THE WIENER TEST AND POTENTIAL ESTIMATES FOR QUASILINEAR ELLIPTIC EQUATIONS

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In this note we follow the approach outlined in [KM2] to show how potential estimates for quasilinear elliptic equations can be used to establish the necessity part of the Wiener test for all $p \in (1, n]$ in the context of the main model operator, the p-Laplacian.

We assume throughout this note that $\mathcal{A} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function, i.e.

$$\begin{aligned} &\text{the function } x \mapsto \mathcal{A}(x,\xi) \text{ is measurable for all } \xi \in \mathbb{R}^n, \text{ and} \\ &\text{the function } \xi \mapsto \mathcal{A}(x,\xi) \text{ is continuous for a.e. } x \in \mathbb{R}^n; \end{aligned}$$

which satisfies the following assumptions for some constants $0 < \alpha \leq \beta < \infty$ and for all $\xi \in \mathbb{R}^n$, a.e. $x \in \mathbb{R}^n$:

$$\mathcal{A}(x,\xi) \cdot \xi \geq \alpha |\xi|^p, \tag{1}$$

$$|\mathcal{A}(x,\xi)| \leq \beta |\xi|^{p-1}, \tag{2}$$

$$(\mathcal{A}(x,\xi) - \mathcal{A}(x,\zeta)) \cdot (\xi - \zeta) > 0 \tag{3}$$

whenever $\xi \neq \zeta$, and

$$\mathcal{A}(x, \lambda\xi) = \lambda|\lambda|^{p-2}\mathcal{A}(x,\xi) \tag{4}$$

for all $\lambda \in \mathbb{R}\backslash\{0\}$.

We define the operator T as

$$Tu(\varphi) = \int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla\varphi \, \mathrm{d}x, \quad \forall\, \varphi \in C_0^\infty(\Omega),$$

where $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$. In other words

$$Tu = -\mathrm{div}\, \mathcal{A}(x, \nabla u)$$

in the sense of distributions. The principal model operator is the p-Laplacian

$$Tu = -\mathrm{div}\, \left(|\nabla u|^{p-2}\nabla u\right),$$

and so the ordinary Laplacian Δ is included in this setting. Next, we define the regularity of boundary points of an open set $\Omega \subset \mathbb{R}^n$.

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Definition 1. *A boundary point x_0 of bounded Ω is* regular *if the solution u to the Dirichlet problem*

$$\begin{cases} Tu = 0 & \text{in } \Omega \\ u - f \in W_0^{1,p}(\Omega) \end{cases}$$

has the limit value $f(x_0)$ at x_0 whenever $f \in W^{1,p}(\Omega)$ is continuous in the closure of Ω.

It was first proved by Wiener in [Wi1, Wi2] that in the case of the Laplacian the regularity of a boundary point $x_0 \in \partial\Omega$ is equivalent to $W_2(\mathbb{R}^n \backslash \Omega, x_0) = +\infty$, where

$$W_p(E, x_0) := \int_0^1 \left(\frac{\operatorname{cap}_p\left(B(x_0,t) \cap E, B(x_0,2t)\right)}{\operatorname{cap}_p\left(B(x_0,t), B(x_0,2t)\right)} \right)^{1/(p-1)} \frac{\mathrm{d}t}{t},$$

and $\operatorname{cap}_p(E, G)$ is the p-capacity of a set E in G,

$$\operatorname{cap}_p(E,\Omega) = \inf_{\substack{G \subset \Omega \text{ open} \\ E \subset G}} \ \sup_{\substack{K \subset G \\ K \text{ compact}}} \ \inf \left\{ \int |\nabla \phi|^p \ \mathrm{d}x : \phi \in C_0^\infty(\Omega), \phi \geq 1 \text{ on } K \right\}.$$

When $K \subset \Omega$ is compact, we have

$$\operatorname{cap}_p(K,\Omega) = \inf \left\{ \int |\nabla \phi|^p \ \mathrm{d}x : \phi \in C_0^\infty(\Omega), \phi \geq 1 \text{ on } K \right\}.$$

Later, many authors have extended, and expanded upon this result to more general linear elliptic equations (and parabolic equations), and to certain nonlinear elliptic equations. In the linear case, the definitive result of Littman-Stampacchia-Wienberger [LSW] is, in itself, remarkable. There the authors show that a point $x_0 \in \partial\Omega$ is a regular point for the equation

$$\partial_i \left(a_{ij}(x) \partial_j u \right) = 0$$

in Ω with $a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$, $\lambda > 0$, and $|a_{ij}(x)| \leq M < \infty$ for all $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n \backslash \{0\}$ if and only if x_0 is a regular point for the Laplace equation in Ω. Linear equations with lower order terms were subsequently considered in [St]. The theory for nonlinear second order elliptic equations began with Maz'ya [Ma] when he found that a sufficient condition for a point $x \in \partial\Omega$ to be a regular point for the p-Laplace equation, it is: $W_p\left(\mathbb{R}^n \backslash \Omega, x\right) = +\infty$. After that, Kilpeläinen-Malý [KM2] establish the necessity part of the Wiener test for all $p \in (1, n]$, which is the main theorem of this note.

Theorem 2 ([KM2, Theorem 1.1])**.** *A finite boundary point $x_0 \in \partial\Omega$ is regular if and only if*

$$W_p\left(\mathbb{R}^n \backslash \Omega, x_0\right) = +\infty.$$

An immediate corollary is:

Corollary 3. *The regularity depends only on n and p, not on the operator T itself.*

The proof of Theorem 2 is related to a variant of the Wiener criterion problem, known among specialists in nonlinear potential theory. A set $E \subset \mathbb{R}^n$ is said to be *p-thin* at a point $x_0 \in \mathbb{R}^n$ if $W_p(E, x_0) < +\infty$. There is a relationship between p-thinness and $\mathcal{A}$-superharmonic functions. So first we need to define what a $\mathcal{A}$-superharmonic is.

Definition 4. *A function $u : \Omega \to \mathbb{R}$ is said to be $\mathcal{A}$-harmonic in Ω if it is a continuous weak solution of*

$$-div\, \mathcal{A}(x, \nabla u) = 0 \tag{5}$$

in Ω. Sometimes we will use the notation

$$\mathcal{H}_{\mathcal{A}}(\Omega) := \{u : u \text{ is } \mathcal{A}\text{-harmonic in } \Omega\}.$$

To develop a potential theory, it is necessary to define superharmonic functions.

