\frac{\partial x'}{\partial y'} = (D^2_{x'}u)^{-1}, \\ &\frac{\partial x_i}{\partial y_n} = U^{ni}(\det D^2_{x'}u)^{-1}, \quad i = 1, 2, \cdots, n-1, \end{aligned}$$

where $\{U^{ij}\}$ is the cofactor matrix of D^2u.

Using these transformations, we find that

$$u^\star_{y_n} = -u_{x_n} \text{ and } u^\star_{y_i} = x_i,\ i = 1, 2, \cdots, n-1,$$

and

$$D^2_{y'}u^\star = (D^2_{x'}u)^{-1}, \quad u^\star_{y_ny_n} = -\sum_{i=1}^{n} \frac{\partial u_{x_n}}{\partial x_i}\frac{\partial x_i}{\partial y_n} = -u_{x_nx_n} - \sum_{i=1}^{n-1} u_{x_ix_n}\frac{U^{ni}}{\det D^2_{x'}u}.$$

Since

$$\det D^2_{y'}u^\star = (\det D^2_{x'}u)^{-1} = 1/U^{nn},$$

it follows that

$$\begin{aligned} \frac{-u^\star_{y_ny_n}}{\det D^2_{y'}u^\star} &= \left(u_{x_nx_n} + \sum_{i=1}^{n-1} u_{x_ix_n}\frac{U^{ni}}{\det D^2_{x'}u}\right)\det D^2_{x'}u \\ &= \sum_{i=1}^{n} u_{x_ix_n}U^{ni} = \det D^2u. \end{aligned}$$

Then combining (1.1), we have

$$f(D_{y'}u^\star, y_n, y' \cdot D_{y'}u^\star - u^\star, y', -u_{y_n}) \det D^2_{y'}u^\star + u^\star_{y_ny_n} = 0. \tag{1.2}$$

The organization of this note is as follows. In Section 2, we present the interior regularity of degenerate Monge-Ampère equations arising from prescribed Gaussian curvatures. Next, in Section 3, we demonstrate the global smoothness of the eigenfunctions of the Monge-Ampère operator. Then, a Bernstein-type theorem is established in Section 4. Finally, in Section 5, we discuss possible extensions to Monge-Ampère type fourth-order equations.

2. Regularity of degenerate Monge-Ampère equations

In this section, we investigate the regularity of solutions to the generalized Monge-Ampère equation,

$$\det D^2 u = f(x,u,Du) \sim \left(|x_n|^{2m} + \psi(x)\right) K(x,u,Du), \tag{2.1}$$

where K is a smooth and positive function defined on $\Omega \times \mathbb{R} \times \mathbb{R}^n$, ψ is smooth and positive on Ω, m is a positive integer, and $\psi^{1/2m}$ is Lipschitz continuous. This type of equation arises in prescribed Gaussian curvature problems, which have been extensively studied using various methods. We briefly review some historical developments here.

When $f > 0$, equation (2.1) is elliptic, and its theory is well developed. For instance, if $f = f(x)$, Caffarelli, Nirenberg, and Spruck [CNS] established the existence of a unique smooth convex solution u to the Dirichlet problem for (2.1) in Ω with smooth data, provided that $\partial\Omega$ has positive Gaussian curvature. However, if f is allowed to vanish in Ω, regularity may fail dramatically. For example, if $u(x) = |x|^{2+\frac{2}{n}}$, then by rotation invariance and homogeneity, (2.1) holds with $f = c_n|x|^2$. Consequently, u is a $C^{2,\frac{2}{n}}$ solution and cannot possess higher regularity, even though f is an analytic function that vanishes to the least possible order. The best possible regularity for the degenerate Dirichlet problem was established by Guan [Gu2] and later refined by Guan, Trudinger, and Wang [GTW]; for nonnegative and smooth f, they proved the existence of a unique convex solution $u \in C^{1,1}(\overline{\Omega})$ to the Dirichlet problem for (2.1) in the Alexandrov sense.

In two dimensions, Guan [Gu1] demonstrated that a $C^{1,1}(\Omega)$ solution u to (2.1) is smooth if f vanishes to finite order in a specific sense and if one principal curvature of u is bounded away from zero. This result highlights the rank of the Hessian of u as a crucial obstacle to regularity, even in subelliptic cases. In higher dimensions, Rios, Sawyer, and Wheeden [RSW1] extended Guan's regularity theorem by employing a higher-dimensional partial Legendre transform. They showed that a convex solution $u \in C^{2,1}(\Omega)$ to (2.1) is smooth if f vanishes to finite order in a certain sense and if at least $n-1$ of the principal curvatures of u remain bounded away from zero (fewer than $n-1$ nonvanishing principal curvatures are insufficient).

In the following, we present the method developed by Rios, Sawyer, and Wheeden, which employs the higher-dimensional partial Legendre transform to establish specific regularity properties of solutions. The main result is stated as follows.

Theorem 2.1. *Let $u \in C^{2,1}(\Omega)$ be a convex solution to (2.1). If we further assume that the determinant of the Hessian with respect to the variables $x' = (x_1, \ldots, x_{n-1})$, denoted by $\det D^2_{x'}u$, is positive everywhere in Ω, then $u \in C^\infty(\Omega)$.*

In proving Theorem 2.1, we follow the approach of Rios, Sawyer, and Wheeden [RSW1], which extends Guan's two-dimensional result [Gu1]. To proceed, we first introduce some definitions and theorems.

Definition 2.2. *Let $A(x) = [a_{ij}(x)]_{i,j=1}^n$ be a symmetric nonnegative matrix with bounded measurable coefficients defined in a domain $\Omega \subset \mathbb{R}^n$. We say that a vector field $T = \sum_{i=1}^n \alpha_i(x) \frac{\partial}{\partial x_i}$, with bounded coefficients α_i, is* subunit *with respect to $A(x)$ in Ω if*

$$\left(\sum_{i=1}^n \alpha_i(x) \xi_i \right)^2 \leq \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j, \quad x \in \Omega, \;\; \xi \in \mathbb{R}^n.$$

Definition 2.3. *Let $A(x) = [a_{ij}(x)]_{i,j=1}^n$ be a symmetric nonnegative Lipschitz matrix defined in a domain $\Omega \subset \mathbb{R}^N$. We say $A(x)$ is* subordinate *in Ω if*

$$\sum_{j=1}^n \left(\sum_{i=1}^n \frac{\partial}{\partial y_\ell} a_{ij}(y) \xi_i \right)^2 \leq C \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j, \quad y \in \Omega, \;\; \xi \in \mathbb{R}^n, \;\; 1 \leq \ell \leq N. \tag{2.2}$$

Note that (2.2) can be rephrased as

$$\left[\partial_\ell A(y)\right]^2 \leq C A(y), \quad y \in \Omega,$$

where $B \leq A$ means $A - B$ is nonnegative semidefinite. We will use (2.2) mainly when $N = n$, in which case $A(x)$ is subordinate in Ω if and only if there is $c > 0$ such that the vector fields associated to the rows of$\partial_\ell A(x)$, namely $c \sum_{i=1}^n \frac{\partial}{\partial x_\ell} a_{ij}(x) \frac{\partial}{\partial x_i}$, are subunit with respect to $A(x)$ in Ω for $1 \leq j \leq n$, $1 \leq \ell \leq n$.

