EXERCISES COURSE

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1. Week 3 (9.19)

Problem 1.1 (1.3)**.** *For* $\forall\, n \in \mathbb{N}$*, let* $A_n = \left[-1+\frac{1}{2n}, 1-\frac{1}{n}\right]$*, prove that* $\bigcup\limits_{n=1}^{\infty} A_n = (-1,1)$.

Proof. Firstly, by definition we have that

$$\forall\, x \in \bigcup_{n=1}^{\infty} A_n, \quad \exists\, n_0 \in \mathbb{N}, \quad s.t.\ x \in A_{n_0} \subset (-1,1).$$

Hence

$$x \in (-1,1),$$

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i.e.

$$\bigcup_{n=1}^{\infty} A_n \subset (-1,1). \tag{1.1}$$

On the other hand, $\forall\, x \in (-1,1)$, set

$$N = \left[\max\left\{\frac{1}{2(x+1)}, \frac{1}{1-x}\right\}\right] + 1,$$

then there is $x \in A_N$, thus $x \in \bigcup\limits_{n=1}^{\infty} A_n$, which means

$$(-1,1) \subset \bigcup_{n=1}^{\infty} A_n. \tag{1.2}$$

Combining (1.1) and (1.2) yields

$$\bigcup_{n=1}^{\infty} A_n = (-1,1).$$

□

Problem 1.2 (1.5)**.** *Prove that* $A = \left\{n \sin \frac{n\pi}{2}; n \in \mathbb{Z}\right\}$ *is unbounded.*

Proof. **Definition:** A set X is unbounded if and only if $\forall\, M > 0$, there exists a $x \in X$, such that $|x| > M$. Then $\forall\, M > 0$, let

$$N = 2([M]+1)+1,$$

hence

$$\left|N \sin \frac{N\pi}{2}\right| = 2([M]+1)+1 > M,$$

which implies that A is unbounded. □

Problem 1.3 (1.23)**.** *Suppose that* $f(x)$ *is defined on* E*, where* $|E| \geq 3$*. Prove that* $f(x)$ *is strictly monotonic on* E *iff for* $\forall\, x_1, x_2, x_3 \in E$*, if* $x_1 < x_2 < x_3$*, then there must be*

$$(f(x_1) - f(x_2))(f(x_2) - f(x_3)) > 0. \tag{1.3}$$

Proof. " only if ": Obviously.

" if ": Suppose that $f(x)$ is not a strictly monotonic function, hence there exist x_1, x_2, x_3, $x_4 \in E$ with $x_1 < x_2$ and $x_3 < x_4$ such that

$$f(x_1) \leq f(x_2) \quad \text{and} \quad f(x_3) \geq f(x_4).$$

If $x_1 < x_2 < x_3 < x_4$, we know that at least one of

$$(f(x_1) - f(x_2))(f(x_2) - f(x_3)) \quad \text{and} \quad (f(x_2) - f(x_3))(f(x_3) - f(x_4))$$

is non-positive, contradicting with (1.3).

If $x_1 < x_3 < x_2 < x_4$, we know that at least one of

$$(f(x_1) - f(x_2))(f(x_2) - f(x_4)) \quad \text{and} \quad (f(x_1) - f(x_3))(f(x_3) - f(x_4))$$

is non-positive, contradicting with (1.3).

Similar for other cases. Then we obtain contradictions for all cases, wihch means that $f(x)$ is strictly monotonic on E. □

Problem 1.4 (1.25)**.** *Prove that* $\sin(x^2 + x)$ *is not a periodic function.*

Proof. Suppose $T > 0$ is the period of $f(x) := \sin(x^2 + x)$, we have that

$$f(T) = f(-T) = f(0) = 0.$$

Hence there are $k_1, k_2 \in \mathbb{Z}$ such that

$$T^2 + T = k_1\pi \quad \text{and} \quad T^2 - T = k_2\pi,$$

which yields

$$T = \frac{k_1 - k_2}{2}\pi.$$

Then

$$\frac{(k_1 - k_2)^2}{4}\pi + \frac{k_1 - k_2}{2} = k_1,$$

i.e.

$$\pi = \frac{2(k_1 + k_2)}{(k_1 - k_2)^2} \in \mathbb{Q},$$

contradiction. □

Problem 1.5 (1.26)**.** *Suppose that* $f(x)$ *is defined on* $(0, +\infty)$, $x_1, x_2 > 0$. *Prove*

(1) *If* $\frac{f(x)}{x}$ *is decreasing, then* $f(x_1 + x_2) \leq f(x_1) + f(x_2)$;

(2) *If* $\frac{f(x)}{x}$ *is increasing, then* $f(x_1 + x_2) \geq f(x_1) + f(x_2)$.

Proof. (1) Since $\frac{f(x)}{x}$ is decreasing, then $f(x_1 + x_2) \leq f(x_1) + f(x_2)$, we have

$$\frac{f(x_1 + x_2)}{x_1 + x_2} \leq \frac{f(x_1)}{x_1} \quad \text{and} \quad \frac{f(x_1 + x_2)}{x_1 + x_2} \leq \frac{f(x_2)}{x_2}.$$

Hence

$$f(x_1 + x_2) = \frac{x_1}{x_1 + x_2} f(x_1 + x_2) + \frac{x_2}{x_1 + x_2} f(x_1 + x_2) \leq f(x_1) + f(x_2).$$

Similar for (2). □

Problem 1.6 (1.27)**.** *Suppose* $f(x)$ *is defined on* $(-\infty, +\infty)$, *and* $f(f(x)) \equiv x$.

(1) *Is* $f(x)$ *unique? If not, please give an example;*

(2) *If $f(x)$ is strictly increasing, is it unique? Why?*

Solution. (1) $f(x)$ is not unique. For example $f_1(x) = x$, $f_2(x) = -x$, $f_3(x) = \frac{1}{x}, x \neq 0$; $f_3(0) = 0$.

(2) We prove that $f(x) = x$. Suppose not, if there is a $x_0 \in \mathbb{R}$, such that $f(x_0) \neq x_0$. Without loss of generality, we assume that $f(x_0) > x_0$. Then $f(f(x_0)) > f(x_0)$ by f is strictly increasing. Hence

$$x_0 = f(f(x_0)) > f(x_0) > x_0,$$

contradiction. □

Problem 1.7. *Prove that $f(x) = \sin x + \sin \sqrt{2}x$, $x \in \mathbb{R}$ is not a periodic function.*

Proof. Note that

$$f''(x) = -\sin x - 2\sin\sqrt{2}x.$$

Hence we have

$$f(x) + f''(x) = -\sin\sqrt{2}x,$$
$$2f(x) + f''(x) = \sin x.$$

If $T > 0$ is the period of $f(x)$, then T is also the period of $f(x)+f''(x)$ and $2f(x)+f''(x)$. Hence

$$T = \frac{2k\pi}{\sqrt{2}} = 2m\pi, \quad \text{for some } k, m \in \mathbb{N},$$

which implies

$$\sqrt{2} = \frac{k}{m} \in \mathbb{Q},$$

contradiction. □

Problem 1.8. *Suppose $f(x)$ is an increasing function defined on closed interval $[a, b]$, and f satisfies $f(a) \geq a$, $f(b) \leq b$. Prove that there exists $x_0 \in [a, b]$, such that $f(x_0) = x_0$.*

Proof. Set

$$A = \{x \in [a, b] : f(x) \geq x\}.$$

Obviously, A is not empty since $f(a) \geq a$, and b is an upper bound of A. Hence $\sup A$ exists. We denote $x_0 := \sup A$. For $\forall\, x \in A$, there is $x \leq x_0$. Since f is increasing, we have $x \leq f(x) \leq f(x_0)$, i.e. $f(x_0)$ is an upper bound of A, thus $x_0 \leq f(x_0)$. If $x_0 < f(x_0)$, then $f(x_0) \leq f(f(x_0))$, which means $f(x_0) \in A$, contradiction. Hence $f(x_0) = x_0$. □

Problem 1.9. *Is there a function, whose period are all rational numbers but none of irrational number?*

Solution. Dirichlet function, i.e.

$$D(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \in \mathbb{R} \backslash \mathbb{Q}. \end{cases}$$

□

Problem 2.1 (2.3(5)). *Using definition to prove*

$$\lim_{n\to\infty} n^3 q^n = 0\,(|q|<1).$$

Proof. It suffices to prove $\lim_{n\to\infty} n^3|q|^n = 0$, and it is obviously valid when $|q| = 0$, hence we assume that $|q| \neq 0$ in the following. Let

$$\frac{1}{|q|} = 1+\alpha.$$

By binomial theorem, we have

$$\frac{1}{|q|^n} = (1+\alpha)^n = \sum_{k=0}^{n} C_n^k \alpha^k \geq \frac{n(n-1)(n-2)(n-3)}{4!}\alpha^4$$

provided with $n \geq 4$. Hence for $n \geq 4$,

$$n^3|q|^n \leq \frac{24n^2}{(n-1)(n-2)(n-3)\alpha^4} < \frac{72}{\alpha^4}\frac{1}{n-3}.$$

Then $\forall\,\varepsilon > 0$, choosing $N = \left[\frac{72}{\alpha^4\varepsilon}\right] + 4$, for any $n > N$, there is $n^3|q|^n < \varepsilon$, i.e. $\lim_{n\to\infty} n^3|q|^n = 0$. ☐

Problem 2.2 (2.4). *Suppose for all $n \in \mathbb{N}$, there is $x_n \leq a \leq y_n$ and $\lim_{n\to\infty}(y_n - x_n) = 0$. Prove*

$$\lim_{n\to\infty} x_n = a = \lim_{n\to\infty} y_n.$$

Proof. Since $x_n \leq a \leq y_n$, we know that

$$0 \leq a - x_n \leq y_n - x_n.$$

By Sandwich Theorem, we have

$$\lim_{n\to\infty} x_n = a.$$

Similar for y_n. ☐

Problem 2.3. *Suppose $\{a_n\}$ is monotonic increasing, $\{b_n\}$ is monotonic decreasing, and $\lim_{n\to\infty}(b_n - a_n) = 0$. Prove that $\lim_{n\to\infty} a_n$, $\lim_{n\to\infty} b_n$ exist, and they are equal.*

Proof. Assume $\lim_{n\to\infty} a_n$ doesn't exist, since $\{a_n\}$ is monotonic increasing, we know there is $\lim_{n\to\infty} a_n = +\infty$. By $\{b_n\}$ is monotonic decreasing, and $\lim_{n\to\infty}(b_n - a_n) = 0$, there must be $\lim_{n\to\infty} b_n = -\infty$. Hence, for $\forall\, M > 0$, there exists a $N \in \mathbb{N}$, such that for $\forall\, n > N$, there are

$$a_n > M \quad \text{and} \quad b_n < -M.$$

Then

$$b_n - a_n < -2M,$$

which contradicts with $\lim_{n\to\infty}(b_n - a_n) = 0$. Thus, $\lim_{n\to\infty} a_n$ exists. It's easy to see that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. □

Problem 2.4 (2.6(3)). *$\{F_n\}$ is the Fibonacci sequence, defined by*

$$F_0 = F_1 = 1, \quad F_{n+1} = F_n + F_{n-1}.$$

Prove $\lim_{n\to\infty} F_n = +\infty$ by definition.

Proof. **First way:** We can calculate the general terms of F_n by 'eigenvalue method' (See Problem 2.10 for details). Then we know

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right].$$

Since $\frac{1+\sqrt{5}}{2} > 1$ and $\frac{\sqrt{5}-1}{2} < 1$, we have $\lim_{n\to\infty} F_n = +\infty$.

Second way: We can prove by induction that $F_n \geq n$ for $n \geq 1$. Since $\lim_{n\to\infty} n = +\infty$, we know $\lim_{n\to\infty} F_n = +\infty$. □

Problem 2.5 (2.10(6)). *Calculate*

$$\lim_{n\to\infty} \sqrt[3]{n}(\sqrt[3]{n+1} - \sqrt[3]{n}).$$

Solution.

$$\lim_{n\to\infty} \sqrt[3]{n}(\sqrt[3]{n+1} - \sqrt[3]{n}) = \lim_{n\to\infty} \frac{\sqrt[3]{n}}{(\sqrt[3]{n+1})^2 + \sqrt[3]{n+1}\sqrt[3]{n} + (\sqrt[3]{n})^2} = 0.$$

□

Problem 2.6 (2.14). *Calculate $\lim_{n\to\infty} x_n$.*

(1) $x_n = \dfrac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}$;

(2) $x_n = \sum_{k=n^2}^{(n+1)^2} \frac{1}{\sqrt{k}}$;

(3) $x_n = \sqrt[n]{n \ln n}$.

Solution. (1)Note that

$$\frac{q}{p} < \frac{q+1}{p+1}, \quad \text{for } 0 < q < p.$$

Then

$$\begin{aligned} x_n &= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \\ &= \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n} \\ &< \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdots \frac{2n-2}{2n-1} \cdot \frac{2n}{2n+1} \\ &= \frac{2}{1} \cdot \frac{4}{3} \cdot \frac{6}{5} \cdots \frac{2n}{2n-1} \cdot \frac{1}{2n+1} \\ &= \frac{1}{x_n} \frac{1}{2n+1}, \end{aligned}$$

which gives us

$$x_n < \frac{1}{\sqrt{2n+1}}.$$

Hence $\lim_{n\to\infty} x_n = 0$.

(2) Note that $(n+1)^2 - n^2 + 1 = 2(n+1)$, we have

$$2 = \frac{2(n+1)}{n+1} \leq \sum_{k=n^2}^{(n+1)^2} \frac{1}{\sqrt{k}} \leq \frac{2(n+1)}{n} \to 2, \quad \text{as } n \to \infty.$$

Hence by Sandwich Theorem, $\lim_{n\to\infty} x_n = 2$.

(3) When $n > 3$, there is $n < n \ln n < n^2$. Then

$$1 \leftarrow \sqrt[n]{n} < \sqrt[n]{n \ln n} < \sqrt[n]{n^2} \to 1, \quad \text{as } n \to \infty.$$

Hence by Sandwich Theorem, $\lim_{n\to\infty} x_n = 1$. □

Problem 2.7 (2.17)**.** *Sequence* $\{q_n\}$ *satisfies*

(2.1) $$0 < q_n < 1, \quad (1-q_n)q_{n+1} > \frac{1}{4}, \quad \forall\, n \in \mathbb{N}.$$

Prove that $\{q_n\}$ *is monotonic increasing and* $\lim_{n\to\infty} q_n = \frac{1}{2}$.

Proof. By (2.1) and the mean value inequality, we have

$$\frac{q_{n+1}}{q_n} > \frac{1}{4q_n(1-q_n)} \geq \frac{1}{4\left(\frac{q_n+1-q_n}{2}\right)^2} = 1,$$

i.e. $q_{n+1} > q_n$. Hence $\{q_n\}$ is monotonic increasing. Then by the monotone bounded convergence theorem, we know $\lim_{n\to\infty} q_n$ exists. Denote $q := \lim_{n\to\infty} q_n$, by (2.1),

$$(1-q)q \geq \frac{1}{4},$$

i.e.

$$\left(q - \frac{1}{2}\right)^2 \leq 0,$$

which means $q = \frac{1}{2}$. □

Problem 2.8 (2.19)**.** *Suppose that* $0 < a_1 < b_1$*, let*

$$a_{n+1} = \sqrt{a_n b_n}, \quad b_{n+1} = \frac{1}{2}(a_n + b_n) \quad (n = 1, 2, \cdots).$$

Prove $\lim_{n\to\infty} a_n$ *and* $\lim_{n\to\infty} b_n$ *exist, and* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

Proof. By the mean value inequality, we have

$$a_{n+1} = \sqrt{a_n b_n} \leq \frac{1}{2}(a_n + b_n) = b_{n+1} \quad \text{for } n \geq 1.$$

Hence

$$a_{n+1} = \sqrt{a_n b_n} \geq a_n \quad \text{and} \quad b_{n+1} = \frac{1}{2}(a_n + b_n) \leq b_n \quad \text{for } n \geq 2.$$

Then

$$a_2 \leq \cdots \leq a_n \leq b_n \leq \cdots \leq b_2 \quad \text{for all } n \geq 2.$$

By the monotone bounded convergence theorem, we know $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist, and it's easy to see that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. □

Problem 2.9. *Suppose that* $a_1 > b_1 > 0$*, let*

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \frac{2a_n b_n}{a_n + b_n} \quad (n = 1, 2, \cdots).$$

Prove $\lim_{n\to\infty} a_n$ *and* $\lim_{n\to\infty} b_n$ *exist, and* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \sqrt{a_1 b_1}$.