Definition 5. *A function $u : \Omega \to \mathbb{R} \cup \{\infty\}$ is $\mathcal{A}$-superharmonic in Ω if*

(1) *u is lower semicontinuous,*
(2) *$u \not\equiv \infty$ in each component of Ω, and*
(3) *for each open $D \subset\subset \Omega$ and each $h \in C(\overline{D}) \cap \mathcal{H}_{\mathcal{A}}(D)$ the inequality $u \geq h$ on ∂D implies $u \geq h$ in D.*

A function v is $\mathcal{A}$-subharmonic if $-v$ is $\mathcal{A}$-superharmonic.

Proposition 6.

(1) *If u is $\mathcal{A}$-superharmonic, then $\lambda u + \tau$ is $\mathcal{A}$-superharmonic whenever λ and τ are real numbers and $\lambda \geq 0$.*
(2) *If u and v are $\mathcal{A}$-superharmonic in Ω, then the function $\min\{u, v\}$ is $\mathcal{A}$-superharmonic.*

The following proposition connects $\mathcal{A}$-superharmonic functions with supersolutions of (5).

Proposition 7.

(1) *If $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is such that $Tu \geq 0$, then there is an $\mathcal{A}$-superharmonic function v such that $u = v$ a.e. Moreover,*

$$v(x) = \operatorname{ess} \liminf_{y \to x} v(y) \quad \text{for all } x \in \Omega.$$

(2) *If v is $\mathcal{A}$-superharmonic, then above equation holds. Moreover, $Tv \geq 0$ if $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$.*
(3) *If v is $\mathcal{A}$-superharmonic and locally bounded, then $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and $Tv \geq 0$.*

Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be an $\mathcal{A}$-superharmonic function in Ω. Then it follows from Proposition 7 that $\mu = Tu$ is a nonnegative Radon measure on Ω. Then we can show

Theorem 8 ([KM2, Theorem 1.3])**.** *Let $E \subset \mathbb{R}^n$ and $x_0 \in \overline{E} \backslash E$. Then E is p-thin at x_0 if and only if there is an $\mathcal{A}$-superharmonic function u in a neighborhood of x_0 such that*

$$\liminf_{x \to x_0,\, x \in E} u(x) > u(x_0).$$

The proofs of Theorems 2 and Theorem 8 are based on pointwise estimates of solutions to

$$Tu = \mu$$

with a Radon measure μ on the right side in terms of the Wolff potential

$$\mathbf{W}^{\mu}_{1,p}(x_0, r) = \int_0^r \left(\frac{\mu(B(x_0,t))}{t^{n-p}} \right)^{1/(p-1)} \frac{\mathrm{d}t}{t}. \tag{6}$$

One easily infers that $\mathbf{W}^{\mu}_{1,2}(x_0, \infty)$ is the Newtonian potential of μ. Indeed, denote $\mu = f\,\mathrm{d}x$, there is

$$\begin{aligned} \mathbf{W}^{\mu}_{1,2}(x_0, \infty) &= \int_0^{+\infty} \left(\int_{B(x_0,t)} f(x)\ \mathrm{d}x \right) \cdot \frac{1}{t^{n-1}}\ \mathrm{d}t \\ &= \int_{\mathbb{R}^n} \left(\int_{|x-x_0|}^{+\infty} \frac{1}{t^{n-1}}\ \mathrm{d}t \right) f(x)\ \mathrm{d}x \\ &= C \int_{\mathbb{R}^n} \frac{f(x)}{|x-x_0|^{n-2}}\ \mathrm{d}x. \end{aligned}$$

Then we have

Theorem 9 ([KM2, Theorem 1.6])**.** *Suppose that u is a nonnegative $\mathcal{A}$-superharmonic function in $B(x_0, 3r)$. If $\mu = Tu$, then*

$$C_1 \mathbf{W}^{\mu}_{1,p}(x_0, r) + \inf_{B(x_0,2r)} u \le u(x_0) \le C_2 \inf_{B(x_0,r)} u + C_3 \mathbf{W}^{\mu}_{1,p}(x_0, 2r),$$

where C_1, C_2, and C_3 are positive constants, depending only on n, p, and the structural constants α and β. In particular, $u(x_0) < \infty$ if and only if $\mathbf{W}^{\mu}_{1,p}(x_0, r) < \infty$.

Let us start by showing the lower estimate following the method in [KM1, Section 3]. We first record an appropriate form of Trudinger's weak Harnack inequality [Tr, Theorem 1.2].

Lemma 10 ([KM1, Lemma 3.2])**.** *Let $B_R := B(0, R)$ and let u be a nonnegative supersolution of* (5) *in B_{3R}. If $q > 0$ is such that $q(n-p) < n(p-1)$, then*

$$R^{-n/q} \left(\int_{B_{2R}} u^q\ \mathrm{d}x \right)^{\frac{1}{q}} \le C \inf_{B_R} u,$$

where $C = C(n, p, q, \alpha, \beta) > 0$.

Proof. By [Tr, Theorem 1.2] such a constant C exists if $u \leq 1$ in B_{3R}. However, as well known, the simpler structure of our equation allows us to obtain the inequality without boundedness restriction. Indeed, set $u_j = \min\{u/j, 1\}$. Then

$$R^{-n/q} \left(\int_{B_{2R}} {u_j}^q \, \mathrm{d}x \right)^{\frac{1}{q}} \leq C \inf_{B_R} u_j,$$

and hence

$$R^{-n/q} \left(\int_{B_{2R}} (\min\{u, j\})^q \, \mathrm{d}x \right)^{\frac{1}{q}} \leq C \inf_{B_R} u.$$

Letting $j \to +\infty$ we obtain the desired estimate. □

Also we need the following well known estimate.

Lemma 11 ([KM1, Lemma 3.3])**.** *Let u be a supersolution of* (5) *in B_{2R} such that $u > 0$ in B_R. Let $\eta \in C_0^\infty(B_R)$ be nonnegative. For all $\varepsilon \in (0, p-1)$, we have*

$$\int_{B_R} |\nabla u|^p u^{-1-\varepsilon} \eta^p \, \mathrm{d}x \leq C \int_{B_R} u^{p-1-\varepsilon} |\nabla \eta|^p \, \mathrm{d}x,$$

where $C = (p\beta/\alpha\varepsilon)^p$.