Definition 2.4. *Let $A(x) = [a_{ij}(x)]_{i,j=1}^n$ be a symmetric nonnegative semidefinite matrix with bounded measurable coefficients defined in a domain $\Omega \subset \mathbb{R}^n$. We say that*

$$Lu := \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} u \right)$$

is α-subelliptic *in Ω for $\alpha > 0$, if there is a positive function $C(\cdot,\cdot,\cdot,\cdot,\cdot)$ defined on $\mathcal{P}(\Omega) \times [0,\infty) \times [0,\infty) \times [0,\infty) \times [0,\infty)$, increasing in each variable separately, such that for all m-tuples $\mathbf{T} = (T_1, \cdots, T_m)$ of bounded subunit (with respect to $A(x)$) vector fields, all bounded functions f, $\mathbf{g}$, and all compact subsets K of Ω, every weak solution $u \in W^{1,2}(\Omega)$ to the divergence form equation*

$$Lu = f + \mathbf{T} \cdot \mathbf{g}$$

satisfies

$$\|u\|_{C^\alpha(K)} \leq C\left(K, \|u\|_{L^2(\Omega)}, \|f\|_{L^\infty(\Omega)}, \|\mathbf{g}\|_{L^\infty(\Omega)}, m\right).$$

Definition 2.5. *We say that* $L = \nabla' A(x) \nabla$ *is* α-elliptic extendible *in* Ω *for* $\alpha > 0$*, if for every* x_0 *and* Ω_1 *with* $x_0 \in \Omega_1 \subset\subset \Omega$*, there exists a symmetric smooth nonnegative subordinate matrix* $B(x)$ *in* Ω *satisfying:*

(1) $B(x)$ *vanishes in a neighborhood* $\mathcal{N} \subset\subset \Omega_1$ *of* x_0*,*
(2) $B(x)$ *is elliptic in* $\Omega \setminus \Omega_1$*,*
(3) *The perturbed operator*

$$L_\varepsilon = \nabla' \left(A(x) + B(x) + \varepsilon I \right) \nabla$$

is α*-subelliptic in* Ω*, uniformly for* $0 < \varepsilon < 1$*.*

We will need the following extension of a theorem in [Gu1].

Theorem 2.6. *Suppose* $\mathbf{p} = (p_\ell)_{1 \le \ell \le N}$*,* $\mathbf{v} = (v_\ell)_{1 \le \ell \le N_0} \in C^{0,1}(\Omega)$*, and that* $\mathbf{p}$ *is a weak solution of the system*

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x, \mathbf{v}, \mathbf{p}) \frac{\partial}{\partial x_j} p_\ell \right) = h_\ell(x, \mathbf{v}, \mathbf{p}, D\mathbf{p}), \quad 1 \le \ell \le N, \tag{2.3}$$

where $a_{ij} \in C^\infty(\Gamma)$*,* Γ *is a subdomain of* $\Omega \times \mathbb{R}^{N_0} \times \mathbb{R}^N$*,* $A(x, \mathbf{v}, \mathbf{p}) = [a_{ij}(x, \mathbf{v}, \mathbf{p})]_{i,j=1}^n$ *is symmetric, nonnegative semidefinite, and subordinate in relatively compact subdomains of* Γ*,* $\mathbf{h} = (h_\ell)_{1 \le \ell \le N} \in C^\infty(\Gamma \times \mathbb{R}^{nN})$ *and where*

$$D\mathbf{v} = \mathbf{\Psi}(x, \mathbf{v}, \mathbf{p}),$$

for $\mathbf{\Psi} \in C^\infty(\Gamma)$*. Let*

$$\widetilde{L} := \nabla' \widetilde{A}(x) \nabla = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(\widetilde{a}_{ij}(x) \frac{\partial}{\partial x_j} \right)$$

be the scalar linear operator with $\widetilde{a}_{ij}(x) = a_{ij}(x, \mathbf{v}(x), \mathbf{p}(x))$*. Suppose that* $\widetilde{L}$ *is* α*-elliptic extendible in* Ω *for some* $\alpha > 0$*, that*

$$\operatorname{trace} \widetilde{A}(x) \ge c > 0 \quad in \quad \Omega, \tag{2.4}$$

and that $\mathbf{h}$ *has the product decomposition*

$$h_\ell(x, \mathbf{v}, \mathbf{p}, D\mathbf{p}) = H_{\ell,0}(x, \mathbf{v}, \mathbf{p}) + \sum_{\mu=1}^{M} H_{\ell,\mu}(x, \mathbf{v}, \mathbf{p}) \Phi_{\ell,\mu}(x, \mathbf{v}, \mathbf{p}, D\mathbf{p}), \quad 1 \le \ell \le N,$$

with $H_{\ell,\mu}$ *and* $\Phi_{\ell,\mu}$ *smooth functions of their arguments, and where the vector fields*

$$H_{\ell,\mu}(x, \mathbf{v}(x), \mathbf{p}(x)) \frac{\partial}{\partial x_k}$$

are subunit with respect to $\widetilde{A}$ *for* $1 \le \mu \le M$*,* $1 \le \ell \le N$*,* $1 \le k \le n$*. Then the functions* $\mathbf{p}$ *and* $\mathbf{v}$ *are both smooth in* Ω*.*

The reverse Hölder norm $\|a\|_{RH_\infty}$ of a nonnegative function a on the real line is given by the least constant C such that

$$\sup_{s\in I} a(s) \leq C\frac{1}{|I|}\int_I a(s)\,\mathrm{d}s,$$

for all intervals I.

Theorem 2.7. *Let $A(x)=[a_{ij}(x)]_{i,j=1}^n$ with $a_{ij}\in L^\infty(\Omega)$, satisfying for some $c, C>0$:*

$$c\left(\xi_n^2+a(x)^2|\xi'|^2\right)\leq\langle A(x)\xi,\xi\rangle\leq C\left(\xi_n^2+a(x)^2|\xi'|^2\right),\quad \forall\, x\in\Omega,\ \xi=(\xi',\xi_n)\in\mathbb{R}^n$$

where the coefficient $a(x)$ satisfies $\|a\|_{C^{0,1}(\Omega)}\leq C$, $\|a(x',\cdot)\|_{RH_\infty}\leq C$, and the non-degeneracy $\|a(x',\cdot)\|_{L^\infty}\geq c>0$ for $x'=(x_1,\cdots,x_{n-1})\in\mathbb{R}^{n-1}$. Then the operator $L=\nabla' A(x)\nabla$ is α-subelliptic in Ω with $\alpha>0$ depending only on c, C.

Proof of Theorem 2.1. Denote $v=\frac{\partial}{\partial y_\ell}u^\star$ for some $\ell\in\{1,2,\dots,n-1\}$. Differentiating (1.2) with respect to y_ℓ yields

$$\frac{\partial}{\partial y_\ell}\left(f\det D^2_{y'}u^\star\right)+v_{y_ny_n}=0.$$

Since

$$\delta_{i\ell}\det D^2_{y'}u^\star=\sum_{k=1}^{n-1}U^{\star ik}_{y'}u^\star_{y_\ell y_k},$$

we obtain

$$\begin{aligned}0&=\frac{\partial}{\partial y_\ell}\left(f\det D^2_{y'}u^\star\right)+v_{y_ny_n}\\&=\frac{\partial}{\partial y_i}\left(f\delta_{i\ell}U^{\star ik}_{y'}u^\star_{y_\ell y_k}\right)+v_{y_ny_n}\\&=\frac{\partial}{\partial y_i}\left(fU^{\star ik}_{y'}v_{y_k}\right)+v_{y_ny_n}.\end{aligned}$$

Hence we know

$$A:=\begin{pmatrix}fU^\star_{y'} & 0\\ 0 & 1\end{pmatrix},$$

where $U^\star_{y'}$ denote the cofactor matrix of $D^2_{y'}u^\star$. Thus, $U^\star_{y'}=(\det D^2_{x'}u)^{-1}D^2_{x'}u$. In order to apply Theorem 2.6 with x there replaced by y, we consider the linear operator

$$\nabla'\widetilde{A}\nabla=\frac{\partial}{\partial y_i}\left(f(D_{y'}u^\star(y),y_n,y'\cdot D_{y'}u^\star(y)-u^\star(y),y',-u_{y_n}(y))U^{\star ij}_{y'}(y)\frac{\partial}{\partial y_j}\right)+\frac{\partial^2}{\partial y_n^2},$$

here $\widetilde{A}$ is given by

$$\widetilde{A}=\begin{pmatrix}f(D_{y'}u^\star(y),y_n,y'\cdot D_{y'}u^\star(y)-u^\star(y),y',-u_{y_n}(y))U^\star_{y'}(y) & 0\\ 0 & 1\end{pmatrix},$$

and the function f satisfies

$$f \sim \left(|y_n|^{2m} + \psi(D_{y'}u(y), y_n)\right) K(D_{y'}u^\star(y), y_n, y' \cdot D_{y'}u^\star(y) - u^\star(y), y', -u_{y_n}(y)).$$

We now verify the hypotheses of Theorem 2.6. The lower bound $c = 1$ in the trace (2.4) is obvious. Note that since $U^\star_{y'} = (\det D^2_{x'}u)^{-1} D^2_{x'}u$ is positive definite by assumption, the quadratic form of A has a lower bound

$$\xi \cdot A\xi \geq cf|\xi'|^2 + \xi_n^2, \quad \forall\, \xi = (\xi', \xi_n) \in \mathbb{R}^n.$$

The standard inequality $|Df| \leq C\sqrt{f}$ now shows that A is subordinate in relatively compact subdomains of its domain. Thus in order to apply Theorem 2.6, it only remains to prove that $\nabla'\widetilde{A}\nabla$ is α-elliptic extendible in $\Omega^\star$.