Proof. The existence is similar to Problem 2.8. Thus, we only need to prove $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \sqrt{a_1 b_1}$, and if we notice that

$$a_{n+1}b_{n+1} = a_n b_n = \cdots = a_1 b_1,$$

and

$$a_{n+1} \geq \sqrt{a_n b_n} \geq b_{n+1},$$

it's easy to obtain the conclusion. □

Problem 2.10. *Suppose that* $a_1 = \alpha$, $b_1 = \beta$. *Let*

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \frac{a_{n+1} + b_n}{2} \quad (n = 1, 2, \cdots).$$

Prove $\lim_{n\to\infty} a_n$ *and* $\lim_{n\to\infty} b_n$ *exist,* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$, *and find the limitation.*

Proof. By $a_{n+1} = (a_n + b_n)/2$, we have $b_n = 2a_{n+1} - a_n$. Hence

$$2a_{n+2} - a_{n+1} = \frac{a_{n+1} + 2a_{n+1} - a_n}{2} = \frac{3}{2}a_{n+1} - \frac{1}{2}a_n,$$

which is

$$a_{n+2} = \frac{5}{4}a_{n+1} - \frac{1}{4}a_n. \tag{2.2}$$

Then the characteristic equation of (2.2) is

$$x^2 - \frac{5}{4}x + \frac{1}{4} = 0.$$

We find that $x = 1$ and $x = \frac{1}{4}$ are the solution. Hence the general form of a_n is

$$a_n = A + B\left(\frac{1}{4}\right)^n.$$

By $a_1 = \alpha$, $a_2 = (a_1 + b_1)/2 = (\alpha + \beta)/2$, we have

$$a_n = \frac{1}{3}\alpha + \frac{2}{3}\beta + \frac{2}{3}(\alpha - \beta)\left(\frac{1}{4}\right)^{n-1}.$$

Hence $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \frac{1}{3}\alpha + \frac{2}{3}\beta$. □

Problem 2.11 (2.20)**.** *Calculate following limitations.*

(2) $\lim_{n\to\infty}\left(1 + \frac{1}{n^2}\right)^n$;

(3) $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^{n^2}$.

Solution. (2)We first note that

$$\left(1+\frac{1}{n^2}\right)^{n^2} < e.$$

(This can be proved by definition of e.) Then we have

$$1 < \left(1+\frac{1}{n^2}\right)^{n} < e^{\frac{1}{n}} \to 1 \quad \text{as } n \to \infty.$$

Hence by Sandwich Theorem

$$\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^{n} = 1.$$

(3)**First way:** By binomial theorem, we know

$$\left(1+\frac{1}{n}\right)^{n^2} = 1 + n^2\frac{1}{n} + \cdots \geq n.$$

Hence

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n^2} = +\infty.$$

Second way: By the definition of e, we know there exists a $N \in \mathbb{N}$, such that for $n > N$, there is

$$\frac{e}{2} < \left(1+\frac{1}{n}\right)^{n} < e.$$

(Via choosing $\varepsilon = \frac{e}{2}$.) Hence we have

$$\left(1+\frac{1}{n}\right)^{n^2} > \left(\frac{e}{2}\right)^{n} \to +\infty \quad \text{as } n \to \infty,$$

i.e.

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n^2} = +\infty.$$

□

Problem 2.12. *Using* $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e$ *to prove* $\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{k!} = e$.

Proof. By binomial theorem, we have

$$\begin{aligned}\left(1+\frac{1}{n}\right)^{n} &= \sum_{k=0}^{n} C_n^k \frac{1}{n^k} = \sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\frac{1}{n^k}\\ &= \sum_{k=0}^{n}\frac{n(n-1)\cdots(n-k+1)}{k!}\frac{1}{n^k}\end{aligned}$$

$$= \sum_{k=0}^{n} \frac{1}{k!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)$$
$$\leq \sum_{k=0}^{n} \frac{1}{k!}.$$

From the above calculation, we know for $\forall\, m > n$, there is

$$\left(1+\frac{1}{m}\right)^m \geq \sum_{k=0}^{n} \frac{1}{k!}\left(1-\frac{1}{m}\right)\cdots\left(1-\frac{k-1}{m}\right).$$

Let $m \to +\infty$, there is

$$e \geq \sum_{k=0}^{n} \frac{1}{k!} \geq \left(1+\frac{1}{n}\right)^n.$$

Hence

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k!} = e.$$

□

Problem 2.13. *Calculate the limitation of*

$$\prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right).$$

Solution.

$$\lim_{n\to\infty} \prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right) = \lim_{n\to\infty} \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdots\frac{n-1}{n}\cdot\frac{n+1}{n} = \frac{1}{2}.$$

□

Problem 2.14. *Suppose* $\lim\limits_{n\to\infty} a_n = a$. *Prove*

$$\lim_{n\to\infty} \frac{p_1 a_n + p_2 a_{n-1} + \cdots + p_n a_1}{p_1 + p_2 + \cdots + p_n} = a,$$

where $p_k > 0$ *and* $\lim\limits_{n\to\infty} \dfrac{p_n}{p_1 + p_2 + \cdots + p_n} = 0$.

Proof. Without loss of generality, we can assume $a = 0$. (Otherwise, we can consider $a_n - a$ instead.) Since $\lim\limits_{n\to\infty} a_n = 0$, we know for $\forall\, \varepsilon > 0$, there exists $N_1 \in \mathbb{N}$, such that for all $n > N_1$, there is $|a_n| < \varepsilon/2$. Then

$$\left|\frac{p_1 a_n + \cdots + p_{n-N_1} a_{N_1+1}}{p_1 + p_2 + \cdots + p_n}\right| \leq \frac{p_1 + \cdots + p_{n-N_1}}{p_1 + p_2 + \cdots + p_n}\cdot\frac{\varepsilon}{2} < \frac{\varepsilon}{2}.$$

Since a_n is convergence, we know $\{a_n\}$ is bounded. We assume $|a_n| \leq M$ for some $M > 0$. By $\lim_{n\to\infty} \frac{p_n}{p_1 + p_2 + \cdots + p_n} = 0$, we have

$$\begin{aligned} \left| \frac{p_{n-N_1+1}a_{N_1} + \cdots + p_n a_1}{p_1 + p_2 + \cdots + p_n} \right| &\leq \left(\frac{p_{n-N_1+1} + \cdots + p_n}{p_1 + p_2 + \cdots + p_n} \right) M \\ &< M \sum_{k=1}^{N_1} \frac{p_{n-N_1+k}}{p_1 + p_2 + \cdots + p_{n-N_1+k}} \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Hence there exists a $N_2 \in \mathbb{N}$ such that for all $n > N_2$, there is

$$\left| \frac{p_{n-N_1+1}a_{N_1} + \cdots + p_n a_1}{p_1 + p_2 + \cdots + p_n} \right| < \frac{\varepsilon}{2}.$$

Then, set $N = \max\{N_1, N_2\}$, we know for all $n > N$, there is

$$\begin{aligned} \left| \frac{p_1 a_n + p_2 a_{n-1} + \cdots + p_n a_1}{p_1 + p_2 + \cdots + p_n} \right| &\leq \left| \frac{p_1 a_n + \cdots + p_{n-N_1} a_{N_1+1}}{p_1 + p_2 + \cdots + p_n} \right| + \left| \frac{p_{n-N_1+1}a_{N_1} + \cdots + p_n a_1}{p_1 + p_2 + \cdots + p_n} \right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

i.e.

$$\lim_{n\to\infty} \frac{p_1 a_n + p_2 a_{n-1} + \cdots + p_n a_1}{p_1 + p_2 + \cdots + p_n} = 0.$$

□

Problem 2.15. *Define sequence $\{a_n\}$ by $a_{n+1} = 2a_n - a_n^2$, where a_0 is given. Discuss the convergence and divergence of $\{a_n\}$ regards the choice of a_0.*

Solution. Firstly, we have

$$1 - a_{n+1} = 1 - 2a_n + a_n^2 = (1 - a_n)^2.$$

By induction, we have

$$1 - a_{n+1} = (1 - a_n)^2 = \cdots = (1 - a_0)^{2^{n+1}},$$

i.e.

$$a_n = 1 - (1 - a_0)^{2^n} \quad \text{for } n \geq 1.$$

Hence,

When $|1 - a_0| < 1$, i.e. $0 < a_0 < 2$, $\{a_n\}$ is convergent, and $\lim_{n\to\infty} a_n = 1$.

When $|1 - a_0| = 1$, i.e. $a_0 = 0$ or $a_0 = 2$, $\{a_n\}$ is convergent, and $\lim_{n\to\infty} a_n = 0$.

When $|1 - a_0| > 1$, i.e. $a_0 < 0$ or $a_0 > 2$, $\{a_n\}$ is divergent, $\lim_{n\to\infty} a_n = -\infty$. □

Problem 3.1 (2.21). *Suppose that* $\{b_n\}$ *is strictly increasing and* $\lim_{n\to\infty} b_n = +\infty$. *Prove that if*

$$\lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = A, \tag{3.1}$$

where A *is finite or* $\pm\infty$. *Then*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = A.$$

Proof. We first prove the case for A is finite. By (3.1), we know $\forall\, \varepsilon > 0$, there exists a $N_1 \in \mathbb{N}$, such that for $\forall\, n > N_1$, there is

$$\left| \frac{a_n - a_{n-1}}{b_n - b_{n-1}} - A \right| < \varepsilon.$$

Since $b_n > b_{n-1}$ for all $n \in \mathbb{N}$, we have

$$(A - \varepsilon)(b_n - b_{n-1}) < a_n - a_{n-1} < (A + \varepsilon)(b_n - b_{n-1}).$$

For given N_1, summing all those inequalities, we obtain

$$(A - \varepsilon)(b_n - b_{N_1}) < a_n - a_{N_1} < (A + \varepsilon)(b_n - b_{N_1}),$$

i.e.

$$\left| \frac{a_n - a_{N_1}}{b_n - b_{N_1}} - A \right| < \varepsilon.$$

Note the identity

$$\frac{a_n}{b_n} - A = \left(1 - \frac{b_{N_1}}{b_n}\right) \cdot \left(\frac{a_n - a_{N_1}}{b_n - b_{N_1}} - A\right) + \frac{a_{N_1} - A b_{N_1}}{b_n},$$

and combining $\lim_{n\to\infty} b_n = +\infty$, we know there exists a $N_2 \in \mathbb{N}$, such that for $n > N_2$, there is

$$0 < 1 - \frac{b_{N_1}}{b_n} < 2 \quad \text{and} \quad \left| \frac{a_{N_1} - A b_{N_1}}{b_n} \right| < \varepsilon.$$

Choosing $N = \max\{N_1, N_2\}$, then for $\forall\, n > N$, we have

$$\left| \frac{a_n}{b_n} - A \right| < 3\varepsilon,$$

i.e.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = A.$$

Next, we prove the case for $A = +\infty$. By (3.1), we know $\forall\, M > 0$, there exists a $N_1 \in \mathbb{N}$, such that for $\forall\, n > N_1$, there is

$$\frac{a_n - a_{n-1}}{b_n - b_{n-1}} > 3M.$$

Since $b_n > b_{n-1}$ for all $n \in \mathbb{N}$, we have

$$a_n - a_{n-1} > 3M(b_n - b_{n-1}).$$

For given N_1, summing all those inequalities, we obtain

$$a_n - a_{N_1} > 3M(b_n - b_{N_1}),$$

i.e.

$$\frac{a_n - a_{N_1}}{b_n - b_{N_1}} > 3M.$$

Note the identity

$$\frac{a_n}{b_n} = \left(1 - \frac{b_{N_1}}{b_n}\right) \cdot \left(\frac{a_n - a_{N_1}}{b_n - b_{N_1}}\right) + \frac{a_{N_1}}{b_n},$$

and combining $\lim_{n\to\infty} b_n = +\infty$, we know there exists a $N_2 \in \mathbb{N}$, such that for $n > N_2$, there is

$$\frac{1}{2} < 1 - \frac{b_{N_1}}{b_n} \quad \text{and} \quad \left|\frac{a_{N_1}}{b_n}\right| < \frac{1}{2}M.$$

Choosing $N = \max\{N_1, N_2\}$, then for $\forall\, n > N$, we have

$$\frac{a_n}{b_n} > \frac{3}{2}M - \frac{1}{2}M = M,$$

i.e.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = +\infty.$$

Similar for $A = -\infty$. □

Problem 3.2 ($\frac{0}{0}$ type of Stolz theorem)**.** *Suppose that* $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} b_n = 0$, *and* $\{b_n\}$ *is strictly decreasing. Prove that if*

$$\lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = A, \tag{3.2}$$

where A *is finite or* $\pm\infty$*. Then*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = A.$$

Proof. We only prove the case for A is finite. By (3.2), we know $\forall\, \varepsilon > 0$, there exists a $N \in \mathbb{N}$, such that for $\forall\, n > N_1$, there is

$$\left|\frac{a_n - a_{n-1}}{b_n - b_{n-1}} - A\right| < \varepsilon.$$

Since $b_n > b_{n+1}$ for all $n \in \mathbb{N}$, we have

$$(A - \varepsilon)(b_n - b_{n+1}) < a_n - a_{n+1} < (A + \varepsilon)(b_n - b_{n+1}).$$

For any $m > n$, summing all those inequalities, we obtain

$$(A - \varepsilon)(b_n - b_m) < a_n - a_m < (A + \varepsilon)(b_n - b_m),$$

i.e.

$$\left|\frac{a_n - a_m}{b_n - b_m} - A\right| < \varepsilon.$$

Let $m \to \infty$, and combining $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} b_n = 0$, we know for $\forall\, n > N$, there is

$$\left|\frac{a_n}{b_n} - A\right| \leq \varepsilon,$$

i.e.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = A.$$

□

Remark 3.3. *If $A = \infty$ in Problem 3.1 and Problem 3.2, the conclusion is not correct. For example $a_n = (-1)^n n$, $b_n = n$ in Problem 3.1 and $a_n = (-1)^n \frac{1}{n}$, $b_n = \frac{1}{n}$ Problem 3.2.*

Problem 3.4. *Using Stolz theorem to prove*

(1) *If $\lim_{n\to\infty} a_n = a$, then $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} a_k$ exists and $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} a_k = a$.*

(2) *If $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} a_k = a$, and $\lim_{n\to\infty} n(a_n - a_{n-1}) = 0$, then $\lim_{n\to\infty} a_n$ exists and $\lim_{n\to\infty} a_n = a$.*

Proof. (1) By Stolz theorem,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} a_k = \lim_{n\to\infty} \frac{\sum\limits_{k=1}^{n} a_k - \sum\limits_{k=1}^{n-1} a_k}{n - (n-1)} = \lim_{n\to\infty} a_n = a.$$

(2) Assume $a_0 = 0$. Let $A_n := a_n - a_{n-1}$, then $a_n = \sum\limits_{k=1}^{n} A_k$. The conditions become

$$a = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} a_k = \lim_{n\to\infty} \frac{nA_1 + (n-1)A_2 + \cdots + A_n}{n},$$

and

$$0 = \lim_{n\to\infty} n(a_n - a_{n-1}) = \lim_{n\to\infty} nA_n.$$

Hence

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{na_n}{n} = \lim_{n\to\infty} \frac{n\sum\limits_{k=1}^{n} A_k}{n} \\
&= \lim_{n\to\infty} \frac{nA_1 + nA_2 + \cdots + nA_n}{n}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty}\left(\frac{nA_1+(n-1)A_2+\cdots+A_n}{n}+\frac{A_2+\cdots+(n-1)A_n}{n}\right)\\
&= \lim_{n\to\infty}\frac{nA_1+(n-1)A_2+\cdots+A_n}{n}+\lim_{n\to\infty}\frac{A_2+\cdots+(n-1)A_n}{n}\\
&= a+\lim_{n\to\infty}\frac{A_2+\cdots+(n-1)A_n}{n}\\
&= a+\lim_{n\to\infty}\frac{(n-1)A_n}{n-(n-1)} \qquad \text{(Stolz theorem)}\\
&= a+\lim_{n\to\infty}\frac{n-1}{n}\cdot\lim_{n\to\infty}nA_n\\
&= a.
\end{aligned}
$$

□

Problem 3.5. *Suppose that* $x_{n+1}=x_n(1-x_n)$, $n=1,2,\cdots$, $0<x_1<1$. *Prove that* $\lim\limits_{n\to\infty}x_n=0$ *and* $\lim\limits_{n\to\infty}nx_n=1$.

Proof. Firstly. note that

$$x_{n+1}-x_n=-x_n^2\le 0.$$

Then $\{x_n\}$ is monotonic decreasing. Since $0<x_1<1$, we can prove $0<x_n<1$ by induction. Hence by monotone bounded convergence theorem, we know $\lim\limits_{n\to\infty}x_n$ exists and $\lim\limits_{n\to\infty}x_n=0$. And by Stolz theorem, we have

$$
\begin{aligned}
\lim_{n\to\infty}nx_n &= \lim_{n\to\infty}\frac{n}{\frac{1}{x_n}}=\lim_{n\to\infty}\frac{n-(n-1)}{\frac{1}{x_n}-\frac{1}{x_{n-1}}}\\
&= \lim_{n\to\infty}\frac{x_{n+1}x_n}{x_n-x_{n+1}}=\lim_{n\to\infty}\frac{x_{n+1}}{x_n}\\
&= \lim_{n\to\infty}(1-x_n)=1.
\end{aligned}
$$

□

Remark 3.6. *If we replace* $x_{n+1}=x_n(1-x_n)$ *by* $x_{n+1}=\ln(1+x_n)$, *we can have a similar conclusion.*

Problem 3.7 (2.23)**.** *Suppose* f *is defined on* (a,b), *and for* $\forall\,\xi\in(a,b)$, *there exists a* $\delta>0$, *such that for* $x\in(\xi-\delta,\xi+\delta)\cap(a,b)$,

(1) *If* $x<\xi$, *there is* $f(x)<f(\xi)$;
(2) *If* $x>\xi$, *there is* $f(x)>f(\xi)$.