Proof. Without loss of generality, we can assume that $u \geq \delta > 0$ in B_R, otherwise we set $\tilde{u} = u + \delta$ and then let $\delta \to 0$. Set $v = u^{-\varepsilon}$ and $w = v\eta^p$. Then $w \in W_0^{1,p}(B_R)$ is nonnegative and hence

$$\begin{aligned} 0 &\leq \int_{B_R} \mathcal{A}(x, \nabla u) \cdot \nabla w \, \mathrm{d}x \\ &= \int_{B_R} \mathcal{A}(x, \nabla u) \eta^p \cdot \nabla v \, \mathrm{d}x + p \int_{B_R} \mathcal{A}(x, \nabla u) v \eta^{p-1} \cdot \nabla \eta \, \mathrm{d}x. \end{aligned}$$

Using (1)-(2) and the Hölder inequality, it follows that

$$\begin{aligned} &\alpha\varepsilon \int_{B_R} |\nabla u|^p u^{-1-\varepsilon} \eta^p \, \mathrm{d}x \\ &\leq \varepsilon \int_{B_R} \mathcal{A}(x, \nabla u) \cdot \nabla u \, u^{-1-\varepsilon} \eta^p \, \mathrm{d}x = -\int_{B_R} \mathcal{A}(x, \nabla u) \eta^p \cdot \nabla v \, \mathrm{d}x \\ &\leq p \int_{B_R} \mathcal{A}(x, \nabla u) v \eta^{p-1} \cdot \nabla \eta \, \mathrm{d}x = p \int_{B_R} \mathcal{A}(x, \nabla u) \cdot \nabla \eta u^{-\varepsilon} \eta^{p-1} \, \mathrm{d}x \\ &\leq p\beta \int_{B_R} |\nabla u|^{p-1} u^{-\varepsilon} |\nabla \eta| \eta^{p-1} \, \mathrm{d}x \\ &\leq p\beta \left(\int_{B_R} |\nabla u|^p u^{-1-\varepsilon} \eta^p \, \mathrm{d}x \right)^{(p-1)/p} \left(\int_{B_R} u^{p-1-\varepsilon} |\nabla \eta|^p \, \mathrm{d}x \right)^{1/p}, \end{aligned}$$

which implies the required estimate. □

The next estimate is a refined version of an estimate of Gariepy and Ziemer [GZ].

Lemma 12 ([KM1, Lemma 3.4])**.** *Let u be a nonnegative supersolution of* (5) *in B_{4R}. Let $\eta \in C_0^\infty(B_{3R})$ be a cut-off function such that $0 \le \eta \le 1$, $\eta = 1$ on B_{2R}, and $|\nabla \eta| \le 10/R$. Then*

$$\int_{B_{3R}} |\nabla u|^{p-1} \eta^{p-1} |\nabla \eta| \, \mathrm{d}x \le C R^{n-p} \inf_{B_{2R}} u^{p-1},$$

where $C = C(n, p, \alpha, \beta) > 0$.

Proof. We use the argument of the proof of Theorem 2.1 in [GZ]. Let $\varepsilon = \frac{1}{2} \min\{p - 1, p/(n-1)\}$ (if $p = n$, let $\varepsilon = (n-1)/2$). Denote $q = p/(p-1)$, $\gamma_1 = p - 1 - \varepsilon$, and $\gamma_2 = (p-1)(1+\varepsilon)$. Using Lemma 10 and Lemma 11, and the Hölder inequality we obtain

$$\begin{aligned}
&\int_{B_{3R}} |\nabla u|^{p-1} \eta^{p-1} |\nabla \eta| \, \mathrm{d}x = \int_{B_{3R}} |\nabla u|^{p-1} \eta^{p-1} u^{-(1+\varepsilon)/q} u^{(1+\varepsilon)/q} |\nabla \eta| \, \mathrm{d}x \\
&\le \left(\int_{B_{3R}} |\nabla u|^p u^{-1-\varepsilon} \eta^p \, \mathrm{d}x \right)^{1/q} \left(\int_{B_{3R}} u^{(p-1)(1+\varepsilon)} |\nabla \eta|^p \, \mathrm{d}x \right)^{1/p} \\
&\le \left(C \int_{B_{3R}} u^{\gamma_1} |\nabla \eta|^p \, \mathrm{d}x \right)^{1/q} \left(\int_{B_{3R}} u^{\gamma_2} |\nabla \eta|^p \, \mathrm{d}x \right)^{1/p} \\
&\le C R^{-p} \left(\int_{B_{3R}} u^{\gamma_1} \, \mathrm{d}x \right)^{1/q} \left(\int_{B_{3R}} u^{\gamma_2} \, \mathrm{d}x \right)^{1/p} \\
&\le C R^{n-p} \left(\inf_{B_{2R}} u \right)^{\frac{\gamma_1}{q} + \frac{\gamma_2}{p}} = C R^{n-p} \inf_{B_{2R}} u^{p-1},
\end{aligned}$$

and the lemma is proved. $\square$

The following estimate takes the measure data into account.

Lemma 13 ([KM1, Lemma 3.5])**.** *Suppose that u is $\mathcal{A}$-superharmonic and $\mu = Tu$ in an open set containing $\overline{B}_R$. Then*

$$R^{p-n} \mu(B_{R/2}) \le C \left(\inf_{B_{R/2}} u - \inf_{B_R} u \right)^{p-1}$$

where $C = C(n, p, \alpha, \beta) > 0$.

Proof. Write $a = \inf_{B_R} u$ and $b = \inf_{B_{R/2}} u$. Without loss of generality, we may assume that u is locally bounded, otherwise choosing a positive integer $j \ge b$ and set $u_j = \min\{u, j\}$, then letting $j \to \infty$. Let $\eta \in C_0^\infty(B_{3R/4})$, $0 \le \eta \le 1$, be a cut-off function such that $\eta = 1$ in $B_{R/2}$ and $|\nabla \eta| \le 10/R$.

Set $v = \min\{u, b\} - a$, and use $(b - a - v)\eta^p \in W_0^{1,p}(B_{3R/4})$ as a test function we have

$$0 \le \int_{B_{3R/4}} \mathcal{A}(x, \nabla u) \cdot (-\nabla v)\eta^p \, \mathrm{d}x + p \int_{B_{3R/4}} \mathcal{A}(x, \nabla u)(b - a - v)\eta^{p-1} \cdot \nabla\eta \, \mathrm{d}x$$
$$\le \int_{B_{3R/4}} \mathcal{A}(x, \nabla u) \cdot (-\nabla v)\eta^p \, \mathrm{d}x + p\beta \int_{B_{3R/4}} |\nabla v|^{p-1}|b - a - v|\eta^{p-1}|\nabla\eta| \, \mathrm{d}x,$$

which implies

$$\int_{B_{3R/4}} \mathcal{A}(x, \nabla u) \cdot \nabla v\eta^p \, \mathrm{d}x \le p\beta \int_{B_{3R/4}} |\nabla v|^{p-1}|b - a - v|\eta^{p-1}|\nabla\eta| \, \mathrm{d}x$$
$$\le p\beta \left(\int_{B_{3R/4}} |\nabla v|^p\eta^p \, \mathrm{d}x\right)^{(p-1)/p} \left(\int_{B_{3R/4}} |b - a - v|^p|\nabla\eta|^p \, \mathrm{d}x\right)^{1/p}$$
$$\le \frac{p\beta}{\alpha^{(p-1)/p}} \left(\int_{B_{3R/4}} \mathcal{A}(x, \nabla u) \cdot \nabla v\eta^p \, \mathrm{d}x\right)^{(p-1)/p} \left(\int_{B_{3R/4}} |b - a - v|^p|\nabla\eta|^p \, \mathrm{d}x\right)^{1/p},$$

hence

$$\int_{B_{3R/4}} \mathcal{A}(x, \nabla u) \cdot \nabla v\eta^p \, \mathrm{d}x \le C \int_{B_{3R/4}} |b - a - v|^p|\nabla\eta|^p \, \mathrm{d}x. \tag{7}$$

(The inequality (7) is referred to in some literature as the Caccioppoli estimate).