So for fixed point $y_0 \in \Omega^\star$, without loss of generality, we may suppose that $f = 0$ at y_0 and that in fact $y_0 = 0$. We choose $\delta > 0$ sufficiently small and a smooth nonnegative function $\eta(y')$, such that $\eta = 0$ for $|y'| < \delta$, $\eta > 0$ for $|y'| > 2\delta$ and $\eta(y')^{\frac{1}{2m}}$ is Lipschitz (i.e. all zeroes of η vanish to order at least $2m$). Then we define

$$B := \begin{pmatrix} \eta(y')I_{n-1} & 0 \\ 0 & 0 \end{pmatrix},$$

where I_{n-1} denotes the $(n-1) \times (n-1)$ identity matrix. Clearly the operator $\nabla'(\widetilde{A} + B)\nabla$ is elliptic in $\Omega^\star \setminus \overline{B_{3\delta}}$ since $|y_n|^{2m} + \eta(y')$ is positive there, and $\nabla'(\widetilde{A} + B)\nabla = \nabla'\widetilde{A}\nabla$ in B_δ. The inequality $|D\eta| \leq C\sqrt{\eta}$ shows that B is subordinate in $\Omega^\star$. We further observe using (2.1) that

$$c\left(\xi_n^2 + a_\varepsilon(y)^2|\xi'|^2\right) \leq \xi'(\widetilde{A} + B + \varepsilon I)\xi \leq C\left(\xi_n^2 + a_\varepsilon(y)^2|\xi'|^2\right)$$

for $0 \leq \varepsilon < 1$, where

$$a_\varepsilon(y) = \sqrt{|y_n|^{2m} + \psi(D_{y'}u^\star(y), y_n) + \eta(y') + \varepsilon} \tag{2.5}$$

since $U^\star_{y'}$ is positive definite and K is positive in $\Omega^\star$. We now claim that $a_\varepsilon(y)$ satisfies the hypotheses of Theorem 2.7 uniformly in $0 \leq \varepsilon < 1$, namely that

$$\begin{aligned} \|a_\varepsilon\|_{C^{0,1}(\Omega^\star)} &\leq C, \\ \|a_\varepsilon(y', \cdot)\|_{RH_\infty} &\leq C, \\ \|a_\varepsilon(y', \cdot)\|_{L^\infty} &\geq c > 0. \end{aligned} \tag{2.6}$$

With this established, Theorem 2.7 completes the proof that $\nabla'\widetilde{A}\nabla$ is α-elliptic extendible in $\Omega^\star$. Then Theorem 2.6 shows that $\partial_\ell u^\star$, $1 \leq \ell \leq n-1$ are smooth in $\Omega^\star$. Since $\frac{\partial y}{\partial x} = \det D^2_{x'}u > 0$, we conclude that u is smooth in Ω, and this completes the proof of Theorem 2.1.

So it remains to prove (2.6). It is enough to prove the case $\varepsilon = 0$ since we may replace ψ by $\psi + \varepsilon$ in (2.5). We now write $a(y)$ for $a_\varepsilon(y)$. The first inequality in (2.6) follows immediately from the fact that $D_{y'}u^\star$ is Lipschitz, since then so also are the functions $|y_n|$, $\psi(D_{y'}u^\star, y_n)^{\frac{1}{2m}}$ and $\eta(y')^{\frac{1}{2m}}$, and hence their ℓ^{2m} length as a vector in $\mathbb{R}^3$; $a(y)$ is the m^{th} power of this length. The RH_∞ inequality,

$$\sup_{s\in I} a(y',s) \le C\frac{1}{|I|}\int_I a(y',s)\,\mathrm{d}s,$$

for all intervals I and points y', is easier to check separately in the two cases

$$\sup_{s\in I}|s|^m \ge \sup_{s\in I}\sqrt{\widetilde{\psi}(y',s)},$$

$$\sup_{s\in I}|s|^m \le \sup_{s\in I}\sqrt{\widetilde{\psi}(y',s)},$$

where we have set $\widetilde{\psi}(y',y_n) = \psi(D_{y'}u^\star(y),y_n) + \eta(y')$. Indeed, in the first case

$$\sup_{s\in I} a(y',s) \le C\sup_{s\in I}|s|^m \le C\frac{1}{|I|}\int_I |s|^m\,\mathrm{d}s \le C\frac{1}{|I|}\int_I a(y',s)\,\mathrm{d}s.$$

In the second case,

$$\sup_{s\in I} a(y',s) \le C\sup_{s\in I}\sqrt{\widetilde{\psi}(y',s)}.$$

Let $s_1\in I$ be such that $\sqrt{\widetilde{\psi}(y',s_1)} = \sup_{s\in I}\sqrt{\widetilde{\psi}(y',s)}$. Then we observe that

$$|I|^m \le C\sup_{s\in I}|s|^m \le C\sqrt{\widetilde{\psi}(y',s_1)}$$

implies

$$\widetilde{\psi}(y',s_1)^{\frac{1}{2m}} \ge c|I|.$$

Since $\widetilde{\psi}(y',y_n)^{\frac{1}{2m}} = [\psi(D_{y'}u^\star(y),y_n)+\eta(y')]^{\frac{1}{2m}}$ is Lipschitz, we have

$$\widetilde{\psi}(y',y_n)^{\frac{1}{2m}} \ge \frac{1}{2}\widetilde{\psi}(y',s_1)^{\frac{1}{2m}}$$

for y_n in an interval J of length at least $c|I|$ that contains s_1 and is contained in I. Then we conclude,

$$\begin{aligned}\frac{1}{|I|}\int_I a(y',s)\,\mathrm{d}s &\ge \frac{1}{|I|}\int_J \sqrt{\widetilde{\psi}(y',s)}\,\mathrm{d}s\\ &\ge \frac{|J|}{|I|}\sqrt{\frac{1}{2^{2m}}\widetilde{\psi}(y',s_1)}\\ &\ge c\frac{|I|}{|I|}\sup_{s\in I} a(y',s)\\ &\ge c\sup_{s\in I} a(y',s).\end{aligned}$$

The nondegeneracy inequality in (2.6) follows from $a(y)\ge |y_n|^m$. □

3. Smoothness of the eigenfunctions of Monge-Ampère equations

In this section, we prove the global smoothness of the eigenfunctions of the Monge-Ampère operator $(\det D^2u)^{1/n}$. The eigenvalue problem for this operator was first investigated by Lions [Li], who demonstrated that there exists a unique (up to positive multiplicative constants) nonzero convex eigenfunction $u \in C^{1,1}(\overline{\Omega}) \cap C^\infty(\Omega)$ satisfying the problem

$$\begin{cases} \left(\det D^2u\right)^{\frac{1}{n}} = \lambda|u| & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

We note that λ, called the Monge-Ampère eigenvalue of Ω, has an interesting stochastic interpretation given by Lions [Li]. The question of obtaining global higher derivative estimates up to the boundary for the eigenfunction u is a well-known open problem, see for example Trudinger and Wang's survey paper [TW3].