Prove that f *is strictly increasing in* (a,b).

Proof. For $\forall\, x_1, x_2 \in (a,b)$ with $x_1 < x_2$, we want to show $f(x_1) < f(x_2)$. By assumption, $\forall\, \xi \in [x_1, x_2]$, there exists a $\delta = \delta(\xi)$ (If necessary, we can shrink δ so that $U(\xi,\delta) \subset (a,b)$), such that for $x \in U(\xi,\delta)$,

(1) If $x < \xi$, there is $f(x) < f(\xi)$;
(2) If $x > \xi$, there is $f(x) > f(\xi)$.

Then we know

$$[x_1, x_2] \subset \bigcup_{\xi \in [x_1,x_2]} U(\xi, \delta(\xi))$$

is an open covering. By Heine–Borel theorem, there exists a finite subcovering, i.e. there are $\xi_1, \cdots, \xi_n \in [x_1, x_2]$ and $\delta_1, \cdots, \delta_n > 0$, such that

$$[x_1, x_2] \subset \bigcup_{i=1}^{n} U(\xi_i, \delta_i).$$

Without loss of generality, we assume that

$$\xi_1 < \xi_2 < \cdots < \xi_n \quad \text{and} \quad U(\xi_i, \delta_i) \cap U(\xi_{i+1}, \delta_{i+1}) \neq \emptyset,\ i = 1, 2, \cdots, n-1.$$

Hence, we have $f(x_1) \le f(\xi_1) < f(\xi_2) < \cdots < f(\xi_n) \le f(x_2)$, i.e. $f(x_1) < f(x_2)$ valids for any $x_1 < x_2$. Thus, f is strictly increasing in (a,b). □

Problem 3.8 (2.25)**.** *Using supremum and infimum principle to prove accumulation point principle.*

Proof. **First way:** Suppose S is a bounded set consists of an infinite number of elements. By supremum and infimum principle, we know that $\sup S$ and $\inf S$ exist. If one of them is not a isolated point of S, it's obviously a accumulation point. Now, we assume none of them is the accumulation point of S. Set

$$E := \{x \in \mathbb{R} \,|\, \text{There are only a finite number of elments in } S \text{ that are less than } x\}.$$

Then E is nonempty and has an upper bound. Let $\eta := \sup E$, we prove η is a accumulation point of S. Indeed, by the construction of E, we know that $\forall\, \varepsilon > 0$, there must be $\eta + \varepsilon \notin E$, i.e. there are an infinite number of elements in S that are less that $\eta + \varepsilon$. Since there eixsts a $x_0 \in E$ such that $\eta - \varepsilon < x_0$, we know that there are only a finite number of elments in S that are less than $\eta - \varepsilon$. Then $(\eta - \varepsilon, \eta + \varepsilon)$ contains an infinite number of elements in S, which means that η is a accumulation point of S.

Second way: We first claim that any sequence in $\mathbb{R}$ has at least a monotonic subsequence (either increasing or decreasing). Indeed, if there is no increasing subsequence in $\{x_n\}$, we know there exists a $n_1 > 0$, such that $\forall\, n > n_1$, there is $x_n < x_{n_1}$. Similarly, there is no increasing subsequence in $\{x_n\}_{n > n_1}$, we know there exists a $n_2 > n_1$, such that $\forall\, n > n_2$, there is $x_n < x_{n_2} < x_{n_1}$. Proceeding like this, we can find a strictly decreasing subsequence $\{x_{n_k}\}$. Then it's easy to see that accumulation point principle is valid. □

Problem 3.9 (2.34)**.** *Prove that if* $x_n > 0$ *and*

$$\varlimsup_{n\to\infty} x_n \cdot \varlimsup_{n\to\infty} \frac{1}{x_n} = 1,$$

then the sequence $\{x_n\}$ *is convergent.*

Proof. Suppose there is a subsequence $\{x_{n_k}\} \subset \{x_n\}$, such that

$$\lim_{k\to\infty} x_{n_k} = \varliminf_{n\to\infty} x_n.$$

By the definition of limit superior and $x_n > 0$, we know

$$\frac{1}{x_{n_k}} \leq \sup_{n\geq n_k} \frac{1}{x_n}.$$

Hence

$$\varliminf_{n\to\infty} x_n = \lim_{k\to\infty} x_{n_k} \geq \lim_{k\to\infty} \frac{1}{\sup\limits_{n\geq n_k} \frac{1}{x_n}} = \frac{1}{\varlimsup\limits_{n\to\infty} \frac{1}{x_n}} = \varlimsup_{n\to\infty} x_n,$$

i.e.

$$\varliminf_{n\to\infty} x_n = \varlimsup_{n\to\infty} x_n.$$

Hence the sequence $\{x_n\}$ is convergent. □

Problem 3.10 (2.35)**.** *Suppose that* $\{x_n\}$ *is bounded, and*

$$\lim_{n\to\infty} (x_{n+1} - x_n) = 0. \tag{3.3}$$

Denote $l = \varliminf_{n\to\infty} x_n$ *and* $L = \varlimsup_{n\to\infty} x_n$*. Prove that any number in* $[l, L]$ *is the limitation of a subsequence of* $\{x_n\}$*.*

Proof. **First way:** By definition, l, L are both accumulation points of $\{x_n\}$. Then we assume $l < L$ and $a \in (l, L)$, and we will prove a is a accumulation point of $\{x_n\}$ in the following. We first prove the claim that for any given $\varepsilon > 0$ and $N \in \mathbb{N}$, there must exist a $\bar{n} > N$ such that

$$|x_{\bar{n}} - a| < \varepsilon.$$

By (3.3), we know there exists a $N' \in \mathbb{N}$, such that for $\forall\, n > N'$, there is

$$|x_{n+1} - x_n| < \varepsilon.$$

Let $N_0 := \max\{N, N'\}$. We know there must exist at least two points $x_{n'}$, $x_{n''}$ in $\{x_n\}_{n\geq N_0}$ such that $x_{n'} < a$, $x_{n''} > a$ (otherwise, if there is no points that are less than a, we must have $\varliminf_{n\to\infty} x_n \geq a$, contradicts with $l < a$; if there is no points that are large than a, we must have $\varlimsup_{n\to\infty} x_n \leq a$, contradicts with $a < L$). Without loss of generality, we assume $n' < n''$. Let $\bar{n}$ be the maximal integer which satisfies $n' \le n \le n''$ and $x_n \le a$. Clearly, $\bar{n} \le n'' - 1$ and $x_{\bar{n}} \le a$, $x_{\bar{n}+1} > a$. Hence $\bar{n} > N$, and

$$|x_{\bar{n}} - a| \le x_{\bar{n}+1} - x_{\bar{n}} < \varepsilon.$$

The claim is proved.

Now choosing $\varepsilon_1 = 1$, $N_1 = 1$, there exists a x_{n_1} $(n_1 > 1)$ such that $|x_{n_1} - a| < 1$. Then choosing $\varepsilon_2 = \frac{1}{2}$, $N_2 = n_1$, there exists a x_{n_2} $(n_2 > n_1)$ such that $|x_{n_2} - a| < \frac{1}{2}$. Next, choosing $\varepsilon_3 = \frac{1}{3}$, $N_3 = n_2$, there exists a x_{n_3} $(n_3 > n_2)$ such that $|x_{n_3} - a| < \frac{1}{3}$. Proceeding like this, we obtain a subsequence $\{x_{n_k}\} \subset \{x_n\}$ satisfies

$$|x_{n_k} - a| < \frac{1}{k}.$$

Hence $\lim\limits_{k\to\infty} x_{n_k} = a$, i.e. a is a accumulation point of $\{x_n\}$.

Second way: By definition, l, L are both accumulation points of $\{x_n\}$. Then we assume $l < L$ and $a \in (l, L)$, and we will prove a is a accumulation point of $\{x_n\}$ in the following. Let $\delta := \min\{L - a, a - l\}$. For $\forall\, i \in \mathbb{N}$, there exists a $k_i \in \mathbb{N}$, when $n \ge k_i$, we have $|x_{n+1} - x_n| < \frac{\delta}{2^i}$.

By definition of l, L, we can choose $l_1 \ge k_1$, such that $x_{l_1} > L - \frac{\delta}{2}$, and choose $m_1 > l_1$, such that $x_{m_1} < l + \frac{\delta}{2}$. Again, choosing $l_2 > \max\{k_2, m_1\}$, such that $x_{l_2} > L - \frac{\delta}{2^2}$, and choose $m_2 > l_2$, such that $x_{m_2} < l + \frac{\delta}{2^2}$. Proceeding like this, we can choose $l_i > \max\{k_i, m_{i-1}\}$, such that $x_{l_i} > L - \frac{\delta}{2^i}$, and choose $m_i > l_i$, such that $x_{m_1} < l + \frac{\delta}{2^i}$. Hence, we know there exists n_i, $l_i < n_i < m_i$ $(i \in \mathbb{N})$, such that

$$x_{n_i} < l + \delta \le a, \quad x_{n_i - 1} \ge a \quad (i \in \mathbb{N}).$$

Then

$$|x_{n_i} - a| < |x_{n_i} - x_{n_i-1}| \quad (i \in \mathbb{N}).$$

By $n_i > l_i > k_i$, we know $n_i - 1 \ge k_i$. Hence

$$|x_{n_i} - a| < |x_{n_i} - x_{n_i-1}| < \frac{\delta}{2^i}.$$

Hence $\lim\limits_{i\to\infty} x_{n_i} = a$, i.e. a is a accumulation point of $\{x_n\}$. □

Problem 3.11 (2.36)**.** *Suppose $\{x_n\}$ and $\{y_n\}$ satisfy*

$$x_{n+1} = y_n + q x_n \,(0 < q < 1), \quad n = 1, 2, \cdots.$$

Prove $\{y_n\}$ converges iff $\{x_n\}$ converges.

Proof. "if": Note that $y_n = x_{n+1} - q x_n$, it's easy to show $\{y_n\}$ converges if $\{x_n\}$ is convergent.

"only if": We first prove that $\{x_n\}$ is bounded. Since $\{y_n\}$ is convergent, we know it is bounded. Assume that $|y_n| \le M$ for some $M > 0$. Then

$$|x_{n+1}| = |y_n + q x_n| = |y_n + q(y_{n-1} + q x_{n-1})|$$

$$
\begin{aligned}
&= \left|y_n + q y_{n-1} + q^2 x_{n-1}\right| \\
&= \left|y_n + q y_{n-1} + \cdots + q^{n-1} y_1 + q^n x_1\right| \\
&\le M\left(1 + q + \cdots + q^{n-1}\right) + q^n |x_1| \\
&< \frac{M}{1-q} + |x_1|.
\end{aligned}
$$

Hence the upper limit and lower limit of $\{x_n\}$ exist. We have from $x_{n+1} = y_n + q x_n$ that

$$\varlimsup_{n\to\infty} x_n = \varlimsup_{n\to\infty} (y_{n-1} + q x_{n-1}) = \lim_{n\to\infty} y_n + q \varlimsup_{n\to\infty} x_{n-1} \le \lim_{n\to\infty} y_n + q \varlimsup_{n\to\infty} x_n,$$

which yields

$$\varlimsup_{n\to\infty} x_n \le (1-q)^{-1} \lim_{n\to\infty} y_n.$$

Similarly, we have

$$\varliminf_{n\to\infty} x_n \ge (1-q)^{-1} \lim_{n\to\infty} y_n.$$

Hence

$$\varlimsup_{n\to\infty} x_n = \varliminf_{n\to\infty} x_n = (1-q)^{-1} \lim_{n\to\infty} y_n,$$

i.e. $\{x_n\}$ converges. □

Problem 3.12.

(1) *(2.37). Suppose $\{x_n\}$ satisfies for $\forall\, n, m \in \mathbb{N}$, there is*

$$0 \le x_{n+m} \le x_n + x_m.$$

Prove that $\left\{\frac{x_n}{n}\right\}$ has a limitation.

(2) *Suppose $\{x_n\}$ satisfies for $\forall\, n, m \in \mathbb{N}$, there is*

$$0 \le x_{n+m} \le x_n \cdot x_m.$$

Prove that $\left\{x_n^{\frac{1}{n}}\right\}$ has a limitation.

Proof. (1) By

$$x_n \le x_{n-1} + x_1 \le x_{n-2} + 2x_1 \le \cdots \le n x_1,$$

we have

$$0 \le \frac{x_n}{n} \le x_1.$$

Hence $\left\{\frac{x_n}{n}\right\}$ is bounded. Denote $\varliminf_{n\to\infty} \frac{x_n}{n} = l$, then $0 \le l \le x_1$. By definition, for $\forall\, \varepsilon > 0$, there exists a $N \in \mathbb{N}$, such that

$$\frac{x_N}{N} < l + \varepsilon.$$

For any $n > N$, we choose $q \in \mathbb{N}$ and $0 \le r < N$, such that $n = qN + r$. Then

$$x_n = x_{qN+r} \le x_{qN} + x_r \le qx_N + rx_1 \le qx_N + Nx_1.$$

Hence

$$\frac{x_n}{n} \le \frac{qx_N}{n} + \frac{Nx_1}{n} \le \frac{x_N}{N} + \frac{Nx_1}{n} < l + \varepsilon + \frac{Nx_1}{n}.$$

Thus

$$\overline{\lim_{n\to\infty}} \frac{x_n}{n} \le l + \varepsilon.$$

Let $\varepsilon \to 0$, we have

$$\overline{\lim_{n\to\infty}} \frac{x_n}{n} \le l = \underline{\lim_{n\to\infty}} \frac{x_n}{n}.$$

Hence

$$\overline{\lim_{n\to\infty}} \frac{x_n}{n} = \underline{\lim_{n\to\infty}} \frac{x_n}{n},$$

i.e. $\lim\limits_{n\to\infty} \frac{x_n}{n}$ exists.

(2) The proof is similar to (1), we leave it to readers. □

Problem 3.13 (3.8)**.** *Calculate the following limitations.*

(6) $\lim\limits_{t\to 1}(1-t)\tan\frac{\pi t}{2}$;

(8) $\lim\limits_{x\to\frac{\pi}{4}} \frac{\tan x - 1}{x - \frac{\pi}{4}}$;

(9) $\lim\limits_{x\to 0} \frac{\cos(n \arccos x)}{x}$ $\quad$ (*n is odd*).

Solution. (6)

$$\lim_{t\to 1}(1-t)\tan\frac{\pi t}{2} = \lim_{t\to 1}\frac{1-t}{\cos\frac{\pi t}{2}} = \lim_{t\to 1}\frac{1-t}{\sin\frac{\pi}{2}(1-t)} = \frac{2}{\pi}.$$

(8)

$$\lim_{x\to\frac{\pi}{4}} \frac{\tan x - 1}{x - \frac{\pi}{4}} = \lim_{x\to\frac{\pi}{4}} \frac{1}{\cos x} \cdot \lim_{x\to\frac{\pi}{4}} \frac{\sin x - \cos x}{x - \frac{\pi}{4}} = 2 \lim_{x\to\frac{\pi}{4}} \frac{\sin(x - \frac{\pi}{4})}{x - \frac{\pi}{4}} = 2.$$

(9)

$$\begin{aligned}
\lim_{x\to 0} \frac{\cos(n \arccos x)}{x} &= \lim_{x\to 0} \frac{(-1)^{\frac{n-1}{2}} \sin(\frac{n\pi}{2} - n \arccos x)}{x} \\
&= \lim_{x\to 0} \frac{(-1)^{\frac{n-1}{2}} \sin\left(n(\frac{\pi}{2} - \arccos x)\right)}{x} \\
&= \lim_{x\to 0} \frac{(-1)^{\frac{n-1}{2}} \sin\left(n(\arcsin x)\right)}{x} \\
&= (-1)^{\frac{n-1}{2}} n,
\end{aligned}$$

where we have used $\arcsin x + \arccos x = \frac{\pi}{2}$. □

Problem 3.14 (3.9(4)). *Calculate the following limitation.*

$$\lim_{x\to\infty}\left(\cos\frac{a}{x}\right)^{x^2} \qquad (a \neq 0).$$

Solution. Note that

$$\begin{aligned}\lim_{x\to\infty}\left(\cos\frac{a}{x}\right)^{x^2} &= \lim_{x\to 0}(\cos ax)^{\frac{1}{x^2}} = \lim_{x\to 0}(1+\cos ax - 1)^{\frac{1}{x^2}}\\ &= \lim_{x\to 0}\left(1-2\sin^2\frac{ax}{2}\right)^{\frac{1}{x^2}}\\ &= \lim_{x\to 0}\left(\left(1-2\sin^2\frac{ax}{2}\right)^{\frac{1}{-2\sin^2\frac{ax}{2}}}\right)^{\frac{-2\sin^2\frac{ax}{2}}{x^2}}.\end{aligned}$$

Since

$$\lim_{x\to 0}\left(1-2\sin^2\frac{ax}{2}\right)^{\frac{1}{-2\sin^2\frac{ax}{2}}} = e,$$

we know that for $\forall\,\varepsilon > 0$, there exists a $\delta > 0$, such that $\forall\, x : |x| < \delta$, there is

$$e-\varepsilon < \left(1-2\sin^2\frac{ax}{2}\right)^{\frac{1}{-2\sin^2\frac{ax}{2}}} < e+\varepsilon.$$

Hence, we have for $|x| < \delta$ that

$$(e+\varepsilon)^{\frac{-2\sin^2\frac{ax}{2}}{x^2}} < \left(\left(1-2\sin^2\frac{ax}{2}\right)^{\frac{1}{-2\sin^2\frac{ax}{2}}}\right)^{\frac{-2\sin^2\frac{ax}{2}}{x^2}} < (e-\varepsilon)^{\frac{-2\sin^2\frac{ax}{2}}{x^2}}.$$

Then

$$\begin{aligned}(e+\varepsilon)^{\frac{-a^2}{2}} &\leq \varliminf_{x\to 0}\left(\left(1-2\sin^2\frac{ax}{2}\right)^{\frac{1}{-2\sin^2\frac{ax}{2}}}\right)^{\frac{-2\sin^2\frac{ax}{2}}{x^2}}\\ &\leq \varlimsup_{x\to 0}\left(\left(1-2\sin^2\frac{ax}{2}\right)^{\frac{1}{-2\sin^2\frac{ax}{2}}}\right)^{\frac{-2\sin^2\frac{ax}{2}}{x^2}}\\ &\leq (e-\varepsilon)^{\frac{-a^2}{2}}.\end{aligned}$$

Since ε is arbitrary, we have

$$\varliminf_{x\to 0}\left(\left(1-2\sin^2\frac{ax}{2}\right)^{\frac{1}{-2\sin^2\frac{ax}{2}}}\right)^{\frac{-2\sin^2\frac{ax}{2}}{x^2}} = \varlimsup_{x\to 0}\left(\left(1-2\sin^2\frac{ax}{2}\right)^{\frac{1}{-2\sin^2\frac{ax}{2}}}\right)^{\frac{-2\sin^2\frac{ax}{2}}{x^2}} = e^{\frac{-a^2}{2}},$$

i.e.

$$\lim_{x\to 0}\left(\left(1-2\sin^2\frac{ax}{2}\right)^{\frac{1}{-2\sin^2\frac{ax}{2}}}\right)^{\frac{-2\sin^2\frac{ax}{2}}{x^2}} = e^{\frac{-a^2}{2}}.$$

□

4. Week 7 (10.17)

Problem 4.1.

