AN ESTIMATE FOR THE LAPLACIAN OF PICK INVARIANT

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In this short notes, we will derive an estimate for the Laplacian of Pick invariant of an affine hypersphere along the line of [LSZ]. It was firstly obtained by W. Blaschke [Bl] in $n = 2$. For higher dimensional affine spheres it was obtained by E. Calabi [Ca] in the case of parabolic affine hyperspheres, and for arbitrary affine hyperspheres by R. Schneider [Sc] (with a minor misprint of a constant) and also by Cheng and Yau [CY]. First, we state the main result as follows.

Theorem 1. *On a locally strongly convex affine hypersphere, we have*

$$\frac{n(n-1)}{2}\Delta J \geq \|\nabla A\|^2 + n(n-1)(n+1)J(J+L_1),$$

where L_1 is the affine mean curvature, $A = \sum A_{ijk}w^i w^j w^k$ is the Fubini-Pick tensor and

$$J = \frac{1}{n(n-1)}\sum G^{il}G^{jm}G^{kr}A_{ijk}A_{lmr} = \frac{1}{n(n-1)}\|A\|^2.$$

To prove Theorem 1, we need some lemmas. We firstly state and prove them in order.

Lemma 2. *A non-degenerate hypersurface is an affine hypersphere if and only if the cubic form satisfies*

$$A_{ijk,l} = A_{ijl,k}$$

with respect to the Levi-Civita connection.

Proof. Note that

$$A_{ijk,l} - A_{ijl,k} = \frac{1}{2}\left(G_{ik}B_{jl} + G_{jk}B_{il} - G_{il}B_{jk} - G_{jl}B_{ik}\right).$$

Hence, if there is $A_{ijk,l} = A_{ijl,k}$, we have $B = L_1 G$, which implies that the hypersurface is an affine hypersphere. The converse is trival since an affine hypersphere satisfies $B = L_1 G$ with L_1, a constant. □

Lemma 3. *On a locally strongly convex affine hypersphere, and with respect to any orthonormal frame field, there is*

$$\frac{n(n-1)}{2}\Delta J = \sum A_{ijk,l}^2 + \sum R_{ij}^2 + \sum R_{ijkl}^2 - (n+1)RL_1, \tag{1}$$

where R_{ij}, R_{ijkl}, and R are Ricci curvature tensor, Riemann curvature tensor and scalar curvature.

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Proof. Let $x : M \to A^{n+1}$ be an affine hypersphere. Choose a local equiaffine frame field $\{x; e_1, \cdots, e_{n+1}\}$ on M such that $G_{ij} = \delta_{ij}$. Since M is an affine hypersphere, we have

$$A_{ijk,l} = A_{ijl,k} \tag{2}$$

$$R_{ijkl} = \sum_m \left(A_{iml}A_{jmk} - A_{imk}A_{jml}\right) - L_1\left(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}\right). \tag{3}$$

Applying the Ricci identity, we have

$$\begin{aligned}
\Delta A_{ijk} &= \sum_l A_{ijk,ll} = \sum_l A_{ijl,kl} \quad ((2)\text{ used})\\
&= \sum_l A_{ijl,lk} + \sum_{l,r} A_{ijr}R_{rlkl} + \sum_{l,r} A_{irl}R_{rikl} + \sum_{l,r} A_{rjl}R_{rikl} \quad (\text{Ricci identity})\\
&= \sum_{l,r} A_{ijr}R_{rlkl} + \sum_{l,r} A_{irl}R_{rjkl} + \sum_{l,r} A_{rjl}R_{rikl} \quad (\text{Apolarity condition}).
\end{aligned}$$

Then the Laplacian of the Pick invariant J is given by

$$\begin{aligned}
\Delta J &= \frac{1}{n(n-1)}\Delta\left(\sum A_{ijk}^2\right)\\
&= \frac{2}{n(n-1)}\left[\sum A_{ijk,l}^2 + \sum A_{ijk}A_{ijk,ll}\right]\\
&= \frac{2}{n(n-1)}\left[\sum A_{ijk,l}^2 + \sum A_{ijk}A_{ijr}R_{rlkl} + \sum\left(A_{ijk}A_{irl} - A_{ijl}A_{irk}\right)R_{rjkl}\right].
\end{aligned} \tag{4}$$

By (3), we know that

$$\sum_{i,j} A_{ijk}A_{ijr} = R_{kr} - (n-1)L_1\delta_{kr},$$

and

$$\sum_i \left(A_{ijk}A_{irl} - A_{ijl}A_{irk}\right) = R_{rjkl} + L_1\left(\delta_{rl}\delta_{jk} - \delta_{rk}\delta_{jl}\right).$$