In the two-dimensional case, Hong, Huang, and Wang [HHW] resolved this question affirmatively. In higher dimensions, Savin [Sa] established the global $C^2(\overline{\Omega})$ estimate for the eigenfunction u. Later, using Savin's C^2-estimate from [Sa] and perturbation arguments [Ca, CC], Le and Savin [LS2] obtained basic boundary Hölder second derivative estimates for solutions to $\det D^2u \sim d^\alpha_{\partial\Omega}$, which in turn led to global $C^{2,\beta}$ estimates up to the boundary for eigenfunctions of the Monge-Ampère operator $(\det D^2u)^{1/n}$. Finally, by employing the higher-dimensional partial Legendre transform, they resolved the problem of global smoothness of the eigenfunction u in all dimensions.

In the following, we will present their arguments related to the higher-dimensional partial Legendre transform. We begin by stating the main theorem.

Theorem 3.1. *Let Ω be a bounded and uniformly convex domain in $\mathbb{R}^n$. Assume $\partial\Omega \in C^\infty$ and u satisfies* (3.1)*. Then $u \in C^\infty(\overline{\Omega})$.*

As stated earlier, we first need to establish the global $C^{2,\beta}$ estimates for the eigenfunction u, which is both essential and intricate. For the details, we refer to the original paper [LS2], and we simply state the result here for later use.

Theorem 3.2 ([LS2, Theorem 1.3])**.** *Let Ω be a bounded and uniformly convex domain in $\mathbb{R}^n$. Assume $\partial\Omega \in C^{2,\beta}$ with $\beta \in \left(0, \frac{2}{n+2}\right)$, and u satisfies* (3.1)*. Then $u \in C^{2,\beta}(\overline{\Omega})$.*

To prove Theorem 3.1, we first perform a Hodograph transform and reduce (3.1) to a similar equation in the upper half-space. Then, we apply the partial Legendre transform in the nondegenerate x' coordinates. The structure of the equation satisfied by the transformed function allows us to utilize the $C^{2,\beta}$ estimates for the Monge-Ampère eigenfunctions obtained in Theorem 3.2, along with Schauder estimates for linear equations with Hölder coefficients modeled by a degenerate Grushin-type operator. These steps yield the desired global C^∞ regularity.

We first write an equation in the upper half-space that is locally equivalent to (3.1). After a dilation we may assume that $\lambda = 1$. The Monge-Ampère eigenfunctions are

C^∞ in the interior of Ω, so it remains to prove their C^∞ smoothness near the boundary $\partial\Omega$. Assume that $0 \in \partial\Omega$ and e_n is the inner normal of $\partial\Omega$ at 0. We make the rotation of coordinates

$$y_n = -x_{n+1}, \quad y_{n+1} = x_n, \quad y_k = x_k \quad (1 \le k \le n-1).$$

In the new coordinates, the graph of u near the origin can be represented as $y_{n+1} = \widetilde{u}(y)$ in the upper half-space $\mathbb{R}^n_+ = \{y \in \mathbb{R}^n : y_n > 0\}$. The tangent plane at 0 is given by

$$x_{n+1} - u_n x_n - u_k x_k = 0.$$

After the above rotation of coordinates, it is given by

$$-y_n - u_n y_{n+1} - u_k y_k = 0, \quad \text{or} \quad y_{n+1} + \frac{y_n}{u_n} + \frac{u_k}{u_n} y_k = 0.$$

Hence

$$\widetilde{u}_{y_n} = -\frac{1}{u_{x_n}} > 0 \,(\text{near } 0), \ \widetilde{u}_{y_k} = -\frac{u_k}{u_n}.$$

Note that the Gauss curvature is invariant under the transform, hence

$$K = \frac{\det D^2\widetilde{u}}{(1+|D\widetilde{u}|^2)^{\frac{n+2}{2}}} = \frac{\det D^2 u}{(1+|Du|^2)^{\frac{n+2}{2}}} = \frac{|u|^n}{(1+|Du|^2)^{\frac{n+2}{2}}}.$$

Now, we obtain the following equation in a neighborhood of the origin in the upper half-space $\{y_n > 0\}$:

$$\det D^2\widetilde{u} = K(1+|D\widetilde{u}|^2)^{\frac{n+2}{2}} = |y_n|^n \left(\frac{1+|D\widetilde{u}|^2}{1+|Du|^2}\right)^{\frac{n+2}{2}} = |y_n|^n \left(\frac{1}{u_n^2}\right)^{\frac{1}{n+2}} = y_n^n \widetilde{u}_n^{n+2}.$$

Near the origin, the boundary $\partial\Omega$ is given by $x_n = \phi(x')$ in the original coordinates. Thus, the boundary condition for $\widetilde{u}$ is $\widetilde{u} = \phi$ on $\{y_n = 0\}$. Thus, locally, we have for some small $r_0 > 0$ (now relabeling y by x)

$$\begin{cases} \det D^2\widetilde{u} = x_n^n \widetilde{u}_n^{n+2} & \text{in } B_{r_0}^+, \\ \qquad \widetilde{u} = \phi & \text{on } \{x_n = 0\} \cap B_{r_0}. \end{cases} \tag{3.2}$$

Since $u \in C^{2,\beta}(\overline{\Omega})$ by Theorem 3.2, we have $\widetilde{u} \in C^{2,\beta}(\overline{B}_{r_0}^+)$ for some small $\beta > 0$, and $\widetilde{u}_n > c$. It remains to show that solutions of (3.2) with $\phi \in C^\infty$ are smooth up to the boundary in a neighborhood of the origin. For simplicity of notation, we relabel $\widetilde{u}$ from (3.2) as u. Now, we apply the partial Legendre transformation to the solutions u of (3.2). Combining (1.2), we know that if u satisfies (3.2), then $u^\star$ (which is convex in y' and concave in y_n) satisfies

$$\begin{cases} y_n^\alpha(-u_n^\star)^{n+2} \det D^2_{y'} u^\star + u_{nn}^\star = 0 & \text{in } B_\delta^+, \\ \qquad\qquad u^\star = \phi^\star & \text{on } \{y_n = 0\} \cap B_\delta, \end{cases} \tag{3.3}$$

where $\alpha = n$. Moreover $u^\star \in C^{2,\beta}(\overline{B}_\delta^+)$, $-u_n^\star > c$ and $\phi^\star \in C^\infty$.

In order to obtain the smoothness of $u^\star$ from (3.3), we should establish Schauder estimates for its linearized equation. We consider linear equations of the form

$$x_n^\alpha \sum_{i,j=1}^{n-1} a^{ij} v_{ij} + v_{nn} = x_n^\alpha f(x) \tag{3.4}$$

with a^{ij} uniformly elliptic

$$\lambda I_{n-1} \le (a^{ij})_{1\le i,j\le n-1} \le \Lambda I_{n-1}.$$

We define the distance d_α between two points y and z in the upper half-space by

$$d_\alpha(y,z) := |y'-z'| + \left| y_n^{\frac{2+\alpha}{2}} - z_n^{\frac{2+\alpha}{2}} \right|.$$

If function w is C^γ respect to d_α (with $\gamma \in \left(0, \frac{2}{2+\alpha}\right)$), we write

$$w \in C^\gamma_\alpha(\overline{B}^+_\delta)$$

and define

$$[w]_{C^\gamma_\alpha(\overline{B}^+_1)} = \sup_{\substack{y,z\in \overline{B}^+_1 \\ y\neq z}} \frac{|w(y)-w(z)|}{(d_\alpha(y,z))^\gamma}, \quad \|w\|_{C^\gamma_\alpha(\overline{B}^+_1)} = \|w\|_{L^\infty(\overline{B}^+_1)} + [w]_{C^\gamma_\alpha(\overline{B}^+_1)}.$$

Then we have

Proposition 3.3 (Schauder estimate)**.** *Assume that v solves* (3.4) *in $\overline{B}^+_\delta$ and*

$$v = \varphi(x') \quad \text{on } \{x_n = 0\} \cap \overline{B}^+_\delta.$$

If $a^{ij}, f \in C^\gamma_\alpha(\overline{B}^+_\delta)$ with $\frac{\gamma}{2} \le \frac{\min\{1,\alpha\}}{2+\alpha}$, and $\varphi \in C^{2,\gamma}$, then

$$Dv,\ D^2 v \in C^\gamma_\alpha(\overline{B}^+_{\delta/2}).$$

The proof of Proposition 3.3 is standard, and we refer to Section 6.3 of [LS2] for the detailed proof. By repeatedly differentiating (3.4) in the x' direction, we readily obtain Schauder estimates for higher derivatives. Below, $m = (m_1, \dots, m_{n-1})$ denotes a multi-index with m_i being nonnegative integers.