(1) *Suppose* $\{x_n\}$ *is a positive sequence. Prove that*

$$\varlimsup_{n\to\infty} n\left(\frac{1+x_{n+1}}{x_n}-1\right) \geq 1.$$

(2) *Suppose* $\{x_n\}$ *is a positive sequence. Prove that*

$$\varlimsup_{n\to\infty} \left(\frac{x_1+x_{n+1}}{x_n}\right)^n \geq e.$$

Proof. (1) Proof by contradiction. Assume that

$$\varlimsup_{n\to\infty} n\left(\frac{1+x_{n+1}}{x_n}-1\right) < 1.$$

Hence there exists a $N \in \mathbb{N}$, such that $\forall\, n \geq N$, there is

$$n\left(\frac{1+x_{n+1}}{x_n}-1\right) < 1.$$

Then we have

$$\frac{1}{n+1} < \frac{x_n}{n} - \frac{x_{n+1}}{n+1}, \quad n \geq N.$$

Summing all inequalities from N to n yields

$$\frac{1}{N+1} + \frac{1}{N+2} + \cdots + \frac{1}{n+1} < \frac{x_N}{N} - \frac{x_{n+1}}{n+1} \leq \frac{x_N}{N}.$$

However, we already konw $\lim\limits_{n\to\infty}\left(\frac{1}{N+1}+\frac{1}{N+2}+\cdots+\frac{1}{n+1}\right) = +\infty$, which makes a contradiction.

(2) Proof by contradiction. Assume that

$$\varlimsup_{n\to\infty} \left(\frac{x_1+x_{n+1}}{x_n}\right)^n < e.$$

Hence there exists a $N \in \mathbb{N}$, such that $\forall\, n \geq N$, there is

$$\left(\frac{x_1+x_{n+1}}{x_n}\right)^n < e < \left(1+\frac{1}{n-1}\right)^n.$$

Then we have

$$\frac{1}{n} < \frac{x_n}{n-1} - \frac{x_{n+1}}{n}, \quad n \geq N.$$

Summing all inequalities from N to n yields

$$\frac{1}{N} + \frac{1}{N+1} + \cdots + \frac{1}{n} < \frac{x_N}{N-1} - \frac{x_{n+1}}{n} \leq \frac{x_N}{N-1}.$$

However, we already konw $\lim_{n\to\infty}\left(\frac{1}{N}+\frac{1}{N+1}+\cdots+\frac{1}{n}\right)=+\infty$, which makes a contradiction. $\square$

Remark 4.2. *Both constants in Problem 4.1 are optimal. Indeed, we can choose* $x_n = n\ln n$.

Exercise 4.3 (Leave to readers)**.** *Let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence of positive numbers. Prove that*

$$\varliminf_{n\to\infty}\left(n^2\left(4a_n(1-a_{n-1})-1\right)\right)\leq\frac{1}{4}.$$

Hint: It suffices to prove the conclusion for $a_n\in(0,1)$. Assume by contradiction that $\varliminf_{n\to\infty}\left(n^2\left(4a_n(1-a_{n-1})-1\right)\right)>\frac{1}{4}$. Then for $\varliminf_{n\to\infty}\left(n^2\left(4a_n(1-a_{n-1})-1\right)\right)>l>\frac{1}{4}$, there exists a $N\in\mathbb{N}$, such that $\forall\, n>N$(without loss of generality, we can assume that $N=1$), there is

$$n^2\left(4a_n(1-a_{n-1})-1\right)>l.$$

Firstly, to prove that $\{a_n\}$ is monotonic increasing and $\lim_{n\to\infty}a_n=\frac{1}{2}$ (by the monotone bounded convergence theorem). Secondly, let $a_n:=\frac{1}{2}-b_n$, where $b_n\geq 0$ and $\lim_{n\to\infty}b_n=0$. Note that

$$b_{n-1}-b_n-2b_nb_{n-1}>\frac{l}{2n^2}.$$

Next, to prove that $\{nb_n\}$ is monotonic decreasing for large enough n. Indeed, there is

$$\begin{aligned}
nb_n-(n-1)b_{n-1}&<\frac{nb_{n-1}-\frac{l}{2n}}{1+2b_{n-1}}-(n-1)b_{n-1}\\
&=\frac{nb_{n-1}-\frac{l}{2n}-(n-1)b_{n-1}-2(n-1)b_{n-1}^2}{1+2b_{n-1}}\\
&=\frac{-2(n-1)\left(b_{n-1}-\frac{1}{4(n-1)}\right)^2+\frac{1}{8(n-1)}-\frac{l}{2n}}{1+2b_{n-1}}\\
&=\frac{-2(n-1)\left(b_{n-1}-\frac{1}{4(n-1)}\right)^2+\frac{(1-4l)n+4l}{8n(n-1)}}{1+2b_{n-1}}\\
&\leq 0,
\end{aligned}$$

provided with $n\geq\frac{4l}{4l-1}$. Hence $\lim_{n\to\infty}nb_n$ exists. Denote that $\lim_{n\to\infty}nb_n=A$. By Stolz theorem (for lower limit), we have

$$A=\lim_{n\to\infty}nb_n=\varliminf_{n\to\infty}\frac{b_n}{\frac{1}{n}}$$

$$
\begin{aligned}
&\geq \varliminf_{n\to\infty} \frac{b_n - b_{n-1}}{\frac{1}{n} - \frac{1}{n-1}} \\
&\geq \varliminf_{n\to\infty} n(n-1)\left(\frac{l}{2n^2} + 2b_n b_{n-1}\right) \\
&= \frac{l}{2} + 2\lim_{n\to\infty} n(n-1)b_n b_{n-1} \\
&= \frac{l}{2} + 2A^2,
\end{aligned}
$$

i.e.

$$2A^2 - A + \frac{l}{2} \leq 0.$$

However, $\Delta = 1^2 - 4 \times 2 \times \frac{l}{2} = 1 - 4l < 0$, which makes a contradiction.

From the above proof, we can see that $\frac{1}{4}$ is the optimal constant. Actually, we can choose $a_n = \frac{1}{2} - \frac{1}{4n}$. □

Problem 4.4. *Suppose that* $x_n > 0$*. Prove that* $\varlimsup_{n\to\infty} \sqrt[n]{x_n} \leq 1$ *iff* $\lim_{n\to\infty} \frac{x_n}{l^n} = 0,\ \forall\, l > 1$.

Proof. "if": If $\lim_{n\to\infty} \frac{x_n}{l^n} = 0$, we know there exists a $N \in \mathbb{N}$ such that $\forall\, n > N$, there is $x_n < \frac{1}{2}l^n$. Hence $\sqrt[n]{x_n} < \left(\frac{1}{2}\right)^{\frac{1}{n}} l$, which means $\varlimsup_{n\to\infty} \sqrt[n]{x_n} \leq l$. Since $l > 1$ is arbitrary, we know $\varlimsup_{n\to\infty} \sqrt[n]{x_n} \leq 1$.

"only if": For $\forall\, l > 1$, by $\varlimsup_{n\to\infty} \sqrt[n]{x_n} \leq 1$ we have there exists a $N \in \mathbb{N}$, such that $\forall\, n > N$, there is

$$\sqrt[n]{x_n} < 1 + \frac{l-1}{2} = \frac{l+1}{2}.$$

Then

$$\frac{x_n}{l^n} < \left(\frac{l+1}{2l}\right)^n \to 0 \quad \text{as } n \to \infty,$$

since $\frac{l+1}{2l} < 1$. Thus, $\lim_{n\to\infty} \frac{x_n}{l^n} = 0,\ \forall\, l > 1$. □

Problem 4.5. *Suppose that* $x_n > 0$*. If* $\varlimsup_{n\to\infty} \sqrt[n]{x_n} = 1$*, prove that* $\varlimsup_{n\to\infty} \sqrt[n]{\sum_{k=1}^{n} x_k} = 1$.

Proof. It is clear that $\varlimsup_{n\to\infty} \sqrt[n]{\sum_{k=1}^{n} x_k} \geq \varlimsup_{n\to\infty} \sqrt[n]{x_n} = 1$. Hence, we only need to prove the inverse inequality. Since $\varlimsup_{n\to\infty} \sqrt[n]{x_n} = 1$, we know for any $l > 1$, there exists a $N \in \mathbb{N}$, such that $\forall\, n > N$, there is $\sqrt[n]{x_n} < l$. Then

$$\begin{aligned}\sum_{k=1}^{n} x_k &\le \sum_{k=1}^{N} x_k + \sum_{k=N+1}^{n} l^k \\ &\le \sum_{k=1}^{N} x_k + \sum_{k=1}^{n} l^k \\ &< \sum_{k=1}^{N} x_k + \frac{l^{n+1}}{l-1}.\end{aligned}$$

Hence

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\sum_{k=1}^{n} x_k} \le \overline{\lim_{n\to\infty}} \sqrt[n]{\sum_{k=1}^{N} x_k + \frac{l^{n+1}}{l-1}} = l.$$

Since $l > 1$ is arbitrary, we know $\overline{\lim\limits_{n\to\infty}} \sqrt[n]{\sum\limits_{k=1}^{n} x_k} \le 1$. Then we are done. □

Problem 4.6. *Suppose that* $x_n > 0$, $\lim\limits_{n\to\infty} \frac{x_n}{n} = 0$ *and* $\overline{\lim\limits_{n\to\infty}} \frac{1}{n}\sum\limits_{k=1}^{n} x_k < +\infty$. *Prove that* $\overline{\lim\limits_{n\to\infty}} \frac{1}{n^2}\sum\limits_{k=1}^{n} x_k^2 = 0$.

Proof. Since $\lim\limits_{n\to\infty} \frac{x_n}{n} = 0$ and $L := \overline{\lim\limits_{n\to\infty}} \frac{1}{n}\sum\limits_{k=1}^{n} x_k < +\infty$, we know $\forall\, \varepsilon > 0$, there exists a $N \in \mathbb{N}$, such that $\forall\, n > N$, there is

$$\frac{x_n}{n} < \varepsilon \quad \text{and} \quad \frac{1}{n}\sum_{k=1}^{n} x_k < L + 1.$$

Then for $\forall\, n > N$,

$$\begin{aligned}\frac{1}{n^2}\sum_{k=1}^{n} x_k^2 &= \frac{1}{n^2}\sum_{k=1}^{N} x_k^2 + \frac{1}{n^2}\sum_{k=N+1}^{n} x_k^2 \\ &< \frac{1}{n^2}\sum_{k=1}^{N} x_k^2 + \frac{1}{n^2}\sum_{k=N+1}^{n} x_k \cdot k\varepsilon \\ &\le \frac{1}{n^2}\sum_{k=1}^{N} x_k^2 + \left(\frac{1}{n}\sum_{k=N+1}^{n} x_k\right)\varepsilon\end{aligned}$$

$$< \frac{1}{n^2}\sum_{k=1}^{N} x_k^2 + (L+1)\varepsilon.$$

Hence

$$\varlimsup_{n\to\infty} \frac{1}{n^2}\sum_{k=1}^{n} x_k^2 \le \varlimsup_{n\to\infty}\left(\frac{1}{n^2}\sum_{k=1}^{N} x_k^2 + (L+1)\varepsilon\right) = (L+1)\varepsilon.$$

Sicne $\varepsilon > 0$ is arbitrary, we know $\varlimsup_{n\to\infty} \frac{1}{n^2}\sum_{k=1}^{n} x_k^2 = 0$, i.e. $\lim_{n\to\infty} \frac{1}{n^2}\sum_{k=1}^{n} x_k^2 = 0$. □

Problem 4.7. *Calculate the following limitations.*

(1) $\lim_{x\to 0} \frac{\sqrt[m]{1+\alpha x} - \sqrt[n]{1+\beta x}}{x}$, *where* $m, n \in \mathbb{N}$, $\alpha, \beta \in \mathbb{R}$ *are constants;*

(2) $\lim_{n\to\infty} \sum_{k=1}^{n} \sin\left(\frac{2k-1}{n^2}x\right)$, *where* $x \in \mathbb{R}$ *is a constant;*

(3) $\lim_{x\to+\infty} \sin\sqrt{x+1} - \sin\sqrt{x}$;

(4) $\lim_{x\to 0} (x + e^x)^{\frac{1}{x}}$;

(5) $\lim_{x\to+\infty} \frac{[xf(x)]}{x}$, *where* $\lim_{x\to+\infty} f(x) = 1$;

(6) $\lim_{x\to+\infty} \left(\frac{1}{p}\sum_{k=1}^{p} a_k^x\right)^{\frac{1}{x}}$, *where* $a_1, \cdots, a_p$ $(p \ge 2)$ *are positive.*

(7) $\lim_{x\to 0+} \left(\frac{1}{p}\sum_{k=1}^{p} a_k^x\right)^{\frac{1}{x}}$, *where* $a_1, \cdots, a_p$ $(p \ge 2)$ *are positive.*

Solution. (1) By replacement with equivalent infinitesimal, we have

$$\begin{aligned}\lim_{x\to 0} \frac{\sqrt[m]{1+\alpha x} - \sqrt[n]{1+\beta x}}{x} &= \lim_{x\to 0} \frac{\sqrt[m]{1+\alpha x} - 1}{x} + \lim_{x\to 0} \frac{1 - \sqrt[n]{1+\beta x}}{x} \\ &= \frac{\alpha}{m} - \frac{\beta}{n}.\end{aligned}$$

(2) It's clear that the limitation is 0 when $x = 0$. Next, we assume that $x \neq 0$. By the Prosthaphaeresis formula, we have

$$\begin{aligned}\lim_{n\to\infty} \sum_{k=1}^{n} \sin\left(\frac{2k-1}{n^2}x\right) &= \lim_{n\to\infty} \frac{1}{2\sin\frac{x}{n^2}} \sum_{k=1}^{n} 2\sin\frac{x}{n^2}\sin\left(\frac{2k-1}{n^2}x\right) \\ &= \lim_{n\to\infty} \frac{1}{2\sin\frac{x}{n^2}} \sum_{k=1}^{n} \left[\cos\left(\frac{2k-2}{n^2}x\right) - \cos\left(\frac{2k}{n^2}x\right)\right]\end{aligned}$$

$$= \lim_{n\to\infty} \frac{1-\cos\left(\frac{2}{n}x\right)}{2\sin\frac{x}{n^2}}$$
$$= x.$$

(3) By the Prosthaphaeresis formula, we have

$$\begin{aligned}
\lim_{x\to+\infty} \sin\sqrt{x+1} - \sin\sqrt{x} &= \lim_{x\to+\infty} 2\cos\left(\frac{\sqrt{x+1}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+1}-\sqrt{x}}{2}\right) \\
&= \lim_{x\to+\infty} 2\cos\left(\frac{\sqrt{x+1}+\sqrt{x}}{2}\right)\sin\left(\frac{1}{2(\sqrt{x+1}+\sqrt{x})}\right) \\
&= 0.
\end{aligned}$$

(4) By replacement with equivalent infinitesimal, we have

$$\begin{aligned}
\lim_{x\to 0} (x+e^x)^{\frac{1}{x}} &= \lim_{x\to 0} e^{\frac{\ln(x+e^x)}{x}} \\
&= \lim_{x\to 0} e^{\frac{\ln(1+x+e^x-1)}{x}} \\
&= \lim_{x\to 0} e^{\frac{x+e^x-1}{x}} \\
&= e^2.
\end{aligned}$$

(5) Note that

$$f(x) - \frac{1}{x} < \frac{[xf(x)]}{x} \le f(x).$$

It's easy to obtain $\lim\limits_{x\to+\infty} \dfrac{[xf(x)]}{x} = 1$ by the Sandwich Theorem.