Hence, we have

$$\begin{aligned}
\sum A_{ijk}A_{ijr}R_{rlkl} &= \sum (R_{kr} - (n-1)L_1\delta_{kr})R_{kr}\\
&= \sum R_{kr}^2 - (n-1)L_1\sum \delta_{kr}R_{kr}\\
&= \sum R_{ij}^2 - (n-1)L_1R,
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
\sum\left(A_{ijk}A_{irl} - A_{ijl}A_{irk}\right)R_{rjkl} &= \sum(R_{rjkl} + L_1\left(\delta_{rl}\delta_{jk} - \delta_{rk}\delta_{jl}\right))R_{rjkl}\\
&= \sum R_{rjkl}^2 + L_1\sum\left(\delta_{rl}\delta_{jk} - \delta_{rk}\delta_{jl}\right)R_{rjkl}\\
&= \sum R_{ijkl}^2 - 2L_1R.
\end{aligned} \tag{6}$$

Inserting (5) and (6) into (4), we get

$$\frac{n(n-1)}{2}\Delta J = \sum A_{ijk,l}^2 + \sum R_{ij}^2 + \sum R_{ijkl}^2 - (n+1)RL_1.$$

□

To get the final inequality we need the following lemma, which is due to Calabi.

Lemma 4. *Let M be a C^∞ n-dimensional Riemannian manifold. With respect to any orthonormal frame field, the Riemannian curvature tensor R_{ijkl}, Ricci curvature tensor R_{ij}, and the scalar curvature R satisfying*

$$\sum R_{ij}^2 \geq \frac{1}{n}R^2, \tag{7}$$

$$\sum R_{ijkl}^2 \geq \frac{2}{n-1}\sum R_{ij}^2. \tag{8}$$

Proof. (7) can be obtained easily by Cauchy-Schwarz's inequality. It suffices to prove (8). Before proving (8), we note that for $n = 2$, both terms of this inequality are identical, then we only need to consider $n \geq 3$. Consider the conformal curvature tensor of Weyl,

$$\begin{aligned} C_{ijkl} = R_{ijkl} &- \frac{1}{n-2}\left(\delta_{ik}R_{jl} + \delta_{jl}R_{ik} - \delta_{il}R_{jk} - \delta_{jk}R_{il}\right) \\ &+ \frac{R}{(n-1)(n-2)}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}). \end{aligned}$$

Then the conformal curvature tensor is traceless, which means that the contraction of any two of the four indices of C_{ijkl} gives the zero tensor. Define

$$R'_{ijkl} = R_{ijkl} - C_{ijkl}.$$

Then we get from the definition of C that

$$\sum_{i,j,k,l}\left(R'_{ijkl}\right)^2 = \frac{4}{n-2}\sum_{i,j} R_{ij}^2 - \frac{2R^2}{(n-1)(n-2)}.$$

On the other hand, from the definition of R' and the traceless of C, we have

$$\begin{aligned} \sum_{i,j,k,l} R_{ijkl}^2 &= \sum_{i,j,k,l}\left(R'_{ijkl}\right)^2 + \sum_{i,j,k,l}\left(C_{ijkl}\right)^2 + 2\sum_{i,j,k,l} C_{ijkl}R'_{ijkl} \\ &= \sum_{i,j,k,l}\left(R'_{ijkl}\right)^2 + \sum_{i,j,k,l}\left(C_{ijkl}\right)^2 \\ &\geq \sum_{i,j,k,l}\left(R'_{ijkl}\right)^2 \\ &= \frac{4}{n-2}\sum_{i,j} R_{ij}^2 - \frac{2R^2}{(n-1)(n-2)} \\ &\geq \frac{2}{n-1}\sum R_{ij}^2. \quad \text{(used (7))} \end{aligned}$$

□

Now, we are ready to give the proof of Theorem 1.

Proof of Theorem 1. Choose a local equiaffine frame field $\{x; e_1, \cdots, e_{n+1}\}$ such that $G_{ij} = \delta_{ij}$. Inserting (7) and (8) into (1), we conclude that

$$\frac{n(n-1)}{2}\Delta J \geq \sum_{i,j,k,l} A^2_{ijk,l} + \frac{n+1}{n(n-1)}R^2 - (n+1)RL_1.$$

By Ricci curvature tensor,

$$R_{ij} = \sum A^k_{il}A^l_{kj} + \frac{n-2}{2}B_{ij} + \frac{n}{2}L_1G_{ij},$$

we have

$$R = G^{ij}R_{ij} = n(n-1)(J + L_1).$$

Hence, we find that

$$\frac{n(n-1)}{2}\Delta J \geq \sum_{i,j,k,l} A^2_{ijk,l} + n(n-1)(n+1)J(J + L_1),$$

which is

$$\frac{n(n-1)}{2}\Delta J \geq \|\nabla A\|^2 + n(n-1)(n+1)J(J + L_1).$$

□

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