Corollary 3.4. *If in Proposition 3.3 $\varphi \in C^{k+2,\gamma}$ for some integer $k \ge 0$ and*

$$D^m_{x'} a^{ij},\ D^m_{x'} f \in C^\gamma_\alpha(\overline{B}^+_\delta) \quad \forall\, m \text{ with } |m| \le k,$$

then

$$DD^m_{x'} v,\ D^2 D^m_{x'} v \in C^\gamma_\alpha(\overline{B}^+_{\delta/2}) \quad \forall\, m \text{ with } |m| \le k.$$

Now, we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Recall $u^\star$ satisfies (3.3):

$$\begin{cases} y_n^\alpha (-u^\star_n)^{n+2} \det D^2_{y'} u^\star + u^\star_{nn} = 0 & \text{in } B^+_\delta, \\ \qquad\qquad\qquad\qquad u^\star = \phi^\star & \text{on } \{y_n = 0\} \cap B_\delta, \end{cases}$$

with $\alpha = n$, $u^\star \in C^{2,\beta}(\overline{B}_\delta^+)$, $-u_n^\star > c$ and $\phi^\star \in C^\infty$.

Fix $k < n$. Then $v = u_k^\star$ solves the linearized equation

$$y_n^\alpha \sum_{i,j=1}^{n-1} a^{ij} v_{ij} + v_{nn} = y_n^\alpha f(y) \quad \text{in } B_\delta^+ \tag{3.5}$$

where

$$a^{ij} = (-u_n^\star)^{n+2} U_{y'}^{\star ij}, \quad f(y) = (n+2)(-u_n^\star)^{n+1} u_{nk}^\star \det D_{y'}^2 u^\star,$$

and $U_{y'}^\star$ denotes the cofactor matrix of $D_{y'}^2 u^\star$. Since $u^\star \in C^{2,\beta}(\overline{B}_\delta^+)$ we obtain $Du^\star$, $D^2 u^\star \in C_\alpha^\gamma(\overline{B}_\delta^+)$ for some small $\gamma > 0$, hence a^{ij}, $f \in C_\alpha^\gamma(\overline{B}_\delta^+)$.

By Proposition 3.3, $D^2 v \in C_\alpha^\gamma$ up to the boundary in $\overline{B}_{\delta/2}^+$ which in turn implies $D_{y'} a^{ij}$, $D_{y'} f \in C_\alpha^\gamma(\overline{B}_{\delta/2}^+)$. Now we may apply Corollary 3.4 and iterate this argument to obtain that $D_{y'}^m D_{y_n}^l u^\star$ with $l \in \{0, 1, 2\}$ are continuous up to the boundary in $\overline{B}_{\delta/2}^+$ for all multi-indices $m \geq 0$. In order to obtain the continuity of these derivatives for all values of l we differentiate the equation for $u^\star$ and use that $\alpha = n$ is a nonnegative integer. Then each derivative $D_{y'}^m D_{y_n}^l u^\star$ with $l \geq 3$ can be expressed as a polynomial involving powers of y_n and derivatives $D_{y'}^q D_{y_n}^s u^\star$ with $s < l$, thus $u^\star \in C^\infty(\overline{B}_{\delta/2}^+)$ as desired. □

4. Bernstein theorem for a singular Monge-Ampère equation

In this section, we prove a Bernstein theorem for the singular Monge-Ampère type equation in the half-space, as established by Huang, Tang, and Wang [HTW]. This type of Bernstein theorem arises from investigating the regularity of the free boundary in the Monge-Ampère obstacle problem. Specifically, Huang, Tang, and Wang [HTW] studied the regularity of free boundaries for the fully nonlinear elliptic equation of Monge–Ampère type:

$$\begin{cases} \det D^2 v = f \chi_{\{v>0\}} & \text{in } \Omega, \\ v = v_0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, f and v_0 are positive functions on Ω, and χ is the characteristic function.

After applying the classical Legendre transform,

$$u(x) := \sup_{y \in \Omega} \{x \cdot y - v(y)\}, \quad x \in \Omega^* := Dv(\Omega),$$

equation (4.1) transforms into the following Monge-Ampère equation with a point singularity:

$$\det D^2 u = g(Du) + c^* \delta_0 \quad \text{in } \Omega^*, \tag{4.2}$$

where $g(Du(x)) = \frac{1}{f(y)}$ at $y = Du(x)$, and $c^* = |\{v = 0\}|$ is a constant. By duality, we know

$$\partial u(0) = \{v = 0\}.$$

Hence, heuristically, to establish the regularity of the free boundary $\partial\{v = 0\}$, it suffices to obtain the regularity of u in the spherical coordinates (since u is most likely singular at 0).

In spherical coordinates (θ, r), we make the change

$$\zeta(\theta, r) := \frac{u(\theta, r)}{r}, \quad s = r^{\frac{n}{2}},$$

which transforms (4.2) into

(4.3)
$$\det\begin{pmatrix} \left(\frac{n}{2}\right)^2\zeta_{ss} + \frac{n(n+2)}{4}\frac{\zeta_s}{s} & \frac{n}{2}\zeta_{s\theta_1} & \cdots & \frac{n}{2}\zeta_{s\theta_{n-1}} \\ \frac{n}{2}\zeta_{s\theta_1} & \zeta_{\theta_1\theta_1} + \zeta + \frac{n}{2}s\zeta_s & \cdots & \zeta_{\theta_1\theta_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{n}{2}\zeta_{s\theta_{n-1}} & \zeta_{\theta_1\theta_{n-1}} & \cdots & \zeta_{\theta_{n-1}\theta_{n-1}} + \zeta + \frac{n}{2}s\zeta_s \end{pmatrix} = g.$$

Then the regularity of the free boundary $\partial\{v = 0\}$ is thus reduced to that of ζ in (4.3). A standard approach to obtain regularity is to use a blowup argument, which simplifies equation (4.3) to the following equation:

$$\det\begin{pmatrix} \psi_{x_nx_n} + \frac{n+2}{n}\frac{\psi_{x_n}}{x_n} & \psi_{x_nx_1} & \cdots & \psi_{x_nx_{n-1}} \\ \psi_{x_nx_1} & \psi_{x_1x_1} & \cdots & \psi_{x_1x_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{x_nx_{n-1}} & \psi_{x_1x_{n-1}} & \cdots & \psi_{x_{n-1}x_{n-1}} \end{pmatrix} = \text{constant} \quad \text{in } \mathbb{R}^n_+,$$

where $\mathbb{R}^n_+ = \mathbb{R}^n \cap \{x_n > 0\}$. Hence, a key step is to classify all solutions to the above equation in the half-space.

For simplicity, we relabel ψ as u and assume the right-hand side constant is 1, i.e., we consider the following singular Monge-Ampère equation in the half-space:

$$\det\begin{pmatrix} u_{x_nx_n} + b\frac{u_{x_n}}{x_n} & u_{x_nx_1} & \cdots & u_{x_nx_{n-1}} \\ u_{x_nx_1} & u_{x_1x_1} & \cdots & u_{x_1x_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ u_{x_nx_{n-1}} & u_{x_1x_{n-1}} & \cdots & u_{x_{n-1}x_{n-1}} \end{pmatrix} = 1 \quad \text{in } \mathbb{R}^n_+. \tag{4.4}$$

We now state the following Bernstein theorem.