(6) Note that

$$\frac{\max\{a_1, \cdots, a_p\}}{p^{\frac{1}{x}}} \le \left(\frac{1}{p}\sum_{k=1}^{p} a_k^x\right)^{\frac{1}{x}} \le \max\{a_1, \cdots, a_p\}.$$

By the Sandwich Theorem, we have

$$\lim_{x\to+\infty} \left(\frac{1}{p}\sum_{k=1}^{p} a_k^x\right)^{\frac{1}{x}} = \max\{a_1, \cdots, a_p\}.$$

(7) Note that

$$\begin{aligned}
\frac{1}{x}\ln\left(\frac{1}{p}\sum_{k=1}^{p} a_k^x\right) &= \frac{1}{x}\ln\left(1 + \frac{1}{p}\sum_{k=1}^{p}(a_k^x - 1)\right) \\
&\sim \frac{1}{p}\sum_{k=1}^{p}\frac{a_k^x - 1}{x} \quad \text{as } x\to 0+
\end{aligned}$$

$$\to \frac{1}{p}\sum_{k=1}^{p}\ln a_k \quad \text{as } x \to 0+ .$$

Hence

$$\lim_{x\to 0+}\left(\frac{1}{p}\sum_{k=1}^{p} a_k^x\right)^{\frac{1}{x}} = e^{\frac{1}{p}\sum\limits_{k=1}^{p}\ln a_k} = \sqrt[p]{a_1 a_2 \cdots a_p}.$$

□

Problem 4.8. *Calculate* $\lim\limits_{n\to\infty}\prod\limits_{k=1}^{n}\cos\dfrac{x}{2^k}$, *and prove the Viète formula*

$$\frac{\pi}{2} = \frac{1}{\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots}.$$

Proof. It's clear that $\lim\limits_{n\to\infty}\prod\limits_{k=1}^{n}\cos\dfrac{x}{2^k} = 1$ when $x = 0$. Next, we assume that $x \neq 0$. By $\sin 2x = 2\sin x\cos x$, we have

$$\lim_{n\to\infty}\prod_{k=1}^{n}\cos\frac{x}{2^k} = \lim_{n\to\infty}\frac{\cos\frac{x}{2}\cos\frac{x}{2^2}\cdots\cos\frac{x}{2^n}\sin\frac{x}{2^n}}{\sin\frac{x}{2^n}} = \lim_{n\to\infty}\frac{\sin x}{2^n\sin\frac{x}{2^n}} = \frac{\sin x}{x}.$$

Note that $\cos 2\theta = 2\cos^2\theta - 1$, we know that $\cos\theta = \sqrt{\dfrac{1}{2}+\dfrac{1}{2}\cos 2\theta}$. Choosing $x = \frac{\pi}{2}$, we know that $\cos\frac{x}{2} = \sqrt{\frac{1}{2}}$, $\cos\frac{x}{4} = \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}$. Hence

$$\frac{\sin x}{x} = \sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots,$$

i.e.

$$\frac{1}{\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots} = \frac{\frac{\pi}{2}}{\sin\frac{\pi}{2}} = \frac{\pi}{2}.$$

□

Problem 4.9. *Suppose that* $f(x)$, $g(x)$ *are periodic function defined on* $\mathbb{R}$, *and satisfy* $\lim\limits_{x\to+\infty}(f(x)-g(x)) = 0$. *Prove that* $f(x) = g(x)$, $\forall\, x \in \mathbb{R}$.

Proof. Denote T_1 as the period of $f(x)$ and T_2 as the period of $g(x)$. We first note that by $\lim_{x\to+\infty}(f(x)-g(x))=0$, there is

$$f(x)-g(x+nT_1)=f(x+nT_1)-g(x+nT_1)\to 0 \quad \text{as } n\to\infty.$$

Similarly, we have

$$g(x)=\lim_{n\to\infty} f(x+nT_2), \quad \forall\, x\in\mathbb{R}.$$

Hence we have

$$\begin{aligned} f(x)-g(x) &= \lim_{n\to\infty}(g(x+nT_1)-f(x+nT_2)) \\ &= \lim_{n\to\infty}(g(x+nT_1+nT_2)-f(x+nT_2+nT_1)) \\ &= 0, \quad \forall\, x\in\mathbb{R}. \end{aligned}$$

□

Problem 4.10. *Suppose that $f(x)$ does not have an upper bound in (a,b). Prove that there exists a sequence $\{x_n\}\subset(a,b)$ such that $\lim_{n\to\infty} f(x_n)=+\infty$.*

Proof. Since $f(x)$ has no upper bound, we know that for any $n\in\mathbb{N}$, there exists a $x_n\in(a,b)$ such that $f(x_n)>n$. Then we have that $\lim_{n\to\infty} f(x_n)=+\infty$. □

Problem 4.11. *Suppose $f(x)$ is defined on $(0,1)$, and $\lim_{x\to 0} f(x)=0$, $\lim_{x\to 0}\dfrac{f(x)-f\left(\frac{x}{2}\right)}{x}=0$. Prove that $\lim_{x\to 0}\dfrac{f(x)}{x}=0$.*

Proof. By $\lim_{x\to 0}\dfrac{f(x)-f\left(\frac{x}{2}\right)}{x}=0$, we know that $\forall\,\varepsilon>0$, there exists a $\delta>0$, such that $\forall\, x: 0<x<\delta$, there is

$$\frac{\left|f(x)-f\left(\frac{x}{2}\right)\right|}{x}<\varepsilon,$$

i.e.

$$\left|f(x)-f\left(\frac{x}{2}\right)\right|<\varepsilon x.$$

Then for $0<x<\delta$, there is

$$\begin{aligned} |f(x)| &= \left|f(x)-f\left(\frac{x}{2}\right)+f\left(\frac{x}{2}\right)-f\left(\frac{x}{2^2}\right)+\cdots+f\left(\frac{x}{2^n}\right)\right| \\ &\le \left|f(x)-f\left(\frac{x}{2}\right)\right|+\left|f\left(\frac{x}{2}\right)-f\left(\frac{x}{2^2}\right)\right|+\cdots+\left|f\left(\frac{x}{2^n}\right)\right| \\ &< \varepsilon\left(x+\frac{x}{2}+\cdots+\frac{x}{2^n}\right)+\left|f\left(\frac{x}{2^n}\right)\right| \end{aligned}$$

$$< 2\varepsilon x + \left| f\left(\frac{x}{2^n}\right) \right| .$$

Since $\lim_{x\to 0} f(x) = 0$, we have by letting $n \to \infty$ that

$$|f(x)| \le 2\varepsilon x, \quad 0 < x < \delta.$$

Thus,

$$\lim_{x\to 0} \frac{f(x)}{x} = 0.$$

□

Problem 4.12. *Suppose $f(x) > 0$ is an increasing function defined on $(0, +\infty)$ satisfies $\lim_{x\to+\infty} \frac{f(2x)}{f(x)} = 1$. Prove that $\lim_{x\to+\infty} \frac{f(ax)}{f(x)} = 1$, $\forall\, a \in (0, +\infty)$.*

Proof. It suffices to prove $\lim_{x\to+\infty} \frac{f(ax)}{f(x)} = 1$ for all $a > 1$. Indeed, for any $0 < a < 1$, we have

$$\lim_{x\to+\infty} \frac{f(ax)}{f(x)} = \lim_{x\to+\infty} \frac{f(ax)}{f(\frac{1}{a}ax)} = \frac{1}{\lim_{ax\to+\infty} \frac{f(\frac{1}{a}ax)}{f(ax)}} = 1.$$

For $a > 1$, we know there exist some $N \in \mathbb{N}$ such that $2^N \le a < 2^{N+1}$. Since $f(x)$ is increasing, we know

$$\frac{f(2^N x)}{f(x)} \le \frac{f(ax)}{f(x)} \le \frac{f(2^{N+1}x)}{f(x)}.$$

Then the problem is reduced to prove $\lim_{x\to+\infty} \frac{f(2^K x)}{f(x)} = 1$ for all $K \in \mathbb{N}$, and it is clear since

$$\lim_{x\to+\infty} \frac{f(2^K x)}{f(x)} = \lim_{x\to+\infty} \frac{f(2^K x)}{f(2^{K-1}x)} \cdot \lim_{x\to+\infty} \frac{f(2^{K-1}x)}{f(2^{K-2}x)} \cdots \lim_{x\to+\infty} \frac{f(2x)}{f(x)} = 1.$$

□

Problem 4.13. *Suppose $f : (0, +\infty) \to \mathbb{R}$ satisfies $\forall\, a > 0$, f is bounded on $(0, a)$. Prove that if $\lim_{x\to+\infty} [f(x+1) - f(x)] = l$, then $\lim_{x\to+\infty} \frac{f(x)}{x} = l$.*

Proof. Without loss of generality, we can assume that $l = 0$ (otherwise, we replace $f(x)$ by $f(x) - lx$). Since $\lim_{x\to+\infty} [f(x+1) - f(x)] = 0$, we know $\forall\, \varepsilon > 0$, there exists a $X > 0$, such that $\forall\, x > X$, there is

$$-\varepsilon < f(x+1) - f(x) < \varepsilon.$$

Summing all inequalities, we have

$$-\left([x - X] + 1\right)\varepsilon < f(x+1) - f\left(x - [x - X]\right) < \left([x - X] + 1\right)\varepsilon.$$

Note that $\lim_{x\to+\infty} \frac{f(x-[x-X])}{x+1} = 0$ since f is bounded on $(0, X+1)$. We have

$$-\varepsilon \leq \varliminf_{x\to+\infty} \frac{f(x+1)}{x+1} \leq \varlimsup_{x\to+\infty} \frac{f(x+1)}{x+1} \leq \varepsilon.$$

Since ε is arbitrary, we obtain $\lim_{x\to+\infty} \frac{f(x)}{x} = 0$. □

Problem 4.14. *Suppose $f, g : (a, +\infty) \to \mathbb{R}$ satisfy $\forall\, b > a$, f, g are bounded on (a, b); g is strictly increasing, and $\lim_{x\to+\infty} g(x) = +\infty$. Prove that if $\lim_{x\to+\infty} \frac{f(x+1)-f(x)}{g(x+1)-g(x)} = l$, then $\lim_{x\to+\infty} \frac{f(x)}{g(x)} = l$.*

Proof. The proof is very similar to Problem 4.13, we omit details here and leave it to readers. □

Remark 4.15. *(1) Problem 4.14 is called the function version of Stolz theorem.*
(2) We can change the constant "1" to any positive constant "T".
(3) l can be chosen as $\pm\infty$, but it's not correct for ∞.

Problem 4.16. *Suppose that f is defined on $\mathbb{R}$, and f is bounded in some neighborhood of $x = 0$. If there exist $a > 1$, $b > 1$, such that $f(ax) = bf(x)$, prove that $f(x)$ is continuous at $x = 0$.*

Proof. First, by $f(ax) = bf(x)$, it's easy to know $f(0) = 0$. Assume that there exist $M > 0$, $\delta_0 > 0$, such that $\forall\, x : |x| < \delta_0$, there is $|f(x)| \leq M$. For $\forall\, \varepsilon > 0$, choosing $\delta = \delta_0/a^{N+1}$, where N satisfies $M/b^N < \varepsilon$. Then for $\forall\, x : |x| < \delta$, there is

$$|f(x)| = \frac{1}{b^N}|f(a^N x)| \leq \frac{M}{b^N} < \varepsilon.$$

Hence

$$\lim_{x\to 0} f(x) = 0 = f(0),$$

i.e. $f(x)$ is continuous at $x = 0$. □

Problem 4.17. *Suppose that $f(x) \in C[0, +\infty)$ is bounded, and $\lim_{x\to+\infty} f(x)$ does not exist. Prove there exists $t \in \mathbb{R}$ such that $f(x) = t$ has an infinite number of solutions.*

Proof. Since $f(x) \in C[0, +\infty)$ is bounded, and $\lim_{x\to+\infty} f(x)$ does not exist, we know

$$-\infty < \varliminf_{x\to+\infty} f(x) < \varlimsup_{x\to+\infty} f(x) < +\infty.$$

Denote $l := \underline{\lim}_{x\to+\infty} f(x)$ and $\overline{\lim}_{x\to+\infty} f(x)$. Let $t = \frac{l+L}{2}$, we will show $f(x) = t$ has an infinite number of solutions. Indeed, by the definition of upper limit and lower limit, we know that for $X_1 = 1$, there exist $x_1, y_1 > X_1$, such that $f(x_1) < t$, $f(y_1) > t$. Hence there exists z_1 between x_1 and y_1 such that $f(z_1) = t$ by $f(x)$ is continuous. For $X_2 = \max\{2, x_1, y_1\}$, there exist $x_2, y_2 > X_2$, such that $f(x_2) < t$, $f(y_2) > t$. Hence there exists z_2 between x_2 and y_2 such that $f(z_2) = t$ by $f(x)$ is continuous. Proceeding like this, we can find an infinite number of z_n such that $f(z_n) = t$, i.e. $f(x) = t$ has an infinite number of solutions. □

Problem 4.18. *Suppose that $f(x)$ is uniformly continuous on $[0, +\infty)$, and $\forall\, h > 0$, $\lim_{n\to\infty} f(nh)$ exists. Prove that $\lim_{x\to+\infty} f(x)$ exists.*

Proof. **First method: Find the limitation** Pick $h = 1$, assume that the sequence $\{f(n)\}_{n=1}^{\infty}$ converges to L. Then for each $m \in \mathbb{N}$, let $h = 1/m$. Then $\{f(n/m\}_{n=1}^{\infty}$ contains the subsequence $\{f(n)\}_{n=1}^{\infty}$. Thus $\{f(n/m\}_{n=1}^{\infty}$ converges to L for all m. Now we show that $\lim_{x\to\infty} f(x) = L$.

Let $\varepsilon > 0$. Since f is uniformly continuous, there is $\delta > 0$ so that if $x, y \in \mathbb{R}_+$ and $|x - y| < \delta$, then

$$|f(x) - f(y)| < \frac{\varepsilon}{2}.$$

Now let $m \in \mathbb{N}$ so that $1/m < \delta$. Since $\{f(n/m\}_{n=1}^{\infty}$ converges to L, there is $N \in \mathbb{N}$ so that

$$\left|f\left(\frac{n}{m}\right) - L\right| < \frac{\varepsilon}{2}$$

for all $n \geq N$. Let $M = N/m$. Then if $x \geq M$, there is $n \geq N$ so that $|x - n/m| < \delta$ (we used $1/m < \delta$ here). Then $|f(x) - f(n/m)| < \varepsilon/2$ and thus

$$|f(x) - L| \leq \left|f(x) - f\left(\frac{n}{m}\right)\right| + \left|f\left(\frac{n}{m}\right) - L\right| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary we conclude $\lim_{x\to\infty} f(x) = L$.