Next, we use the test function $w = v\eta^p$ in B_R. Then $0 \le w \le b - a$ and $w = b - a$ on $B_{R/2}$. Using (7) and Lemma 12 for $u - a$, we obtain

$$\int_{B_R} \mathcal{A}(x, \nabla u) \cdot \nabla w \, \mathrm{d}x$$
$$= \int_{B_{3R/4}} \mathcal{A}(x, \nabla u) \cdot \nabla v\eta^p \, \mathrm{d}x + p \int_{B_{3R/4}} \mathcal{A}(x, \nabla u)v\eta^{p-1} \cdot \nabla\eta \, \mathrm{d}x$$
$$\le C \left(\int_{B_{3R/4}} |b - a - v|^p|\nabla\eta|^p \, \mathrm{d}x + (b - a) \int_{B_{3R/4}} |\nabla u|^{p-1}\eta^{p-1}|\nabla\eta| \, \mathrm{d}x\right)$$
$$\le C \left((b - a)^p \int_{B_{3R/4}} |\nabla\eta|^p \mathrm{d}x + R^{n-p}(b - a)(b - a)^{p-1}\right)$$
$$\le CR^{n-p}(b - a)^p.$$

Now it follows that

$$(b - a)\mu(B_{R/2}) \le \int_{B_R} w \, \mathrm{d}\mu = \int_{B_R} \mathcal{A}(x, \nabla u) \cdot \nabla w \, \mathrm{d}x \le CR^{n-p}(b - a)^p,$$

i.e.

$$\mu(B_{R/2}) \le CR^{n-p}(b - a)^{p-1},$$

this concludes the proof. □

Now we are ready to prove the lower estimate in Theorem 9.

Proof of lower estimate in Theorem 9. We may assume that $x_0 = 0$. Let $r_j = 2^{1-j}r$ and $a_j = \inf_{B_{r_j}} u$. Then by Lemma 13 we have

$$\begin{aligned} c\sum_{j=1}^{\infty}(r^{p-n}2^{j(n-p)}\mu(B_{r_j}))^{1/(p-1)} &\le \sum_{j=1}^{\infty}(a_j - a_{j-1}) \\ &= \lim_k (a_k - a_0) = u(0) - a_0 \le u(0) - \inf_{B_{2r}} u. \end{aligned}$$

The desired estimate follows, since

$$\begin{aligned} \mathbf{W}^{\mu}_{1,p}(0,r) &= \int_0^r \left(t^{p-n}\mu(B_t)\right)^{1/(p-1)} \frac{\mathrm{d}t}{t} \\ &\le c\sum_{j=1}^{\infty}(r^{p-n}2^{j(n-p)}\mu(B_{r_j}))^{1/(p-1)}. \end{aligned}$$

□

Next, we derive the upper estimate in Theorem 9. We start with an auxiliary estimate.

Lemma 14 ([KM2, Lemma 4.1])**.** *Suppose that u is $\mathcal{A}$-superharmonic in a ball $2B := B(x_0, 2r)$ and $\mu = Tu$. If a is a real constant, $d > 0$, and $p - 1 < \gamma < n(p-1)/(n-p+1)$, then there are constants $q = q(p,\gamma) > p$ and $C = C(n,p,\alpha,\beta,\gamma) > 0$ such that*

$$\left(d^{-\gamma}r^{-n}\int_{B\cap\{u>a\}}(u-a)^{\gamma}\mathrm{d}x\right)^{p/q} \le Cd^{-\gamma}r^{-n}\int_{2B\cap\{u>a\}}(u-a)^{\gamma}\mathrm{d}x + Cd^{1-p}r^{p-n}\mu(2B),$$

provided that

$$|2B\cap\{u>a\}| < \frac{1}{2}d^{-\gamma}\int_{B\cap\{u>a\}}(u-a)^{\gamma}\ \mathrm{d}x. \tag{8}$$

Proof. Without loss of generality, we may assume that $a = 0$ and u is locally bounded, and hence $u \in W^{1,p}_{\mathrm{loc}}(2B)$. Using (8) we obtain

$$d^{-\gamma}\int_{B\cap\{0<u<d\}}u^{\gamma}\ \mathrm{d}x \le |B\cap\{u>0\}| \le |2B\cap\{u>0\}| < \frac{1}{2}d^{-\gamma}\int_{B\cap\{u>0\}}u^{\gamma}\ \mathrm{d}x,$$

therefore

$$d^{-\gamma}\int_{B\cap\{u>0\}}u^{\gamma}\ \mathrm{d}x \leqslant 2d^{-\gamma}\int_{B\cap\{u\geqslant d\}}u^{\gamma}\ \mathrm{d}x \leqslant C\int_B w^q\ \mathrm{d}x, \tag{9}$$

where

$$w = \left(1 + \frac{u^+}{d}\right)^{\gamma/q} - 1,$$

and q is a constant to be determined later. Pick a cut-off function $\eta \in C_0^\infty(2B)$ such that $0 \le \eta \le 1$, $\eta = 1$ on B and $|\nabla \eta| \le 2/r$. The Hölder inequality and the Sobolev inequality yield

$$\begin{aligned}
\left(r^{-n}\int_B w^q \,\mathrm{d}x\right)^{p/q} &\le \left(r^{-n}\int_{2B} (w\eta)^q \,\mathrm{d}x\right)^{p/q} \\
&\le \left[r^{-n}\left(\int_{2B} (w\eta)^{p^*} \,\mathrm{d}x\right)^{q/p^*} |2B|^{(1-q/p^*)}\right]^{p/q} \\
&= C\left[r^{-n}\cdot r^{n(1-q/p^*)}\left(\int_{2B} (w\eta)^{p^*} \,\mathrm{d}x\right)^{q/p^*}\right]^{p/q} \\
&= Cr^{p-n}\left(\int_{2B} (w\eta)^{p^*} \,\mathrm{d}x\right)^{p/p^*} \\
&\le Cr^{p-n}\int_{2B} |\nabla w|^p \eta^p \,\mathrm{d}x + cr^{p-n}\int_{2B} w^p |\nabla \eta|^p \,\mathrm{d}x,
\end{aligned} \tag{10}$$