Theorem 4.1. *Let $u \in C^{1,1}(\overline{\mathbb{R}^n_+})$ be a solution to (4.4) with constant $b > -1$. Assume that $Du(0) = 0$, $u_{x_n}(x', 0) = 0$ $\forall\, x' \in \mathbb{R}^{n-1}$, and equation (4.4) is uniformly elliptic. Then u is a quadratic polynomial of the form*

$$u(x) = \frac{1}{2}\sum_{i,j=1}^{n-1} c_{ij}x_ix_j + \frac{1}{2}c_{nn}x_n^2, \tag{4.5}$$

where $\{c_{ij}\}_{i,j=1}^{n-1}$ is positive definite and $c_{nn} > 0$.

To prove the Bernstein theorem, we should make use of the Hölder continuity for the following degenerate elliptic equation:

$$\partial_n(x_n\partial_n v) + \sum_{i,j=1}^{n-1} \partial_i(a_{ij}(x)\partial_j v) + \sum_{i=1}^{n} b_i(x)\partial_i v = f(x) \quad \text{in } \mathbb{R}^n_+. \tag{4.6}$$

We assume that the coefficients a_{ij} and b_i satisfy the following conditions:

(i) a_{ij} are measurable and satisfy

$$C_*^{-1}|\xi|^2 \le a_{ij}\xi_i\xi_j \le C_*|\xi|^2, \quad \forall\, \xi \in \mathbb{R}^{n-1},$$

where C_* is a positive constant.

(ii) $b_1 = \cdots = b_{n-1} = 0$ and b_n is a positive constant.

Proposition 4.2. *Let $v \in C^2(B_1^+) \cap L^\infty(B_1^+)$ be a solution to* (4.6)*. Assume conditions (i), (ii), and $f \in L^q(B_1^+)$ for some $q > (n+1)/2$. Then v is continuous up to $\{x_n = 0\}$, and there exists $\alpha \in (0,1)$ such that*

$$|u(x) - u(\tilde{x})| \le C\left(\sup_{B_1^+}|u| + \|f\|_{L^q(B_1^+)}\right)|x-\tilde{x}|^\alpha \quad \forall\, x, \tilde{x} \in B^+_{1/2}, \tag{4.7}$$

where α and C are positive constants depending only on n, b, q, C_*.

The Hölder continuity of solutions for degenerate elliptic equations has been studied by many authors. For proofs of Proposition 4.2, we refer the reader to [FP] and [HHH]. To apply Proposition 4.2 to the singular Monge-Ampère equation (4.4), we make a higher-dimensional partial Legendre transform to change equation (4.4) to the form (4.6).

We know that $u^\star$ satisfies

$$u^\star_{y_ny_n} + b\frac{u^\star_{y_n}}{y_n} + \det D^2_{y'}u^\star = 0 \quad \text{in } \mathbb{R}^n_+. \tag{4.8}$$

In equation (4.8), the singular term $\frac{u^\star_{y_n}}{y_n}$ is separate from the nonlinear part $\det D^2_{y'}u^\star$. This is a very helpful property. Moreover, the Monge-Ampère operator $\det D^2_{y'}u^\star$ is of divergence form. Hence equation (4.8) is of the same form as (4.6). Moreover, we assume that $u \in C^{1,1}$ such that (4.4) is uniformly elliptic. We have the following key estimate.

Lemma 4.3. *Let $u^\star \in C^{1,1}(\overline{\mathbb{R}^n_+})$ be a solution to* (4.8) *with $b > -1$. Assume that $u^\star_{y_n}(y', 0) = 0$, $\forall\, y' \in \mathbb{R}^{n-1}$, and $D^2_{y'}u^\star$ is positive definite. Then $\frac{u^\star_{y_n}}{y_n} \in C^\alpha(\overline{\mathbb{R}^n_+})$ for some $\alpha \in (0,1)$, and we have the estimate*

$$\left\|\frac{u^\star_{y_n}}{y_n}\right\|_{C^\alpha(\mathbb{R}^{n-1}\times[0,1])} \le C \tag{4.9}$$

for a constant C depending only on b, n, $\left\|D^2_{y'}u^\star\right\|_{L^\infty(\mathbb{R}^n_+)}$, and $\left\|(D^2_{y'}u^\star)^{-1}\right\|_{L^\infty(\mathbb{R}^n_+)}$.

Proof. Let $z_n = \frac{1}{4}y_n^2$, $z' = y'$. Then equation (4.8) is changed to

$$z_n u^\star_{z_n z_n} + \frac{b+1}{2} u^\star_{z_n} + \det D_z^2 u^\star = 0 \quad \text{in } \mathbb{R}^n_+.$$

Denote $v = u^\star_{z_n}$. Differentiating the above equation in z_n gives

$$\partial_{z_n}(z_n v_{z_n}) + \sum_{i,j=1}^{n-1} \partial_{z_i}(a_{ij} v_{z_j}) + \frac{b+1}{2} v_{z_n} = 0 \quad \text{in } \mathbb{R}^n_+.$$

Here $\{a_{ij}\}_{i,j=1}^{n-1}$ is the cofactor matrix of $D^2_{z'}u^\star$. By assumption, $D^2_{y'}u^\star$ is positive definite. Hence $\lambda I \le \{a_{ij}\} \le \Lambda I$ for two positive constants λ, Λ depending only on $\left\|D^2_{y'}u^\star\right\|_{L^\infty(\mathbb{R}^n_+)}$ and $\left\|(D^2_{y'}u^\star)^{-1}\right\|_{L^\infty(\mathbb{R}^n_+)}$. Moreover,

$$v(z) = u^\star_{z_n} = \frac{2u^\star_{y_n}}{y_n} = 2\int_0^1 u^\star_{y_n y_n}(y', t y_n)\,\mathrm{d}t \in L^\infty(\mathbb{R}^n_+).$$

Therefore, all the conditions in Proposition 4.2 are satisfied.

By Proposition 4.2, we obtain the Hölder continuity of v. Note that $v(z) = \frac{2u^\star_{y_n}}{y_n}$, hence we obtain (4.9). $\square$

Lemma 4.4. *Let $u \in C^{1,1}(\overline{\mathbb{R}^n_+})$ be a solution to (4.4) with constant $b > -1$. Assume that $u_{x_n}(x', 0) = 0$, $\forall\, x' \in \mathbb{R}^{n-1}$, and equation (4.4) is uniformly elliptic. Then $u \in C^{2,\alpha}(\overline{\mathbb{R}^n_+})$ for some $\alpha \in (0,1)$.*

Proof. Let $u^\star$ be the partial Legendre transform of u. Then $u^\star$ satisfies equation (4.8) and the assumptions of Lemma 4.3. Hence by Lemma 4.3, $\frac{u^\star_{y_n}}{y_n} \in C^\alpha(\overline{\mathbb{R}^n_+})$. Recall that $\frac{u_{x_n}}{x_n} = -\frac{u^\star_{y_n}}{y_n}$. We therefore have

$$\left|\frac{u_{x_n}(x)}{x_n} - \frac{u_{x_n}(\tilde{x})}{\tilde{x}_n}\right| = \left|\frac{u^\star_{y_n}(y)}{y_n} - \frac{u^\star_{y_n}(\tilde{y})}{\tilde{y}_n}\right| \le C|y - \tilde{y}|^\alpha.$$

By the partial Legendre transform, $y_n = x_n$, $y' = D_{x'}u$. It follows that

$$|y' - \tilde{y}'| = |D_{x'}u(x) - D_{x'}u(\tilde{x})| \le \|D^2 u\|_{L^\infty(\mathbb{R}^n_+)}|x - \tilde{x}|.$$

Hence $\frac{u_{x_n}}{x_n} \in C^\alpha(\overline{\mathbb{R}^n_+})$, and we have the estimate

$$\left\|\frac{u_{x_n}}{x_n}\right\|_{C^\alpha(\mathbb{R}^{n-1}\times[0,1])} \le C$$

for a constant C depending only on b, n, and $\|D^2 u\|_{L^\infty(\mathbb{R}^n_+)}$.