Second method: Cauchy principle Let $\varepsilon > 0$. Since f is uniformly continuous, there is $\delta > 0$ so that if $x, y \in \mathbb{R}_+$ and $|x - y| < \delta$, then

$$|f(x) - f(y)| < \frac{\varepsilon}{3}.$$

For ε, δ ginven as above, since $\lim_{n\to\infty} f(n\delta)$ exists, we know there exists $N \in \mathbb{N}$, such that $\forall\, m, n > N$, there is

$$|f(m\delta) - f(n\delta)| < \frac{\varepsilon}{3}.$$

Then, let $X = (N + 1)\delta$, we know that $\forall\, x_1, x_2 > X$, there are

$$\left[\frac{x_i}{\delta}\right] > \frac{x_i}{\delta} - 1 > N \quad (i = 1, 2)$$

and

$$\left|x_i - \left[\frac{x_i}{\delta}\right]\delta\right| = \delta\left|\frac{x_i}{\delta} - \left[\frac{x_i}{\delta}\right]\right| < \delta \quad (i = 1, 2).$$

Hence

$$\begin{aligned}|f(x_1) - f(x_2)| \leq & \left|f(x_1) - f\left(\left[\frac{x_1}{\delta}\right]\delta\right)\right| \\ & + \left|f\left(\left[\frac{x_1}{\delta}\right]\delta\right) - f\left(\left[\frac{x_2}{\delta}\right]\delta\right)\right| \\ & + \left|f\left(\left[\frac{x_2}{\delta}\right]\delta\right) - f(x_2)\right| \\ < & \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ = & \varepsilon.\end{aligned}$$

□

Remark 4.19. *We can make a weak assumption that $f(x)$ is just continuous on $[0, +\infty)$, but to prove this conclusion is so difficult, we omit the detail here and leave it to someone who interested.*

Problem 4.20. *Suppose that $f(x)$ is uniformly continuous on $[0, +\infty)$, and $\forall\, x > 0$, $\lim\limits_{n\to\infty} f(x + n) = 0$. Prove that $\lim\limits_{x\to+\infty} f(x) = 0$.*

Proof. By $f(x)$ is uniformly continuous on $[0, +\infty)$, we know $\forall\, \varepsilon > 0$, there is $\delta > 0$, such that $\forall\, x, y \geq 0 : |x - y| < \delta$, there is

$$|f(x) - f(y)| < \frac{\varepsilon}{2}.$$

Take $k > \frac{1}{\delta}$, and cut $[0, 1]$ uniformly into k pieces. Let $x_i = \frac{i}{k}$ $(i = 1, 2, \cdots, k)$ be the cut points. Note that $x_i - x_{i-1} = \frac{1}{k} < \delta$. Since for every x_i, there is $\lim\limits_{n\to\infty} f(x_i + n) = 0$. We know there exists $N_i > 0$, such that $\forall\, n > N_i$, there is $|f(x_i + n)| < \dfrac{\varepsilon}{2}$. Let $N = \max\{N_1, N_2, \cdots, N_k\}$, then $\forall\, n > N$, there is

$$|f(x_i + n)| < \frac{\varepsilon}{2} \quad (i = 1, 2, \cdots, k).$$

Choose $X = N + 1$, for $\forall\, x > X$, there is $[x] > N$. Since $x - [x] \in [0, 1)$, we know there exists $i \in \{1, 2, \cdots, k\}$, such that $|(x - [x]) - x_i| < \delta$, i.e. $|x - (x_i + [x])| < \delta$. Hence, there is

$$\begin{aligned}|f(x)| \leq & |f(x) - f(x_i + [x])| + |f(x_i + [x])| \\ < & \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ = & \varepsilon.\end{aligned}$$

i.e. $\lim_{x\to+\infty} f(x) = 0$. □

Remark 4.21. *(1) We can use the same method to prove that if $f(x)$ is uniformly continuous on $[0,+\infty)$, and $\forall\, x > 0$, $\lim_{n\to\infty} f(x+n) = A$, then $\lim_{x\to+\infty} f(x) = A$.*
(2) From the proof, we can see that the conclusion is still right for a weak assumption, i.e. $\forall\, x \in [0,1]$, $\lim_{n\to\infty} f(x+n) = 0$.
(3) Find a counterexample if we do not assume $f(x)$ is uniformly continuous, but just continuous. Indeed, define

$$f_n(x) = \begin{cases} 2nx, & 0 \le x \le \dfrac{1}{2n}, \\ 2 - 2nx, & \dfrac{1}{2n} < x \le \dfrac{1}{n}, \\ 0, & \textit{otherwise.} \end{cases}$$

Let $f(x) = \sum_{n=1}^{\infty} f_n(x-n)$, then $f(x)$ is a counterexample.

Exercise 4.22 (Leave to readers)**.** *Find x such that $\lim_{m\to\infty} \sqrt{1+\sqrt{x+\sqrt{x^2+\cdots+\sqrt{x^m}}}} = 2$.*

Hint: Let me describe a sketch of proof that $x = 4$.

A. Observe that if $f(x) = \lim_{n\to\infty} \sqrt{1+\sqrt{x+\sqrt{x^2+\cdots\sqrt{x^n}}}}$, then f is strictly increasing.

B. We shall show that $f(4) = 2$, and hence $x = 4$ is the unique answer.

B_1. Fix $m \in \mathbb{N}$ and show that, for $n = m, m-1, m-2, \cdots$ (induction backwards)

$$2^n < \sqrt{4^n + \sqrt{4^{n+1} + \cdots \sqrt{4^{m-1} + \sqrt{4^m}}}} < 2^n + 1,$$

while

$$\sqrt{4^n + \sqrt{4^{n+1} + \cdots \sqrt{4^{m-1} + \sqrt{4^m} + 1}}} = 2^n + 1.$$

B_2. Next estimate the difference

$$\begin{aligned} &(2^n + 1) - \sqrt{4^n + \sqrt{4^{n+1} + \cdots \sqrt{4^{m-1} + \sqrt{4^m}}}} \\ =&\sqrt{4^n + \sqrt{4^{n+1} + \cdots \sqrt{4^{m-1} + \sqrt{4^m} + 1}}} - \sqrt{4^n + \sqrt{4^{n+1} + \cdots \sqrt{4^{m-1} + \sqrt{4^m}}}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\sqrt{4^{n+1}+\cdots\sqrt{4^{m-1}+\sqrt{4^m}+1}}-\sqrt{4^{n+1}+\cdots\sqrt{4^{m-1}+\sqrt{4^m}}}}{\sqrt{4^n+\sqrt{4^{n+1}+\cdots\sqrt{4^{m-1}+\sqrt{4^m}+1}}}+\sqrt{4^n+\sqrt{4^{n+1}+\cdots\sqrt{4^{m-1}+\sqrt{4^m}}}}} \\
&< \frac{\sqrt{4^{n+1}+\cdots\sqrt{4^{m-1}+\sqrt{4^m}+1}}-\sqrt{4^{n+1}+\cdots\sqrt{4^{m-1}+\sqrt{4^m}}}}{2\cdot 2^n} \\
&< \cdots < \frac{(\sqrt{4^m}+1)-\sqrt{4^m}}{2^{m-n}\cdots 2^{n+(n+1)+\cdots+(m-1)}} \\
&= 2^{-\frac{(m-n)(n+m+1)}{2}}.
\end{aligned}
$$

Thus

$$\lim_{m\to\infty}\sqrt{4^n+\sqrt{4^{n+1}+\cdots\sqrt{4^{m-1}+\sqrt{4^m}}}} = 2^n+1.$$

For $n=0$ we have

$$\lim_{m\to\infty}\sqrt{1+\sqrt{4+\cdots\sqrt{4^{m-1}+\sqrt{4^m}}}} = 2^0+1 = 2.$$

□

A similar but easier question: Find x in:

$$\sqrt{x^2+\sqrt{4x^2+\sqrt{16x^2+\sqrt{64x^2+\cdots}}}} = 5.$$

Hint: $x+1=\sqrt{x^2+2x+1}=\sqrt{x^2+\sqrt{4x^2+4x+1}}=\cdots$.

Exercise 4.23 (Challenge!). *Assume $f\in C[0,+\infty)$, and for all $a>0$, we have*

$$\lim_{x\to+\infty}(f(x+a)-f(x))=0.$$

Prove that $f(x)$ is uniformly continuous.

Hint: Fix $\varepsilon>0$, we want to find $\delta>0$ such that

$$|x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon. \tag{4.1}$$

For every $N\in\mathbb{N}$, let $E_N := \{a \mid x\ge N \Rightarrow |f(x+a)-f(x)|\le \varepsilon/4\}$. E_N is closed (by continuity of f) and $\bigcup_{N\in\mathbb{N}} E_N=[0,\infty)$. By Baire Category Theorem, at least one of them, say, E_N contains a closed interval $[b,c]$. For $x,y\ge N+c$, without loss of generality, say $y\ge x$, if $|y-x|<c-b$, there always exists $z\ge N$ such that $[x,y]\subset[z+b,z+c]$. Then $|f(x)-f(y)|\le|f(x)-f(z)|+|f(y)-f(z)|=|f(z+d)-f(z)|+|f(z+e)-f(z)|\le\varepsilon/2$ where $d,e\in[b,c]$. For $x,y\le N+c$, as $[0,N+c]$ is compact, f restricted to $[0,N+c]$ is uniformly continuous, hence there exists $\delta'>0$ satisfing the requirements in (4.1). Let $\delta=\min(c-b,\delta')$, then we are done. □

Problem 5.1. *Suppose that $f(x)$ is uniformly continuous on $\mathbb{R}$. Prove that there exist $a > 0$, $b > 0$ such that $|f(x)| \leq a|x| + b$, $\forall\, x \in \mathbb{R}$.*

Proof. Since $f(x)$ is uniformly continuous on $\mathbb{R}$, we have that $\forall\, \varepsilon > 0$, there exists $\delta > 0$, such that $\forall\, x, y \in \mathbb{R} : |x - y| < \delta$, there is $|f(x) - f(y)| < \varepsilon$. Now, fix ε and δ. For $\forall\, x \in \mathbb{R}$, there exists $n \in \mathbb{Z}$, such that $x = n\delta + x_0$, where $x_0 \in (-\delta, \delta)$. Note that $f(x)$ is bounded on $[-\delta, \delta]$, i.e. $\exists M > 0$, such that $|f(x)| \leq M$ $(\forall\, x \in)[-\delta, \delta]$. Hence,

$$\begin{aligned}
|f(x)| &= \left|\sum_{k=1}^{n} [f(k\delta + x_0) - f((k-1)\delta + x_0)] + f(x_0)\right| \\
&\leq \sum_{k=1}^{n} |f(k\delta + x_0) - f((k-1)\delta + x_0)| + |f(x_0)| \\
&\leq |n|\varepsilon + M \\
&= \frac{\varepsilon}{\delta}|x - x_0| + M \quad \left(\text{since } \left|\frac{x - x_0}{\delta}\right| = |n|\right) \\
&\leq \frac{\varepsilon}{\delta}|x| + \left(M + \frac{\varepsilon}{\delta}|x_0|\right) \\
&\leq \frac{\varepsilon}{\delta}|x| + (M + \varepsilon).
\end{aligned}$$

Denote $a = \varepsilon/\delta$, $b = M + \varepsilon$, hence

$$|f(x)| \leq a|x| + b \quad (\forall\, x \in (-\infty, +\infty)).$$

□

Problem 5.2 (4.24)**.** *Prove that at any point of the curve*

$$\begin{cases} x = a(\cos t + t \sin t), \\ y = a(\sin t - t\cos t), \end{cases} \quad (a > 0),$$

the distance of the normal line to the origin is equal to a.

Proof. Differentiating respect to t yields

$$\begin{cases} dx = at\cos t\, dt, \\ dy = at\sin t\, dt, \end{cases} \quad (a > 0)$$

Then the normal line at point (x, y), $t \neq k\pi$ is

$$Y - a(\sin t - t\cos t) = -\frac{dx}{dy}(X - a(\cos t + t\sin t)),$$

i.e.

$$Y - a(\sin t - t\cos t) = -\cot t(X - a(\cos t + t\sin t)).$$

Hence

$$\begin{aligned} d &= \frac{|a\cot t(\cos t + t\sin t) + a(\sin t - t\cos t)|}{\sqrt{1+\cot^2 t}} \\ &= |a\cos^2 t + at\sin t\cos t + a\sin^2 t - at\sin t\cos t| \\ &= a. \end{aligned}$$

When $t = k\pi$, we know the normal line is $x = a(-1)^k$, thus $d = a$.

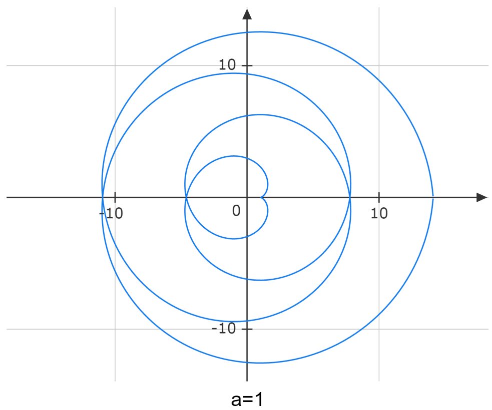

FIGURE 1. Graph of the curve for $a = 1$

□

Problem 5.3. *Calculate the derivative of* $f(x) = x^{\arcsin x}$.

Solution. Note that

$$f(x) = x^{\arcsin x} = e^{\ln x \cdot \arcsin x}.$$

Denote $g(x) = \ln x \cdot \arcsin x$, we have

$$g'(x) = \frac{\arcsin x}{x} + \frac{\ln x}{\sqrt{1-x^2}}.$$

Hence

$$f'(x) = \left(e^{g(x)}\right)' = e^{g(x)}g'(x) = x^{\arcsin x}\left(\frac{\arcsin x}{x} + \frac{\ln x}{\sqrt{1-x^2}}\right).$$

□

Problem 5.4. *Calculate the left right derivative of* $f(x) = \begin{cases} \dfrac{x}{e^{1/x}+1}, & x \neq 0, \\ 0, & x = 0, \end{cases}$ *at* $x = 0$*, and determine whether* $f(x)$ *is differentiable at* $x = 0$*.*

Solution. Left derivative:

$$f'_-(0) = \lim_{x \to 0-0} \frac{\frac{x}{e^{1/x}+1} - 0}{x - 0} = \lim_{x \to 0-0} \frac{1}{e^{1/x}+1} = 1.$$

Right derivative:

$$f'_+(0) = \lim_{x \to 0+0} \frac{\frac{x}{e^{1/x}+1} - 0}{x - 0} = \lim_{x \to 0+0} \frac{1}{e^{1/x}+1} = 0.$$

Hence $f(x)$ is not differentiable at $x = 0$. The graph of $f(x)$ is as follows:

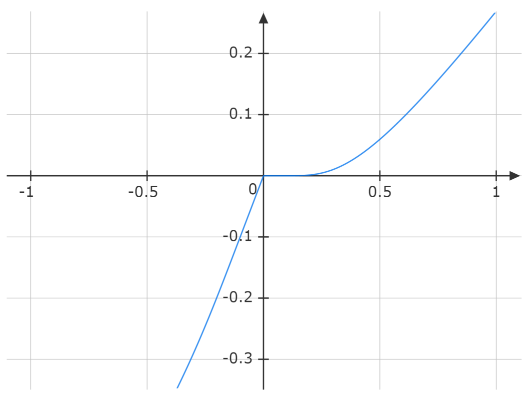

FIGURE 2. Graph of $f(x)$

□

Problem 5.5. *Suppose that* $f(x)$ *is differentiable on* $\mathbb{R}$ *and satisfies* $f(x) \geq x$, $f(x) \geq 1 - x$, $\forall x \in \mathbb{R}$*. Prove that* $f(\frac{1}{2}) > \frac{1}{2}$*.*

Proof. Suppose that $f(\frac{1}{2}) \leq \frac{1}{2}$. Then we have

$$\begin{aligned} f'_-\left(\frac{1}{2}\right) &= \lim_{x \to \frac{1}{2}-0} \frac{f(x) - f(\frac{1}{2})}{x - \frac{1}{2}} \\ &= \lim_{x \to \frac{1}{2}-0} \frac{f(\frac{1}{2}) - f(x)}{\frac{1}{2} - x} \\ &\leq \lim_{x \to \frac{1}{2}-0} \frac{\frac{1}{2} - (1-x)}{\frac{1}{2} - x} \end{aligned}$$

$$= -1,$$

and

$$f'_+\left(\frac{1}{2}\right) = \lim_{x \to \frac{1}{2}+0} \frac{f(x) - f(\frac{1}{2})}{x - \frac{1}{2}}$$
$$\geq \lim_{x \to \frac{1}{2}+0} \frac{x - \frac{1}{2}}{x - \frac{1}{2}}$$
$$= 1,$$

contradicts with $f(x)$ is differentiable at $x = \frac{1}{2}$. Hence $f(\frac{1}{2}) > \frac{1}{2}$. □

Remark 5.6. *From the proof of Problem 5.5, we know that it only needs to assume $f(x)$ is differentiable at $x = \frac{1}{2}$.*

Problem 5.7. *Suppose that $f(x)$ is differentiable. Prove that $F(x) = f(x)(1+|\sin x|)$ is differentiable at $x = 0$ if and only if $f(0) = 0$.*

Proof. Calculating the left derivative of F at $x = 0$ yields

$$F'_-(0) = \lim_{x \to 0-0} \frac{f(x)(1 + |\sin x|) - f(0)}{x}$$
$$= \lim_{x \to 0-0} \frac{f(x) - f(0)}{x} + \lim_{x \to 0-0} \frac{f(0)|\sin x|}{x}$$
$$= f'(0) - f(0).$$

Similarly, the right left derivative of F at $x = 0$ is

$$F'_+(0) = \lim_{x \to 0+0} \frac{f(x)(1 + |\sin x|) - f(0)}{x}$$
$$= \lim_{x \to 0+0} \frac{f(x) - f(0)}{x} + \lim_{x \to 0+0} \frac{f(0)|\sin x|}{x}$$
$$= f'(0) + f(0).$$

Hence $F(x)$ is differentiable at $x = 0$ if and only if $F'_-(0) = F'_+(0)$, if and only if $f'(0) - f(0) = f'(0) + f(0)$, if and only if $f(0) = 0$. □

Problem 5.8. *Suppose that $y = y(x)$ is determined by parametric equation $\begin{cases} x = 2t + |t| \\ y = t^2 + 2t|t| \end{cases}$, $t \in \mathbb{R}$. Prove that $y(x)$ is differentiable at $x = 0$, and find $y'(0)$.*

Proof. **First way:** By $x = 2t + |t|$, we have

$$t = \begin{cases} \dfrac{x}{3}, & x \geq 0, \\ x, & x < 0. \end{cases}$$

Hence by $y = t^2 + 2t|t|$, we know

$$y = \begin{cases} \frac{1}{3}x^2, & x \geq 0, \\ -x^2, & x < 0. \end{cases}$$

It's easy to see that $y(x)$ is differentiable at $x = 0$, and $y'(0) = 0$.