where $p^* := np/(n-p)$ and we will choose $q < p^*$ later. Choose the test function

$$v = \left(1 - \left(1 + \frac{u^+}{d}\right)^{1-\tau}\right)\eta^p,$$

where $\tau > 1$ to be determined later. Note that

$$\nabla v = p\left(1 - \left(1 + \frac{u^+}{d}\right)^{1-\tau}\right)\eta^{p-1}\nabla\eta - \frac{1-\tau}{d}\left(1 + \frac{u^+}{d}\right)^{-\tau}\eta^p \nabla u^+,$$

we have

$$\begin{aligned}
&\int_{2B\cap\{u>0\}} \frac{|\nabla u|^p}{(1+u/d)^\tau}\eta^p \,\mathrm{d}x \\
&\quad\le \alpha^{-1}\int_{2B\cap\{u>0\}} \mathcal{A}(x,\nabla u)\cdot \frac{\nabla u}{(1+u/d)^\tau}\eta^p \,\mathrm{d}x \\
&\quad= -\frac{pd}{\alpha(\tau-1)}\int_{2B\cap\{u>0\}} \mathcal{A}(x,\nabla u)\cdot\left(1 - \left(1+\frac{u}{d}\right)^{1-\tau}\right)\eta^{p-1}\nabla\eta \,\mathrm{d}x \\
&\qquad+ \frac{d}{\alpha(\tau-1)}\int_{2B\cap\{u>0\}} \mathcal{A}(x,\nabla u)\cdot\nabla v \,\mathrm{d}x \\
&\quad= -\frac{pd}{\alpha(\tau-1)}\int_{2B\cap\{u>0\}} \mathcal{A}(x,\nabla u)\cdot\left(1 - \left(1+\frac{u}{d}\right)^{1-\tau}\right)\eta^{p-1}\nabla\eta \,\mathrm{d}x \\
&\qquad+ \frac{d}{\alpha(\tau-1)}\int_{2B} v \,\mathrm{d}\mu \\
&\quad\le Cd\int_{2B\cap\{u>0\}} |\nabla u|^{p-1}\eta^{p-1}|\nabla\eta| \,\mathrm{d}x + Cd\int_{2B}\eta^p \,\mathrm{d}\mu
\end{aligned} \tag{11}$$

$$\leq \frac{1}{2}\int_{2B\cap\{u>0\}} \frac{|\nabla u|^p}{(1+u/d)^\tau}\eta^p \,\mathrm{d}x + C\Big(\frac{d}{r}\Big)^p \int_{2B\cap\{u>0\}} \Big(1+\frac{u}{d}\Big)^{(p-1)\tau} \,\mathrm{d}x$$
$$+ Cd\int_{2B} \eta^p \,\mathrm{d}\mu,$$

where in the last inequality we employed Young's inequality. In the second term of the last inequality, we want the exponent inside the integral to be γ, so we can choose

$$\tau = \frac{\gamma}{p-1}.$$

Hence, to guarantee $\tau > 1$, we need $\gamma > p-1$. Note that

$$\nabla w = \frac{\gamma}{qd}\Big(1+\frac{u^+}{d}\Big)^{\gamma/q-1}\nabla u^+,$$

then

$$|\nabla w|^p = Cd^{-p}\frac{|\nabla u^+|^p}{(1+u^+/d)^{p(\gamma/q-1)}}.$$

To apply (11), we need $p(\gamma/q-1)=\tau$. Therefore, we set

$$q = \frac{p\gamma}{p-\gamma/(p-1)}.$$

To ensure that $q<p^*$, γ must satisfy $\gamma < n(p-1)/(n-p+1)$. On the other hand, $\gamma > p-1$ implies $q>p$. Hence by (11), we have

$$\begin{aligned}\int_{2B}|\nabla w|^p\eta^p \,\mathrm{d}x &\leq Cd^{-p}\int_{2B\cap\{u>0\}}\frac{|\nabla u|^p}{(1+u/d)^\tau}\eta^p \,\mathrm{d}x \\ &\leq Cr^{-p}\int_{2B\cap\{u>0\}}\Big(1+\frac{u}{d}\Big)^\gamma \,\mathrm{d}x + Cd^{1-p}\mu(\operatorname{supp}\eta).\end{aligned} \tag{12}$$

Keeping (8) in mind we obtain

$$\int_{2B\cap\{u>0\}}\Big(1+\frac{u}{d}\Big)^\gamma \,\mathrm{d}x \leq C\int_{2B\cap\{u>0\}}\Big(1+\Big(\frac{u}{d}\Big)^\gamma\Big) \,\mathrm{d}x \leq Cd^{-\gamma}\int_{2B\cap\{u>0\}} u^\gamma \,\mathrm{d}x, \tag{13}$$

and, consequently, by $w^q \leq (1+u^+/d)^\gamma$, Hölder's inequality and (8), we have

$$\begin{aligned} r^p\int_{2B} w^p|\nabla\eta|^p \,\mathrm{d}x &\leq C\int_{2B} w^p \,\mathrm{d}x \\ &\leq C\Big(\int_{2B} w^q \,\mathrm{d}x\Big)^{p/q}|2B\cap\{u>0\}|^{1-p/q} \\ &\leq Cd^{-\gamma}\int_{2B\cap\{u>0\}} u^\gamma \,\mathrm{d}x.\end{aligned} \tag{14}$$

Hence by collecting the estimates (9)-(14), we arrive at the estimate

$$\left(d^{-\gamma}r^{-n}\int_{B\cap\{u>0\}}u^\gamma\ \mathrm{d}x\right)^{p/q}\le Cd^{-\gamma}r^{-n}\int_{2B\cap\{u>0\}}u^\gamma\ \mathrm{d}x+Cd^{1-p}r^{p-n}\mu(\operatorname{supp}\eta),$$

where $C=C(n,p,\alpha,\beta,\gamma)>0$. □

Lemma 15. *Suppose u is a nonnegative $\mathcal{A}$-superharmonic function in $B(x_0,2r)$. If $\mu=Tu$, then for all $\gamma>p-1$ we have that*

$$u(x_0)\le C\left(r^{-n}\int_{B(x_0,r)}u^\gamma\ \mathrm{d}x\right)^{1/\gamma}+C\mathbf{W}^\mu_{1,p}(x_0,2r),$$

where $C=C(n,p,\alpha,\beta,\gamma)>0$.