We make an even extension of $u(x)$ with respect to the variable x_n and still denote it by $u(x)$. Regard $\frac{u_{x_n}}{x_n}$ as a known function, which is Hölder continuous. Then we can write equation (4.4) in the form

$$\mathcal{F}(x, D^2 u) = 1.$$

By our assumption, $\mathcal{F}$ is fully nonlinear, uniformly elliptic, and is C^α smooth in x. Since $\mathcal{F}^{\frac{1}{n}}$ is concave in D^2u, by the Evans-Krylov estimate, we also conclude that $u \in C^{2,\alpha}(\mathbb{R}^n)$. □

With the aid of Lemma 4.4, we can now prove Theorem 4.1.

Proof of Theorem 4.1. Let u be the solution in Theorem 4.1. Let

$$u^m(x) := \frac{u(mx)}{m^2}, \quad m = 1, 2, \cdots$$

be a blowdown sequence of u. Since (4.4) is uniformly elliptic for u, it is also uniformly elliptic for u^m with the same ellipticity constants. The uniform ellipticity implies that there is a constant $\hat{C} > 0$, independent of m, such that

$$\hat{C}^{-1} I \le \mathcal{M}_{u^m} \le \hat{C} I, \tag{4.10}$$

where I is the unit matrix and $\mathcal{M}_u$ denotes matrix in equation (4.4). Hence the first entry in the matrix $\mathcal{M}_{u^m}$ satisfies

$$u^m_{x_n x_n} + b\frac{u^m_{x_n}}{x_n} = \hat{f},$$

for a function $\hat{f}$ satisfying $\hat{C}^{-1} \le f \le \hat{C}$. We can solve the above equation, regarding it as an ode with variable x_n,

$$u^m(x', x_n) = u^m(x', 0) + \int_0^{x_n} r^{-b} \left(\int_0^r s^b \hat{f}(x', s)\, \mathrm{d}s \right) \mathrm{d}r. \tag{4.11}$$

In (4.11) we have used the initial condition $u^m(x', 0) = 0$. Note that (4.10) implies that $u^m(x', 0) = O(|x'|^2)$. Hence from (4.11) we have $u^m(x) = O(|x'|^2)$ near 0.

Hence by the assumptions in Theorem 4.1, u^m satisfies the conditions in Lemma 4.4, uniformly in m. Therefore, by Lemma 4.4 we have

$$\left|D^2u(x) - D^2u(0)\right| = \lim_{m\to+\infty} \left|D^2u^m\left(\frac{x}{m}\right) - D^2u^m(0)\right| = 0$$

for any given point $x \in \mathbb{R}^n_+$. That is, $D^2u(x) = D^2u(0)$, $\forall\, x \in \mathbb{R}^n_+$. Hence u is a quadratic polynomial. By the assumption $u_{x_n}(x', 0) = 0$, $\forall\, x' \in \mathbb{R}^{n-1}$, we have $c_{in} = 0$ in the polynomial (4.5). □

The following example shows that the Bernstein Theorem 4.1 is not unconditionally true.

Example 4.5. *Let*

$$u(x) = \frac{1}{2}(x_2^2 + \cdots + x_{n-1}^2) + \frac{1}{2}x_1^2 x_n^{b-1} + \frac{x_n^{3-b}}{2(3-b)},$$

where $b > 1$. By direct computation, u satisfies equation (4.4).

5. Further discussions

In this section, we will explore potential applications of the higher-dimensional partial Legendre transform to Monge-Ampère type fourth-order equations and linearized Monge-Ampère equations. It is worth noting that successful attempts have already been made in the two-dimensional case, as demonstrated in works such as [WZ1, Wa]. Building on these results, we aim to extend the application of this transform to higher dimensions, providing new insights and tools for analyzing these equations.

We study the regularity of the following fourth-order equations of Monge-Ampère type

$$\sum_{i,j=1}^{n} U^{ij} w_{ij} = 0, \tag{5.1}$$

where $\{U^{ij}\}$ is the cofactor matrix of D^2u for an unknown convex function, $w_{ij} := \frac{\partial^2 w}{\partial x_i \partial x_j}$, and

$$w = \begin{cases} [\det D^2 u]^{-(1-\theta)}, & \theta \in [0,1), \\ \log \det D^2 u, & \theta = 1. \end{cases} \tag{5.2}$$

When $\theta = 1/(n+2)$, this equation corresponds to the *affine maximal hypersurface equation* in affine geometry [Ch]. When $\theta = 0$, it reduces to *Abreu's equation*, which arises in the study of extremal metrics on toric manifolds in Kähler geometry [Ab] and is equivalent to

$$\sum_{i,j} \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j} = 0,$$

where $\{u^{ij}\}$ is the inverse matrix of D^2u.

The regularity of (5.1) has been extensively studied (see [TW1, TW2, Do, Zh1, Zh2, CHLS, Le1, Le2, CW]) and is typically analyzed as a system coupling a Monge-Ampère equation with a linearized Monge-Ampère equation. As a result, previous studies heavily rely on the fundamental interior regularity results of Caffarelli and Gutiérrez [CG] for the linearized Monge-Ampère equation, which were later extended to boundary regularity and higher-order estimates in [LS1, GN1, GN2].

In this section, we explore the possibility of employing the higher-dimensional partial Legendre transform to establish regularity results. Our primary focus is on the case $\theta \in [0,1]$ due to its rich geometric significance.

In order to derive the equation under the partial Legendre transform, we consider the associated functionals of (5.1)

$$A_\theta(u) = \begin{cases} \displaystyle\int_\Omega [\det D^2 u]^\theta \,\mathrm{d}x, & \theta \in (0,1), \\ \displaystyle\int_\Omega \ln\det D^2 u \,\mathrm{d}x, & \theta = 0, \\ \displaystyle\int_\Omega (\det D^2 u)\ln\det D^2 u \,\mathrm{d}x, & \theta = 1. \end{cases}$$

Proposition 5.1. *Let u be a uniformly convex solution to (5.1) in Ω. Then in $\Omega^\star = \mathcal{P}(\Omega)$, its partial Legendre transform $u^\star$ satisfies*

$$\frac{\partial}{\partial y_i}\left[\left(-\frac{u^\star_{y_n y_n}}{\det D^2_{y'} u^\star}\right) U^{\star ij}_{y'} \frac{\partial}{\partial y_j} w^\star\right] + \frac{\partial^2}{\partial y_n^2} w^\star = 0, \tag{5.3}$$

where $w^\star = \left(-\frac{u^\star_{y_n y_n}}{\det D^2_{y'} u^\star}\right)^{\theta-1}$.

Proof. As

$$\det D^2 u = -\frac{u^\star_{y_n y_n}}{\det D^2_{y'} u^\star}, \quad \mathrm{d}x = \det D^2_{y'} u^\star \,\mathrm{d}y,$$

we have

$$\begin{aligned} A_\theta(u) &= \int_{\Omega^\star} \left(-\frac{u^\star_{y_n y_n}}{\det D^2_{y'} u^\star}\right)^\theta \det D^2_{y'} u^\star \,\mathrm{d}y \\ &= \int_{\Omega^\star} (-u^\star_{y_n y_n})^\theta (\det D^2_{y'} u^\star)^{1-\theta} \,\mathrm{d}y =: A^\star_\theta(u^\star), \ \theta \in (0,1); \\ A_0(u) &= \int_{\Omega^\star} \ln\left(-\frac{u^\star_{y_n y_n}}{\det D^2_{y'} u^\star}\right) \det D^2_{y'} u^\star \,\mathrm{d}y =: A^\star_0(u^\star); \\ A_1(u) &= \int_{\Omega^\star} \left(-\frac{u^\star_{y_n y_n}}{\det D^2_{y'} u^\star}\right) \ln\left(-\frac{u^\star_{y_n y_n}}{\det D^2_{y'} u^\star}\right) \det D^2_{y'} u^\star \,\mathrm{d}y =: A^\star_1(u^\star). \end{aligned}$$

Since u is maximal with respect to the functional A_θ, $u^\star$ is maximal with respect to the functional $A^\star_\theta$. It suffices to derive the Euler-Lagrange equation of $A^\star_\theta$.