Second way: A direct differentiating yields

$$\begin{cases} dx = \left(2 + \frac{t}{|t|}\right) dt \\ dy = (2t + 4|t|)\, dt \end{cases}$$

Hence

$$\frac{dy}{dx} = \frac{2t + 4|t|}{2 + \frac{t}{|t|}} = 2|t|.$$

It's easy to see that $y(x)$ is differentiable at $x = 0$, and $y'(0) = 0$. □

Problem 5.9. *Suppose that $x^2y^2 + x^2 + y^2 = 1$, $(xy > 0)$. Prove that*

$$\frac{dx}{\sqrt{1-x^4}} + \frac{dy}{\sqrt{1-y^4}}.$$

Proof. A direct differentiating yields

$$x(1+y^2)dx + y(1+x^2)dy = 0.$$

Note by $x^2y^2 + x^2 + y^2 = 1$ that

$$x^2(1+y^2)^2 = 1 - y^4,$$

and

$$y^2(1+x^2)^2 = 1 - x^4.$$

Since $xy > 0$, we know $x(1+y^2)$ and $y(1+x^2)$ have the same sign. Then

$$\sqrt{1-y^4}dx + \sqrt{1-x^4}dy = 0,$$

i.e.

$$\frac{dx}{\sqrt{1-x^4}} + \frac{dy}{\sqrt{1-y^4}}.$$

The graph of the curve is as follows:

□

Problem 5.10. *Prove the following identities:*

(1) $\displaystyle\sum_{k=1}^{n} kC_n^k = n2^{n-1}$, $n \in \mathbb{N}_+$;

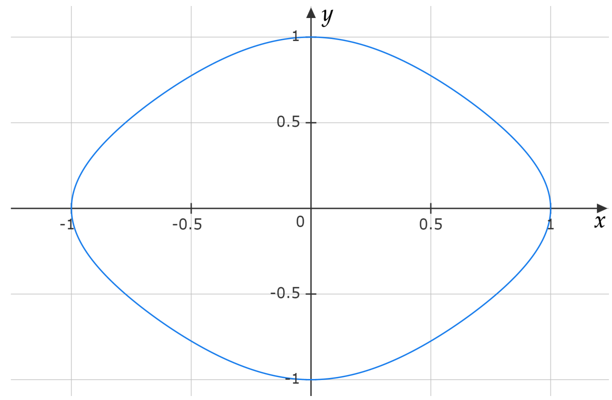

FIGURE 3. Graph of the curve

(2) $\sum_{k=1}^{n} k^2 C_n^k = n(n+1)2^{n-2}$, $n \in \mathbb{N}_+$.

Proof. (1) Note that

$$\sum_{k=1}^{n} kC_n^k x^{k-1} = \left(\sum_{k=1}^{n} C_n^k x^k\right)' = ((1+x)^n - 1)' = n(1+x)^{n-1}.$$

Set $x = 1$, and there is

$$\sum_{k=1}^{n} kC_n^k = n2^{n-1}.$$

(2) **First way:** Note that

$$\begin{aligned}\sum_{k=1}^{n} k^2 C_n^k x^{k-1} &= \left(\sum_{k=1}^{n} kC_n^k x^k\right)' \\ &= \left(x\sum_{k=1}^{n} kC_n^k x^{k-1}\right)' \\ &= \left(nx(1+x)^{n-1}\right)' \\ &= n(1+x)^{n-1} + n(n-1)x(1+x)^{n-2}.\end{aligned}$$

Set $x = 1$, and there is

$$\sum_{k=1}^{n} k^2 C_n^k = n(n+1)2^{n-2}.$$

Second way: Firstly, we have

$$k^2 C_n^k = k^2 \frac{n!}{k!(n-k)!} = nkC_{n-1}^{k-1}.$$

Then

$$\begin{aligned}\sum_{k=1}^{n} k^2 C_n^k x^{k-1} &= n\sum_{k=1}^{n} k C_{n-1}^{k-1} x^{k-1} \\ &= \left(n\sum_{k=1}^{n} C_{n-1}^{k-1} x^k\right)' \\ &= \left(nx(1+x)^{n-1}\right)' \\ &= n(1+x)^{n-1} + n(n-1)x(1+x)^{n-2}.\end{aligned}$$

Set $x = 1$, and there is

$$\sum_{k=1}^{n} k^2 C_n^k = n(n+1)2^{n-2}.$$

□

Problem 5.11. *Suppose that $f : [a, b] \to [a, b]$ satisfies*

$$|f(x) - f(y)| \le |x - y|, \quad \forall\, x, y \in [a, b].$$

Define $x_{n+1} = \frac{1}{2}(x_n + f(x_n))$ for any given $x_1 \in [a, b]$. Prove that $\lim\limits_{n\to\infty} x_n$ exists.

Proof. Note that

$$\begin{aligned}x_{n+1} - x_n &= \frac{1}{2}(x_n + f(x_n)) - \frac{1}{2}(x_{n-1} + f(x_{n-1})) \\ &= \frac{1}{2}(f(x_n) - f(x_{n-1})) + \frac{1}{2}(x_n - x_{n-1}).\end{aligned}$$

Then we have

$$\begin{aligned}(x_{n+1} - x_n)(x_n - x_{n-1}) &= \frac{1}{2}(f(x_n) - f(x_{n-1}))(x_n - x_{n-1}) + \frac{1}{2}(x_n - x_{n-1})^2 \\ &\ge -\frac{1}{2}|x_n - x_{n-1}|^2 + \frac{1}{2}(x_n - x_{n-1})^2 = 0,\end{aligned}$$

since

$$|f(x_n) - f(x_{n-1})||x_n - x_{n-1}| \le |x_n - x_{n-1}|^2.$$

Hence, $\{x_n\}$ is monotonic. Clearly, $\{x_n\}$ is bounded, it's easy to know that $\lim\limits_{n\to\infty} x_n$ exists. □

Problem 5.12. *Suppose that $f(x)$ is differentiable on $[0, 1]$, and $\{x \in [0, 1] \big| f(x) = 0,\ f'(x) = 0\} = \emptyset$. Prove that f has a finite number of zero pionts in $[0, 1]$.*

Proof. Assume that f has an infinite number of zero points. Let $Z := \{x \in [0,1] \big| f(x) = 0\}$. Since Z is a bounded set with an infinite number of elements, we know by the Bolzano-Weierstrass theorem that there is a sequence $\{x_n\} \subset Z$ converges, say $\lim\limits_{n\to\infty} x_n = x_0$. By the continuity of f, we know that $f(x_0) = \lim\limits_{n\to\infty} f(x_n) = 0$. Since $f(x)$ is differentiable, we have

$$f'(x_0) = \lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} = 0,$$

i.e. $x_0 \in \{x \in [0,1] \big| f(x) = 0,\ f'(x) = 0\}$, contradiction. □

Problem 5.13.

(1) *Suppose that $f \in C[0,1]$, $f(0) = f(1)$. Prove that for $0 < \alpha < 1$, if $\frac{1}{\alpha} \in \mathbb{N}$, then there exists $\xi \in [0, 1-\alpha]$ such that $f(\xi) = f(\xi + \alpha)$;*

(2) *Prove that for $0 < \alpha < 1$, $\frac{1}{\alpha} \notin \mathbb{N}$, there always exists $f \in C[0,1]$, $f(0) = f(1)$ such that $\forall\, x \in [0, 1-\alpha]$, there is $f(x) \neq f(x+\alpha)$.*

Proof. (1) Let $g(x) = f(x) - f(x+\alpha)$. Since $\frac{1}{\alpha} \in \mathbb{N}$, we know

$$\sum_{k=0}^{\frac{1}{\alpha}-1} g(k\alpha) = f(0) - f(1) = 0.$$

Then there must be $i, j \in \{0, 1, \cdots, \frac{1}{\alpha} - 1\}$, $i \neq j$, such that $g(i\alpha)g(j\alpha) \leq 0$. Hence there exists $\xi \in [i\alpha, j\alpha] \subset [0, 1-\alpha]$ such that $f(\xi) = f(\xi + \alpha)$.

(2) For $0 < \alpha < 1$, we define

$$f(x) = \sin^2\left(\frac{\pi x}{\alpha}\right) - x \sin^2\left(\frac{\pi}{\alpha}\right).$$

Clearly, f is continuous and $f(0) = 0 = f(1)$. If there is some $x_0 \in [0, 1-\alpha]$, such that $f(x_0) = f(x_0 + \alpha)$, we know there is $\alpha \sin^2\left(\frac{\pi}{\alpha}\right) = 0$. However, since $\frac{1}{\alpha} \notin \mathbb{N}$, we know it's impossible for $\sin^2\left(\frac{\pi}{\alpha}\right) = 0$. Hence, we have that $\forall\, x \in [0, 1-\alpha]$, there is $f(x) \neq f(x+\alpha)$. □

Problem 5.14. *Define $f \in C(\mathbb{R})$ satisfying*

$$f(f(x)) = -x^3 + \sin(x^2 + \ln(1 + |x|)). \tag{5.1}$$

Prove that this equation has no continuous solution.

Proof. Assume by contradiction that there is a continuous function f satisfies (5.1). Then we know that

$$\lim_{x\to+\infty} f(f(x)) = -\infty \quad \text{and} \quad \lim_{x\to-\infty} f(f(x)) = +\infty.$$

We conclude that there must be $\lim_{x\to+\infty} f(x) = -\infty$. Indeed, if there exists a sequence $\{x_n\}$ satisfying $\lim_{n\to\infty} x_n = +\infty$ but $f(x_n)$ is bounded, we know that $f(f(x_n))$ must be bounded since f is continuous, which contradicts with $\lim_{x\to+\infty} f(f(x)) = -\infty$. Hence by the continuity of f, we have that $\lim_{x\to+\infty} f(x) = +\infty$ or $\lim_{x\to+\infty} f(x) = -\infty$. If $\lim_{x\to+\infty} f(x) = +\infty$, then we'll get $\lim_{x\to+\infty} f(f(x)) = +\infty$ which generates contradiction. So we must have $\lim_{x\to+\infty} f(x) = -\infty$. Similarly we must have $\lim_{x\to-\infty} f(x) = +\infty$. But using $\lim_{x\to+\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = +\infty$ we have $\lim_{x\to+\infty} f(f(x)) = +\infty$, which also generates contradiction. Hence, (5.1) has no continuous solution. □

Remark 5.15. *We can prove a general conclusion: For any $f \in C(\mathbb{R})$, $\lim_{x\to+\infty} f(f(x)) = -\infty$ and $\lim_{x\to-\infty} f(f(x)) = +\infty$ cannot be simultaneously true.*

Problem 5.16. *Suppose that $g(x)$ is defined on $[0,1]$, and $g(0) = 1$, $g(1) = 0$. If there exists a continuous function $h(x)$ such that $g(x) + h(x)$ is monotonic increasing on $[0,1]$, prove that $[0,1] \subset g([0,1])$.*

Proof. Denote that $f = g + h$. Since f is monotonic increasing, we have that $f(x-0) \le f(x) \le f(x+0)$, $(0 < x < 1)$. Since h is continuous, we know that $g(x-0) \le g(x) \le g(x+0)$, $(0 < x < 1)$. For $\forall\, y \in (0,1)$, we define

$$E_y = \{t \in [0,1] \big| g(x) > y, \forall\, x \in [0,t]\}.$$

Since $g(0) = 1 > y$, we know that E_y is not empty. Then the supremum of E_y exists. Let $x_0 = \sup E_y$, then $g(x_0 - 0) \ge y \ge g(x_0 + 0)$. Combining above, we obtain that $g(x_0 - 0) = g(x_0) = g(x_0 + 0) = y$, i.e. $y \in g([0,1])$. Hence $(0,1) \subset g([0,1])$. It is clear that $0, 1 \in g([0,1])$, we have that $[0,1] \subset g([0,1])$. □

Exercise 5.17. *Let $f(x)$ be continuous on $\mathbb{R}$. Suppose that f is periodic with the minimal postive period $\mu > 0$, μ is irrational. Show that $\lim_{n\to\infty} f(n)$ does not exist.*

Hint: By Kronecker's Approximation Theorem, we know that the sequence of numbers $\{n\mu - [n\mu]\}$ is dense in the unit interval. Hence we know that for any $x_0 \in [0,1]$, there exists a sequence $\{n_j\mu\}$ such that $n_j\mu - [n_j\mu] \to x_0$, $j \to \infty$. Then

$$f([n_j\mu]) = f([n_j\mu] - n_j\mu) \to f(-x_0), \quad j \to \infty,$$

which means $\{f(n)\}$ does not converge. □

Exercise 5.18. *Suppose that* $x_0 = 1$, $x_n = x_{n-1} + \cos x_{n-1}$, $(n = 1, 2, \cdots)$. *Prove that* $x_n - \frac{\pi}{2} = o\left(\frac{1}{n^n}\right)$ *as* $n \to \infty$.

Hint: Let $y_n = \frac{\pi}{2} - x_n$, $n = 0, 1, 2, \cdots$. Then $y_n = y_{n-1} - \sin y_{n-1}$. By the inequality

$$x - \frac{x^3}{6} < \sin x < x, \quad x \in (0, +\infty),$$

we have

$$0 < y_n = y_{n-1} - \sin y_{n-1} < \frac{y_{n-1}^3}{6} < y_{n-1}^3, \quad n \in \mathbb{N}_+.$$

Hence

$$y_n < y_0^{3^n}.$$

Note that $0 < y_0 < 1$. We konw that there is $N \in \mathbb{N}$, such that $\forall\, n > N$,

$$0 < y_n n^n < y_0^{3^n} n^n < n^n y_0^{n^2} = \left(\frac{n}{\left(\frac{1}{y_0}\right)^n}\right)^n < \frac{1}{2^n}.$$

Hence, we have

$$y_n = o\left(\frac{1}{n^n}\right), \quad n \to \infty.$$

□

Exercise 5.19. *Suppose that* $f \in C[0,1]$, $\lim\limits_{x \to 0+0} \frac{f(x) - f(0)}{x} = \alpha < \beta = \lim\limits_{x \to 1-0} \frac{f(x) - f(1)}{x - 1}$. *Prove that* $\forall\, \lambda \in (\alpha, \beta)$, $\exists\, x_1, x_2 \in [0,1]$, *such that* $\lambda = \frac{f(x_1) - f(x_2)}{x_1 - x_2}$.

Hint: Let $g(x) = f(x) - \lambda x$. Since

$$\lim_{x \to 0+0} \frac{g(x) - g(0)}{x} = \alpha - \lambda < 0,$$

we know there exists $\delta_1 > 0$, such that $0 < x < \delta_1$, there is $g(x) < g(0)$. Similarly, since

$$\lim_{x \to 1-0} \frac{g(x) - g(1)}{x - 1} = \beta - \lambda > 0,$$

we know there exists $\delta_2 > 0$, such that $0 < x < \delta_2$, there is $g(x) < g(1)$. Then we know that the minimum of g is achieved in $(0,1)$. Assume that $g(x_0) = \min\limits_{x \in [0,1]} g(x)$, then $g(x_0) < g(0)$, $g(x_0) < g(1)$. Note that if there exist $x_1, x_2 \in [0,1]$, $x_1 \neq x_2$, such that $g(x_1) = g(x_2)$, then we have $f(x_1) - \lambda x_1 = f(x_2) - \lambda x_2$, i.e.

$$\frac{f(x_1) - f(x_2)}{x_1 - x_2} = \lambda.$$

Indeed, if $g(0) = g(1)$, we are done. Next, we assume that $g(0) \neq g(1)$. Without loss of generality, we assume that $g(0) < g(1)$, then there is $g(x_0) < g(0) < g(1)$. Since $g(x)$ is continuous on $[x_0, 1]$, we know there exists $\xi \in (x_0, 1)$ such that $g(\xi) = g(0)$, then we are done. $\square$

Exercise 5.20. *Let a, b be two nonzero real numbers and a function $f : \mathbb{R} \longrightarrow [0, \infty)$ satisfying the functional equation*

$$f(x + a + b) + f(x) = f(x + a) + f(x + b). \tag{5.2}$$

(1) *Prove that f is periodic if a/b is rational.*
(2) *If a/b is not rational, could f be nonperiodic?*

Hint: (1) From (5.2), easy induction gives

$$f(x + a + nb) - f(x + nb) = f(x + a) - f(x), \quad \forall\, n \in \mathbb{Z}.$$

From $f(x + a + nb) - f(x + a) = f(x + nb) - f(x)$, easy induction gives

$$f(x + ma + nb) - f(x + ma) = f(x + nb) - f(x), \quad \forall\, m \in \mathbb{Z}.$$

So

$$f(x + ma + nb) + f(x) = f(x + ma) + f(x + nb), \quad \forall\, x \in \mathbb{R} \text{ and } \forall\, m, n \in \mathbb{Z}. \tag{5.3}$$

If $\frac{a}{b}$ is rational, we can choose m, n such that $ma + nb = 0$. Let then $u = |ma| > 0$ and (5.3) becomes

$$2f(x) = f(x + u) + f(x - u),$$

or also

$$f(x + u) - f(x) = f(x) - f(x - u).$$

From there, we easily get

$$f(x + nu) = f(x) + n(f(x + u) - f(x)), \quad \forall\, x \in \mathbb{R},\ \forall\, n \in \mathbb{Z}. \tag{5.4}$$

Then, if $f(x + u) - f(x) \neq 0$, setting $n \to +\infty$ or $n \to -\infty$ (depending on sign), (5.4) implies $f(x + nu) < 0$, contradicts with $f(x) \geq 0$. Hence

$$f(x + u) = f(x), \quad \forall\, x \in \mathbb{R} \text{ and for some } u > 0.$$

(2) Define

$$f(x) = \frac{1}{2}\left(\cos\frac{2\pi x}{a} + \cos\frac{2\pi x}{b}\right) + 1.$$

It's easy to verify that f satisfies (5.3). But f is periodic if and only if a/b is rational. This can be seen by assuming $f(T) = f(0) = 2$, which implies both $\cos\frac{2\pi T}{a}$ and $\cos\frac{2\pi T}{b}$ have to be 1, i.e. T/a and T/b have to be integers. $\square$

Remark 5.21. *From (5.4) in the proof of Exercise 5.20, we can see that f is only needed to be bounded above or below.*

Exercise 5.22. *Suppose that $f(x)$ is a uniformly continuous function on $[1, +\infty)$. Prove that* $\overline{\lim}_{x \to +\infty} \frac{f(x)}{x} < +\infty$.