Proof. By Hölder's inequality we may assume that

$$\gamma<\frac{n(p-1)}{n-p+1}.$$

We fix a constant $\delta\in(0,1)$ to be specified later. Let $B_j=B(x_0,r_j)$, where $r_j=2^{1-j}r$. We define a sequence a_j recursively. Let $a_0=0$ and for $j\ge0$ let

$$a_{j+1}=a_j+\delta^{-1}\left(r_j^{-n}\int_{B_{j+1}\cap\{u>a_j\}}(u-a_j)^\gamma\ \mathrm{d}x\right)^{1/\gamma}.$$

Note that $a_j<\infty$ for all j (see Lemma 10). We first derive the estimate

$$\delta^{p\gamma/q}\le C\delta^\gamma\left(\frac{a_j-a_{j-1}}{a_{j+1}-a_j}\right)^\gamma+C(a_{j+1}-a_j)^{1-p}\frac{\mu(B_j)}{r_j^{n-p}},\tag{15}$$

if $j\ge1$ is such that$a_{j+l}>a_j$ and $q=p\gamma/(p-\gamma/(p-1))$ is as in the proof of Lemma 14. From now on we assume that $\delta>0$ is so small that

$$\delta^\gamma\le2^{-n-1}r_j^{-n}|B_j|.\tag{16}$$

Since

$$\begin{aligned}|B_j\cap\{u>a_j\}|&\le(a_j-a_{j-1})^{-\gamma}\int_{B_j\cap\{u>a_j\}}(u-a_{j-1})^\gamma\ \mathrm{d}x\\&\le(a_j-a_{j-1})^{-\gamma}\int_{B_j\cap\{u>a_{j-1}\}}(u-a_{j-1})^\gamma\ \mathrm{d}x=\delta^\gamma r_{j-1}^n\\&=2^nr_j^n\delta^\gamma=2^n(a_{j+1}-a_j)^{-\gamma}\int_{B_{j+1}\cap\{u>a_j\}}(u-a_j)^\gamma\ \mathrm{d}x.\end{aligned}\tag{17}$$

Hence, by (16), we have

$$|B_j\cap\{u>a_j\}|\le\frac12|B_j|.\tag{18}$$

Let

$$d_j=2^{-(n+2)/\gamma}(a_{j+1}-a_j),$$

we have by (17) that

$$|B_j \cap \{u > a_j\}| \le \frac{1}{4} d_j^{-\gamma} \int_{B_{j+1}\cap\{u>a_j\}} (u - a_j)^\gamma \, \mathrm{d}x,$$

which satisfies the hypothesis (8). Hence Lemma 14 yields

$$\begin{aligned} &\left(d_j^{-\gamma} r_j^{-n} \int_{B_{j+1}\cap\{u>a_j\}} (u-a_j)^\gamma \, \mathrm{d}x\right)^{p/q} \\ &\qquad\qquad \le C d_j^{-\gamma} r_j^{-n} \int_{B_j\cap\{u>a_j\}} (u-a_j)^\gamma \, \mathrm{d}x + C d_j^{1-p} r_j^{p-n} \mu(B_j). \end{aligned}$$

Finally, because

$$\begin{aligned} d_j^{-\gamma} r_j^{-n} \int_{B_j\cap\{u>a_j\}} (u-a_j)^\gamma \, \mathrm{d}x &\le d_j^{-\gamma} r_j^{-n} \int_{B_j\cap\{u>a_{j-1}\}} (u-a_{j-1})^\gamma \, \mathrm{d}x \\ &= d_j^{-\gamma} r_j^{-n} \left[(a_j - a_{j-1})\delta\right]^\gamma r_{j-1}^n \\ &= C \left(\frac{d_{j-1}}{d_j}\right)^\gamma \delta^\gamma. \end{aligned}$$

By definition of d_j, we have

$$\begin{aligned} \delta^{p\gamma/q} &\le C\left(d_j^{-\gamma} r_j^{-n} \int_{B_{j+1}\cap\{u>a_j\}} (u-a_j)^\gamma \, \mathrm{d}x\right)^{p/q} \\ &\le C d_j^{-\gamma} r_j^{-n} \int_{B_j\cap\{u>a_j\}} (u-a_j)^\gamma \, \mathrm{d}x + C d_j^{1-p} r_j^{p-n} \mu(B_j) \\ &\le C \delta^\gamma \left(\frac{d_{j-1}}{d_j}\right)^\gamma + C d_j^{1-p} r_j^{p-n} \mu(B_j), \end{aligned}$$

and (15) follows.

Next we show that

$$a_{j+1} - a_j \le \frac{1}{2}(a_j - a_{j-1}) + C \left(\frac{\mu(B_j)}{r_j^{n-p}}\right)^{1/(p-1)}. \tag{19}$$

If $a_{j+1} - a_j \le \frac{1}{2}(a_j - a_{j-1})$, the estimate (19) is trivial. If $a_{j+1} - a_j > \frac{1}{2}(a_j - a_{j-1})$, then (15) implies that

$$\delta^{p\gamma/q} \le C\delta^\gamma + C(a_{j+1} - a_j)^{1-p} \frac{\mu(B_j)}{r_j^{n-p}}.$$

Now choosing $0 < \delta = \delta(n, p, \alpha, \beta, \gamma) \le 1$ small enough we obtain

$$\delta^{p\gamma/q} > 2C\delta^\gamma,$$

hence

$$(a_{j+1} - a_j)^{p-1} \le C \frac{\mu(B_j)}{r_j^{n-p}},$$

i.e. (19) holds.

Now we are ready to conclude the proof. First we deduce from (19) that

$$\begin{aligned} a_k - a_1 \le a_{k+1} - a_1 &= \sum_{j=1}^{k} (a_{j+1} - a_j) \\ &\le \frac{1}{2} \sum_{j=1}^{k} (a_j - a_{j-1}) + C \sum_{j=1}^{k} \left(\frac{\mu(B_j)}{r_j^{n-p}} \right)^{1/(p-1)} \\ &= \frac{1}{2} a_k + C \sum_{j=1}^{k} \left(\frac{\mu(B_j)}{r_j^{n-p}} \right)^{1/(p-1)} . \end{aligned}$$

Hence

$$\begin{aligned} \lim_{k\to\infty} a_k &\le 2a_1 + C \sum_{j=1}^{\infty} \left(\frac{\mu(B_j)}{r_j^{n-p}} \right)^{1/(p-1)} \\ &\le C \left(r^{-n} \int_{B(x_0,r)} u^{\gamma} \, \mathrm{d}x \right)^{1/\gamma} + C \int_0^{2r} \left(\frac{\mu(B_{x_0}, t)}{t^{n-p}} \right)^{1/(p-1)} \frac{\mathrm{d}t}{t} \\ &= C \left(r^{-n} \int_{B(x_0,r)} u^{\gamma} \, \mathrm{d}x \right)^{1/\gamma} + C \mathbf{W}_{1,p}^{\mu}(x_0, 2r). \end{aligned}$$

Now the theorem follows because by (18)

$$\inf_{B_j} u \le a_j$$

for $j = 1, 2, \cdots$, and for u is lower semicontinuous we have

$$u(x_0) \le \lim_{j\to\infty} \inf_{B_j} u \le \liminf_{j\to\infty} a_j,$$

which completes the proof. □

Now, we can give the proof of upper estimate in Theorem 9.