First, we consider $\theta \in (0,1)$. For $\varphi \in C^\infty_0(\Omega^\star)$, by integration by parts,

$$\begin{aligned} &\left.\frac{\mathrm{d}A^\star_\theta(u^\star + t\varphi)}{\mathrm{d}t}\right|_{t=0} \\ &= \int_{\Omega^\star} \left[(1-\theta)\left(-\frac{u^\star_{y_n y_n}}{\det D^2_{y'} u^\star}\right)^\theta U^{\star ij}_{y'} \varphi_{ij} - \theta\left(-\frac{u^\star_{y_n y_n}}{\det D^2_{y'} u^\star}\right)^{\theta-1} \varphi_{y_n y_n}\right] \mathrm{d}y \end{aligned}$$

$$= \int_{\Omega^\star} \left\{ U^{\star ij}_{y'} \left[(1-\theta) \left(-\frac{u^\star_{y_n y_n}}{\det D^2_{y'} u^\star} \right)^{\theta} \right]_{ij} - \left[\theta \left(-\frac{u^\star_{y_n y_n}}{\det D^2_{y'} u^\star} \right)^{\theta - 1} \right]_{y_n y_n} \right\} \varphi \, \mathrm{d}y.$$

Denote $w^\star = \left(-\frac{u^\star_{y_n y_n}}{\det D^2_{y'} u^\star} \right)^{\theta-1}$. Then the equation, after the transformation, becomes

$$\frac{\partial}{\partial y_i} \left((w^\star)^{\frac{1}{\theta-1}} U^{\star ij}_{y'} \frac{\partial}{\partial y_j} w^\star \right) + \frac{\partial^2}{\partial y_n^2} w^\star = 0,$$

i.e. (5.3). Similarly, for $\varphi \in C_0^\infty(\Omega^*)$,

$$\left. \frac{\mathrm{d}A_0^\star(u^\star + t\varphi)}{\mathrm{d}t} \right|_{t=0} = \int_{\Omega^\star} -\frac{\det D^2_{y'} u^\star}{u^\star_{y_n y_n}} \left(-\frac{\varphi_{y_n y_n} \det D^2_{y'} u^\star - u^\star_{y_n y_n} U^{\star ij}_{y'} \varphi_{ij}}{(\det D^2_{y'} u^\star)^2} \right) \det D^2_{y'} u^\star$$
$$+ \ln \left(-\frac{u^\star_{y_n y_n}}{\det D^2_{y'} u^\star} \right) U^{\star ij}_{y'} \varphi_{ij} \, \mathrm{d}y$$
$$= \int_{\Omega^\star} \frac{\varphi_{y_n y_n} \det D^2_{y'} u^\star - u^\star_{y_n y_n} U^{\star ij}_{y'} \varphi_{ij}}{u^\star_{y_n y_n}} + \ln \left(-\frac{u^\star_{y_n y_n}}{\det D^2_{y'} u^\star} \right) U^{\star ij}_{y'} \varphi_{ij} \, \mathrm{d}y,$$

and the equation, after the transformation, becomes

$$\frac{\partial}{\partial y_i} \left((w^\star)^{-1} U^{\star ij}_{y'} \frac{\partial}{\partial y_j} w^\star \right) + \frac{\partial^2}{\partial y_n^2} w^\star = 0.$$

i.e. (5.3). Finally,

$$\left. \frac{\mathrm{d}A_1^\star(u^\star + t\varphi)}{\mathrm{d}t} \right|_{t=0}$$
$$= \int_{\Omega^\star} \left(1 + \ln \left(-\frac{u^\star_{y_n y_n}}{\det D^2_{y'} u^\star} \right) \right) \left(-\frac{\varphi_{y_n y_n} \det D^2_{y'} u^\star - u^\star_{y_n y_n} U^{\star ij}_{y'} \varphi_{ij}}{(\det D^2_{y'} u^\star)^2} \right) \det D^2_{y'} u^\star$$
$$+ \left(-\frac{u^\star_{y_n y_n}}{\det D^2_{y'} u^\star} \right) \ln \left(-\frac{u^\star_{y_n y_n}}{\det D^2_{y'} u^\star} \right) U^{\star ij}_{y'} \varphi_{ij} \, \mathrm{d}y$$
$$= -\int_{\Omega^\star} \left(1 + \ln \left(-\frac{u^\star_{y_n y_n}}{\det D^2_{y'} u^\star} \right) \right) \left(\left(-\frac{u^\star_{y_n y_n}}{\det D^2_{y'} u^\star} \right) U^{\star ij}_{y'} \varphi_{ij} - \varphi_{\eta\eta} \right)$$
$$+ \left(-\frac{u^\star_{y_n y_n}}{\det D^2_{y'} u^\star} \right) \ln \left(-\frac{u^\star_{y_n y_n}}{\det D^2_{y'} u^\star} \right) U^{\star ij}_{y'} \varphi_{ij} \, \mathrm{d}y.$$

Then the equation after transformation is

$$\frac{\partial}{\partial y_i} \left(e^{w^\star} U^{\star ij}_{y'} \frac{\partial}{\partial y_j} w^\star \right) + \frac{\partial^2}{\partial y_n^2} w^\star = 0,$$

which is equivalent to (5.3). □

Indeed, if we assume a condition analogous to that in Theorem 2.1, we can derive an interior estimate for fourth-order equations in higher dimensions. This result can be regarded as a natural extension of [WZ1, Theorem 1.1], which established such estimates in the two-dimensional setting.

Theorem 5.2. *Let $u \in C^{2,1}(\Omega)$ be a convex solution to (5.1) satisfying*

$$0 < \lambda \le \det D^2 u \le \Lambda. \tag{5.4}$$

If we further assume that the determinant of the Hessian with respect to the variables $x' = (x_1, \ldots, x_{n-1})$, denoted by $\det D^2_{x'} u$, is positive everywhere in Ω. Then for any $\Omega' \subset\subset \Omega$, there exists a constant $C > 0$ depending on $\sup_\Omega |u|$, λ, Λ, θ and $\operatorname{dist}(\Omega', \partial\Omega)$, such that

$$\|u\|_{C^{4,\alpha}(\Omega')} \le C.$$

Sketch proof of Theorem 5.2. Since

$$-\frac{u^\star_{y_n y_n}}{\det D^2_{y'} u^\star} = \det D^2 u \quad \text{and} \quad U^\star_{y'} = \frac{1}{\det D^2_{x'} u} D^2_{x'} u,$$

we know by the assumptions that (5.3) becomes a uniformly elliptic equation. Hence, by the classical De Giorgi-Nash-Moser theory, we conclude that $w^\star$ is Hölder continuous in $\Omega^\star$. Moreover, since

$$\frac{\partial y}{\partial x} = \det D^2_{x'} u > 0,$$

we deduce that $\det D^2 u$ is also Hölder continuous in Ω.

By combining the Schauder regularity theory for classical Monge-Ampère equations [Ca] with Caffarelli-Gutiérrez's Hölder estimates for linearized Monge-Ampère equations [CG], we obtain that $u \in C^{4,\alpha}_{loc}(\Omega)$. □

Finally, we remark that for higher-dimensional linearized Monge-Ampère equations, a result analogous to [Wa, Theorem 1.1] may hold. However, the situation is more complicated than in the two-dimensional case, as we do not yet know how to transform the cofactor matrix of the potential function ϕ in higher dimensions. This may make it difficult to derive an explicit formula for the equation after the partial Legendre transform. Nevertheless, inspired by the connection between Monge-Ampère type fourth-order equations and linearized Monge-Ampère equations, as well as the two-dimensional transformed equation (see [Wa, (2.7)]), we conjecture that the leading terms in the higher-dimensional transformed equations might take the form

$$\frac{\partial}{\partial y_i}\left[\left(-\frac{\phi^\star_{y_n y_n}}{\det D^2_{y'} \phi^\star}\right) \Phi^{\star ij}_{y'} \frac{\partial}{\partial y_j} w^\star\right] + \frac{\partial^2}{\partial y_n^2} w^\star$$

for some suitable expression of $w^\star$. We leave this question to be explored in future work.

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