Proof of upper estimate in Theorem 9. By Lemma 15 we have

$$u(x_0) \le C \left(r^{-n} \int_{B(x_0,r)} u^{\gamma} \, \mathrm{d}x \right)^{1/\gamma} + C \mathbf{W}_{1,p}^{\mu}(x_0, 2r).$$

By the weak Harnack inequality in Lemma 10, we may pick $\gamma = \gamma(n, p) > p - 1$ such that

$$\left(r^{-n} \int_{B(x_0,r)} u^{\gamma} \, \mathrm{d}x \right)^{1/\gamma} \le \left(r^{-n} \int_{B(x_0,2r)} u^{\gamma} \, \mathrm{d}x \right)^{1/\gamma} \le C \inf_{B(x_0,r)} u.$$

□

Remark 16. *Because $\mathcal{A}$-superharmonic functions are lower semicontinuous and satisfy the minimum principle, we can replace $\inf_{B(x_0,r)} u$ by $\inf_{\partial B(x_0,r)} u$.*

Before giving the proofs of Theorem 2 and Theorem 8, we first show a direct application of Theorem 9, i.e. the following Harnack inequality for the equations $\mu = Tu$.

Theorem 17. *Suppose that u is a nonnegative $\mathcal{A}$-superharmonic function in $B(x_0, 7r)$ and let $\mu = Tu$. If there are $\varepsilon > 0$ and $M > 0$ such that*

$$\mu(B(x, \varrho)) \leq M \varrho^{n-p+\varepsilon}$$

whenever $x \in B(x_0, r)$ and $0 < \varrho < 4r$, then

$$\sup_{B(x_0,r)} u \leq C_1 \inf_{B(x_0,r)} u + C_2 r^{\varepsilon/(p-1)},$$

where $C_1 = C_l(n, p, \alpha, \beta)$ and $C_2 = C_2(n, p, \alpha, \beta, M, \varepsilon)$ are positive constants.

Proof. For $\forall\, x \in B(x_0, r)$, by Theorem 9 we have

$$u(x) \leq C \inf_{B(x,2r)} u + C\mathbf{W}^{\mu}_{1,p}(x, 4r).$$

Note that

$$\begin{aligned} \mathbf{W}^{\mu}_{1,p}(x, 4r) &= \int_0^{4r} \left(\frac{\mu(B(x, \varrho))}{\varrho^{n-p}} \right)^{1/(p-1)} \frac{\mathrm{d}\varrho}{\varrho} \\ &\leq M^{1/(p-1)} \int_0^{4r} \varrho^{\varepsilon/(p-1)} \frac{\mathrm{d}\varrho}{\varrho} = Cr^{\varepsilon/(p-1)}. \end{aligned}$$

Then

$$u(x) \leq C \inf_{B(x,2r)} u + Cr^{\varepsilon/(p-1)}.$$

It is easy to see that $B(x_0, r) \subset B(x, 2r)$, then we obtain that

$$\sup_{B(x_0,r)} u \leq C \inf_{B(x_0,r)} u + Cr^{\varepsilon/(p-1)}.$$

□

By a standard iteration, it follows from Harnack's inequality in the above theorem that certain $\mathcal{A}$-superharmonic functions are Hölder continuous. Remarkably, we can show that if the solution of $\mu = Tu$ is Hölder continuous, then μ must have the same growth restriction.

Theorem 18. *Suppose that u is $\mathcal{A}$-superharmonic in $B(x_0, r)$. If there are positive constants M and γ such that*

$$|u(x) - u(y)| \leq M|x-y|^{\gamma}$$

for every x and y in $B(x_0, r)$, then

$$\mu(B(x_0, \varrho)) \leq CM^{p-1} \varrho^{n-p+\gamma(p-1)}, \quad \forall\, \varrho \in (0, r/3),$$

where $C = C(n, p, \alpha, \beta) > 0$.

Proof. We apply the estimate in Theorem 9 to $\mathcal{A}$-superharmonic function $u - \inf_{B(x_0,3\varrho)} u$ and obtain

$$\begin{aligned}\left(\frac{\mu(B(x_0,\varrho))}{\varrho^{n-p}}\right)^{1/(p-1)} &\le C\int_{\varrho}^{2\varrho}\left(\frac{\mu(B(x_0,t))}{t^{n-p}}\right)^{1/(p-1)}\frac{\mathrm{d}t}{t}\\ &\le C\big(u(x_0)-\inf_{B(x_0,3\varrho)} u\big)\le CM\varrho^{\gamma},\end{aligned}$$

and the theorem follows. □

To proceed, we need the notion of $\mathcal{A}$-potential. Suppose that E be a subset of Ω. For $x \in \Omega$ let

$$\mathcal{R}^1_E(\Omega;\mathcal{A})(x) = \inf u(x),$$

where the infimum is taken over all nonnegative $\mathcal{A}$-superharmonic functions u in Ω such that $u \ge 1$ on E. The lower semicontinuous regularization

$$\widehat{\mathcal{R}}^1_E(\Omega;\mathcal{A})(x) = \lim_{r\to 0}\inf_{B(x,r)}\mathcal{R}^1_E(\Omega;\mathcal{A})$$

of $\mathcal{R}^1_E(\Omega;\mathcal{A})$ is called the $\mathcal{A}$-potential of E in Ω. The $\mathcal{A}$-potential $\widehat{\mathcal{R}}^1_E(\Omega;\mathcal{A})$ is $\mathcal{A}$-superharmonic in Ω and $\mathcal{A}$-harmonic in $\Omega\backslash\overline{E}$. The following lemma will be used later, and the proof can be found in [KM2, Lemma 3.7].

Lemma 19. *Suppose that Ω is bounded and $E \subset\subset \Omega$. Let $u = \widehat{\mathcal{R}}^1_E(\Omega;\mathcal{A})$ be the $\mathcal{A}$-potential of E in Ω and $\mu = Tu$. Then*

$$\mu(U) \le \frac{\beta^p}{\alpha^{p-1}}\operatorname{cap}_{\boldsymbol{p}}(E\cap U,\Omega)$$

whenever $U \subset \Omega$ is open.

Now, we are ready to prove Theorem 8.

Proof of Theorem 8. The sufficiency part was established in [HK, Section 4]. We are going to prove the necessity. Let E be p-thin at $x_0 \notin E$. We may assume that E is open [HK]. Write $B_j = B(x_0, r_j)$, $r_j = 2^{-j}$, and $E_j = E \cap B_j$. Let $k \ge 2$ be an integer to be specified later. Let $u = \widehat{\mathcal{R}}^1_{E_k}(B_{k-2};\mathcal{A})$ be the $\mathcal{A}$-potential of E_k in B_{k-2} and $\mu = Tu$. Then $u \ge 1$ on E_k and it remains to prove that (for some k) $u(x_0) < 1$. Denote $\lambda = \inf_{B_k} u$, we have

$$\lambda^{p-1}r_k^{n-p} \le C\lambda^{p-1}\operatorname{cap}_p(B_k,B_{k-2}) \le C\lambda^{p-1}\operatorname{cap}_p(\{u>\lambda\},B_{k-2}). \tag{20}$$

Note that for any Ω, there is

$$\begin{aligned}\alpha\int_\Omega |\nabla\min(u,\lambda)|^p\,\mathrm{d}x &\le \int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla\min(u,\lambda)\,\mathrm{d}x\\ &= \int_\Omega \min(u,\lambda)\,\mathrm{d}\mu \le \lambda\mu(\Omega),\end{aligned}$$

which gives us that

$$\lambda^{p-1}\operatorname{cap}_p(\{x\in\Omega : u(x)>\lambda\},\Omega)\le \frac{\mu(\Omega)}{\alpha}.$$

Hence, we obtain from (20) that

$$\inf_{B_k} u \le C\left(\frac{\mu(B_{k-1})}{r_{k-1}^{n-p}}\right)^{1/(p-1)}.$$

Moreover, it follows from Lemma 19 that for $j>k-2$

$$\mu(B_j)\le C\operatorname{cap}_p(E_j,B_{k-2})\le C\operatorname{cap}_p(E_j,B_{j-1}).$$

Hence, by Theorem 9, we have that

$$\begin{aligned} u(x_0) &\le C\inf_{B_k} u + C\mathbf{W}^{\mu}_{1,p}(x_0,r_{k-1}) \\ &\le C\sum_{j=k-1}^{\infty}\left(\frac{\operatorname{cap}_p(E_j,B_{j-1})}{r_j^{n-p}}\right)^{1/(p-1)} \le \frac{1}{2}, \end{aligned}$$

where $C=C(n,p,\alpha,\beta)>0$ and the last inequality follows by choosing k large enough. This completes the proof. □

Next, we prove Theorem 2.

Proof of Theorem 2. That the divergence of the Wiener integral $W_p(\mathbb{R}^n\backslash\Omega,x_0)$ implies the regularity of x_0 was proved by Maz'ya [Ma] if Ω is bounded; the general case was treated in [Ki]. See also [HKM, 6.16 and Chapter 9], where a somewhat simpler proof for Maz'ya's estimate is given.

For the converse, suppose that

$$W_p(\mathbb{R}^n\backslash\Omega,x_0)<\infty.$$

If x_0 is an isolated boundary point, it never is regular as easily follows by using the maximum principle and the removability of singletons for bounded $\mathcal{A}$-harmonic functions [Ki]. Hence we are free to assume that x_0 is an accumulation point of $E=\mathbb{R}^n\backslash\Omega$. Because E is p-thin at x_0, we now infer from Theorem 8 that there are balls $B_i=B_i(x_0,r_i)$, $i=1,2$, such that $r_1<r_2$ and an $\mathcal{A}$-superharmonic function u in B_2 such that $0\le u\le 1$, $u=1$ in $B_2\cap E\backslash\{x_0\}$ and $u(x_0)\le\frac{1}{2}$. Next, choose a function $\varphi\in C^\infty(\mathbb{R}^n)$ such that $\varphi\le u$ in $E\cap\overline{B}_1\backslash\{x_0\}$ and that $\varphi=1$ in a neighborhood of x_0. Consider the upper Perron solution $\overline{H}_\varphi$ taken in the open set $B_1\cap\Omega$, which is defined as

$$\overline{H}_\varphi := \inf\{u : u\in\mathcal{U}_\varphi\},$$

where $\mathcal{U}_\varphi$ consists of all $\mathcal{A}$-superharmonic functions u in $B_1\cap\Omega$ such that u is bounded below and that $\liminf_{x\to y} u(x)\ge\varphi(y)$ for all $y\in\partial(B_1\cap\Omega)$. Note that $\overline{H}_\varphi\in W^{1,p}(B_1\cap\Omega)$ ([HKM, 9.29] or [Ki, 6.2]), it follows from the generalized comparison principle that

$$\overline{H}_\varphi\le u \quad\text{in}\quad B_1\cap\Omega.$$

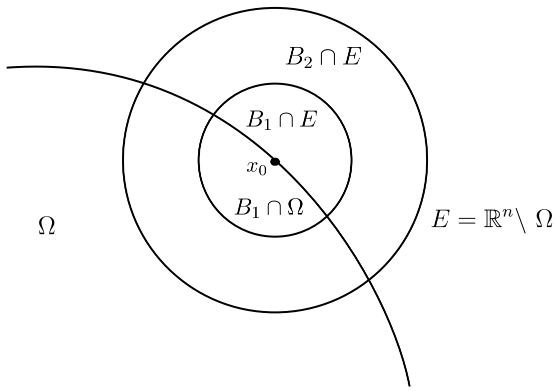

FIGURE 1.

In particular,

$$\liminf_{x\to x_0} \overline{H}_\varphi(x) \le \liminf_{x\to x_0,\, x\in\Omega} u(x) = u(x_0) \le \frac{1}{2} < 1 = \varphi(x_0).$$

Hence x_0 is not regular boundary point of $B_1\backslash\Omega$. Since the barrier characterization for regularity [Ki] implies that the regularity is a local property, it follows that x_0 is not a regular boundary point of Ω. Theorem 2 is proved. □

To conclude, we provide an equivalent characterization of the regularity of boundary points. A boundary point x_0 of a bounded domain Ω is said to be *exposed* if there exists a continuous function $h : \overline{\Omega} \to \mathbb{R}$, $\mathcal{A}$-harmonic in Ω, such that $h(x_0) = 0$ and $h > 0$ on $\overline{\Omega}\backslash\{x_0\}$. We then have the following:

Theorem 20. *A boundary point x_0 of a bounded open set Ω is regular if and only if it is exposed.*

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