Hint: By Problem 5.1, we know that there exist $a, b > 0$ such that $|f(x)| \le ax + b$. Hence

$$\overline{\lim_{x \to +\infty}} \frac{f(x)}{x} \le \overline{\lim_{x \to +\infty}} \frac{|f(x)|}{x} \le \overline{\lim_{x \to +\infty}} \left(a + \frac{b}{x} \right) = a < +\infty.$$

□

Problem 6.1 (Mid 1). *Calculate limitations.*

(1) $\lim_{n\to\infty}[\sin(\ln(n+1)) - \sin(\ln n)]$;

(2) $\lim_{x\to 0}\frac{\sqrt[3]{1+x\sin x}-1}{\arctan x^2}$;

(3) $\lim_{x\to 0}(1+2x)^{\frac{(x+1)^2}{x}}$;

(4) $\lim_{n\to\infty}[(n+\ln n)^a - n^a]$, *where* $0 < a < 1$;

(5) $\lim_{x\to+\infty}\left(\frac{2^{1/x}+8^{1/x}}{2}\right)^x$.

Solution. (1)

$$\lim_{n\to\infty}|\sin(\ln(n+1)) - \sin(\ln n)| = \lim_{n\to\infty}\left|2\cos\left(\frac{\ln(n+1)+\ln n}{2}\right)\sin\left(\frac{\ln(n+1)-\ln n}{2}\right)\right|$$
$$\le \lim_{n\to\infty}(\ln(n+1)-\ln n) = 0.$$

(2)

$$\lim_{x\to 0}\frac{\sqrt[3]{1+x\sin x}-1}{\arctan x^2} = \lim_{x\to 0}\frac{1+\frac{1}{3}x\sin x - 1}{x^2} = \frac{1}{3}.$$

(3)

$$\lim_{x\to 0}(1+2x)^{\frac{(x+1)^2}{x}} = \lim_{x\to 0}(1+2x)^{x+\frac{1}{x}+2} = e^2.$$

(4)

$$\lim_{n\to\infty}[(n+\ln n)^a - n^a] = \lim_{n\to\infty}n^a\left[\left(1+\frac{\ln n}{n}\right)^a - 1\right]$$
$$= \lim_{n\to\infty}\frac{a\ln n}{n^{1-a}} = 0.$$

(5) 4. (Problem 4.7 (7)). □

Problem 6.2 (Mid 2). *Discuss the continuity of the following functions.*

(1) $f(x) = [|\cos x|]$;

(2) $f(x) = \frac{1}{1-e^{\frac{x}{1-x}}}$.

Solution. (1) It is easy to see that

$$0 \leq |\cos x| < 1, \quad \forall\, x \in (n\pi, (n+1)\pi),\, n \in \mathbb{Z}.,$$

and

$$|\cos x| = 1, \quad x = n\pi,\, n \in \mathbb{Z}.$$

Then f is discontinuous at $x = n\pi$, $\forall\, n \in \mathbb{Z}$, and it's the removable discontinuity.

(2) Note that

$$\lim_{x \to 1-0} \frac{1}{1 - e^{\frac{x}{1-x}}} = 0, \quad \lim_{x \to 1+0} \frac{1}{1 - e^{\frac{x}{1-x}}} = 1,$$

and

$$\lim_{x \to 0-0} \frac{1}{1 - e^{\frac{x}{1-x}}} = +\infty, \quad \lim_{x \to 0+0} \frac{1}{1 - e^{\frac{x}{1-x}}} = -\infty,$$

we know that $x = 1$ is the jump discontinuity and $x = 0$ is the discontinuity of second kind. The graph of $f(x)$ is as follows: □

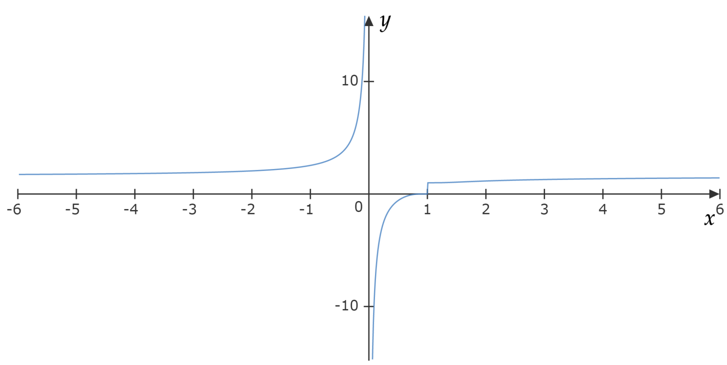

FIGURE 4. Graph of $f(x)$

Problem 6.3 (Mid 3)**.** *Find $n \in \mathbb{N}_+$, such that when $x \to 0$, $e^{x^n} - 1$ is a infinitesimals whose order is lower than $x(\cos\sqrt{x} - 1)(\sqrt[3]{x+1} - 1)$ but higher than $\sqrt{x}\ln(1 + \sqrt[3]{x})$.*

Solution. Note that when $x \to 0$, there are

$$x(\cos\sqrt{x} - 1)(\sqrt[3]{x+1} - 1) \sim -\frac{1}{6}x^3,$$

and

$$\sqrt{x}\ln(1 + \sqrt[3]{x}) \sim x^{\frac{5}{6}}.$$

Hence $n = 1, 2$. □

Problem 6.4 (Mid 5). *Suppose that $f(x)$ satisfying*

$$\lim_{x\to 0}\left(1+x+\frac{f(x)}{x}\right)^{\frac{1}{x}}=e^3.$$

Prove that $\lim\limits_{x\to 0}\frac{f(x)}{x^2}$ *exists, and find the limitation.*

Proof. By $\lim\limits_{x\to 0}\left(1+x+\frac{f(x)}{x}\right)^{\frac{1}{x}}=e^3$, we know that there must be $\lim\limits_{x\to 0}\left(x+\frac{f(x)}{x}\right)=0$. What's more, we have

$$\lim_{x\to 0}\frac{1}{x}\ln\left(1+x+\frac{f(x)}{x}\right)=3.$$

Hence

$$\begin{aligned}\lim_{x\to 0}\frac{f(x)}{x^2}&=\lim_{x\to 0}\frac{1}{x}\left(x+\frac{f(x)}{x}\right)-1\\&=\lim_{x\to 0}\frac{1}{x}\ln\left(1+x+\frac{f(x)}{x}\right)\cdot\lim_{x\to 0}\frac{x+\frac{f(x)}{x}}{\ln\left(1+x+\frac{f(x)}{x}\right)}-1\\&=3-1=2.\end{aligned}$$

□

Problem 6.5 (Mid 6). *Suppose sequence $\{x_n\}$ satisfies $x_0=0$, $x_{2k}=\frac{x_{2k-1}}{2}$ and $x_{2k+1}=x_{2k}+\frac{1}{2}$. Find the upper and lower limit of $\{x_n\}$.*

Solution. Firstly, we have

$$x_{2k+1}=x_{2k}+\frac{1}{2}=\frac{x_{2k-1}}{2}+\frac{1}{2},$$

which gives us

$$x_{2k+1}=\frac{2^{k+1}-1}{2^{k+1}}\quad\text{and}\quad x_{2k}=\frac{2^k-1}{2^{k+1}}.$$

Hence, we know

$$\varliminf_{n\to\infty}x_n=\frac{1}{2},\quad\varlimsup_{n\to\infty}x_n=1.$$

□

Problem 6.6 (Mid 7). *Soppose that $x_0=1$, $x_{n+1}=\frac{1}{x_n^3+4}$. Prove that $\{x_n\}$ converges to the unique positive zero point of equation $x^4+4x-1=0$.*

Proof. Let

$$g(x) = x^4 + 4x - 1.$$

It's easy to see that $g(x)$ is increasing on $[0, +\infty)$. Since $g(0) = -1 < 0$, $g(1) = 4 > 0$, we know that $g(x)$ has only one zero point in $[0, +\infty)$, say α. In particular, $0 < \alpha < 1$. Since $g(\alpha) = 0$, we have that $\alpha = \dfrac{1}{\alpha^3 + 4}$. By induction, we know that $0 < x_n < 1$, $n \geq 1$. Hence, there is

$$\begin{aligned}
|x_{n+1} - \alpha| &= \left| \frac{1}{x_n^3 + 4} - \frac{1}{\alpha^3 + 4} \right| \\
&= \frac{|x_n^3 - \alpha^3|}{(x_n^3 + 4)(\alpha^3 + 4)} \\
&= \frac{|x_n - \alpha||x_n^2 + \alpha x_n + \alpha^2|}{(x_n^3 + 4)(\alpha^3 + 4)} \\
&\leq \frac{3}{16}|x_n - \alpha| \leq \cdots \\
&\leq \left(\frac{3}{16}\right)^n |x_1 - \alpha| \\
&\to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

□

Problem 6.7 (Mid 10)**.** *Suppose that $f(x)$ is Lipschitz on $[1, +\infty)$, i.e. there exists a constant $C > 0$, such that $\forall\, x, y \in [1, +\infty)$, there is $|f(x) - f(y)| \leq C|x - y|$. Prove that $\dfrac{f(x)}{x}$ is uniformly continuous on $[1, +\infty)$.*

Proof. **First way:** By Exercise 5.22, we know that $\dfrac{f(x)}{x}$ is bounded on $[1, +\infty)$, i.e. there exists $M > 0$, such that $\left|\dfrac{f(x)}{x}\right| \leq M$. Note that $\forall\, x, y \in [1, +\infty)$, there is

$$\begin{aligned}
\left| \frac{f(x)}{x} - \frac{f(y)}{y} \right| &= \left| \frac{f(x)}{x} - \frac{f(y)}{x} + \frac{f(y)}{x} - \frac{f(y)}{y} \right| \\
&\leq \frac{|f(x) - f(y)|}{x} + \frac{1}{x}\left|\frac{f(y)}{y}\right| |x - y| \\
&\leq (C + M)|x - y|.
\end{aligned}$$

i.e. $\dfrac{f(x)}{x}$ is also Lipschitz. Thus, it is uniformly continuous.

Second way: We will show $\frac{f(x)}{x}$ is Lipschitz, directly. Indeed,

$$\begin{aligned}\left|\frac{f(x)}{x}-\frac{f(y)}{y}\right| &= \left|\frac{f(x)}{x}-\frac{f(y)}{x}+\frac{f(y)}{x}-\frac{f(y)}{y}\right| \\ &\leq \frac{|f(x)-f(y)|}{x}+\frac{|f(y)|}{xy}|x-y| \\ &\leq \frac{|f(x)-f(y)|}{x}+\frac{|f(y)-f(1)|+|f(1)|}{xy}|x-y| \\ &\leq \frac{C|x-y|}{x}+\frac{C(y-1)+|f(1)|}{xy}|x-y| \\ &\leq (2C+|f(1)|)|x-y|.\end{aligned}$$

$\square$

Problem 7.1 (4.36)**.**

(4) $y = \sin^3 x$;

(6) $y = \dfrac{x^n}{1-x}$;

(8) $\dfrac{\ln x}{x}$.

Solution. (4) Note that

$$\begin{aligned}\sin^3 x &= \sin x(1-\cos^x) = \sin x - \sin x \cos^2 x \\ &= \frac{1}{2}\sin x - \frac{1}{2}\sin x \cos 2x \\ &= \frac{1}{2}\sin x - \frac{1}{4}\sin 3x + \frac{1}{4}\sin x \\ &= \frac{3}{4}\sin x - \frac{1}{4}\sin 3x.\end{aligned}$$

Hence

$$y^{(n)} = \frac{3}{4}\sin\left(x + \frac{n\pi}{2}\right) - \frac{3^n}{4}\sin\left(3x + \frac{n\pi}{2}\right).$$

(6)**First way:** Note that

$$x^n - 1 = (x-1)(x^{n-1} + x^{n-1} + \cdots + 1).$$

Second way: Note that

$$x^n = (x-1+1)^n = 1 + \sum_{k=1}^{n} C_n^k (x-1)^k.$$

(8) By the Leibniz formula, we have

$$\begin{aligned}y^{(n)} &= \left(\frac{\ln x}{x}\right)^{(n)} \\ &= \ln x \left(\frac{1}{x}\right)^{(n)} + \sum_{k=1}^{n} C_n^k (\ln x)^{(k)} \left(\frac{1}{x}\right)^{(n-k)} \\ &= \frac{(-1)^n n!}{x^{n+1}} \ln x + \sum_{k=1}^{n} \left(\frac{n!}{k!(n-k)!}\frac{(-1)^{k-1}(k-1)!}{x^k}\frac{(-1)^{n-k}(n-k)!}{x^{n-k+1}}\right) \\ &= \frac{(-1)^n n!}{x^{n+1}}\left(\ln x - \sum_{k=1}^{n}\frac{1}{k}\right).\end{aligned}$$

□

Problem 7.2 (4.41(3))**.** *If* $y = x^{n-1}e^{\frac{1}{x}}$*, then* $y^{(n)} = \dfrac{(-1)^n}{x^{n+1}}e^{\frac{1}{x}}$.

Proof. Prove by induction. Let $y_n := x^{n-1}e^{\frac{1}{x}}$. Then $y_1 = e^{\frac{1}{x}}$, we have

$$y_1' = \frac{-1}{x^2}e^{\frac{1}{x}}.$$

Suppose that $y_n^{(n)} = \dfrac{(-1)^n}{x^{n+1}}e^{\frac{1}{x}}$, we calculate $y_{n+1}^{(n+1)}$. By the Leibniz formula, we have

$$\begin{aligned}
y_{n+1}^{(n+1)} &= (y_{n+1}^{(n)})' = ((xy_n)^{(n)}) \\
&= (xy_n^{(n)} + ny_n^{(n-1)})' \\
&= xy_n^{(n+1)} + (n+1)y_n^{(n)} \\
&= \frac{(-1)^{n+1}}{x^{n+2}}e^{\frac{1}{x}} + \frac{(-1)^{n+1}(n+1)}{x^{n+1}}e^{\frac{1}{x}} + \frac{(-1)^n(n+1)}{x^{n+1}}e^{\frac{1}{x}} \\
&= \frac{(-1)^{n+1}}{x^{n+2}}e^{\frac{1}{x}}.
\end{aligned}$$

□

Problem 7.3 (4.44)**.** *Suppose that* $f(x)$ *is continuous at* $x = 0$*, and satisfying*

$$\lim_{x\to 0}\frac{f(2x) - f(x)}{x} = m.$$

Prove that $f'(0) = m$.

Proof. The proof is very similar to that of Problem 4.11, we omit the detail here. □

Problem 7.4. *Suppose that* $f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0, \\ 0, & x = 0. \end{cases}$ *Prove that* $f^{(n)}(0) = 0$, $\forall\, n \in \mathbb{N}_+$.

Proof. We firstly calculate $f'(0)$. By definition, there is

$$f'(0) = \lim_{x\to 0}\frac{1}{x}\cdot e^{-\frac{1}{x^2}} = \lim_{y\to\infty} ye^{-y^2} = 0.$$

Next, a direct calculation yields the formula of $f'(x)$ when $x \neq 0$

$$f'(x) = \frac{2}{x^3}e^{-\frac{1}{x^2}}.$$

By definition, again, we have

$$f''(0) = \lim_{x \to 0} \frac{f'(x) - f'(0)}{x} = \lim_{x \to 0} \frac{2}{x^4} \cdot e^{-\frac{1}{x^2}} = 0.$$

Then, a direct calculation yields the formula of $f''(x)$ when $x \neq 0$

$$f''(x) = \left(\frac{4}{x^6} - \frac{6}{x^4} \right) e^{-\frac{1}{x^2}}.$$

Hence, we claim that the nth-order derivative of f when $x \neq 0$ is

$$f^{(n)}(x) = P_n \left(\frac{1}{x} \right) \cdot e^{-\frac{1}{x^2}}, \tag{7.1}$$

where $P_n(y)$ is a polynomial of degree n. We prove this claim by induction. We already know that (7.1) is true for $n = 1, 2$. We suppose that (7.1) is valid for $f^{(k)}(x)$, we calculate $f^{(k+1)}(x)$ in the following.

$$\begin{aligned} f^{(k+1)}(x) &= (f^{(k)}(x))' \\ &= \left[P_k \left(\frac{1}{x} \right) \cdot e^{-\frac{1}{x^2}} \right]' \\ &= P_k' \left(\frac{1}{x} \right) \left(-\frac{1}{x^2} \right) \cdot e^{-\frac{1}{x^2}} + P_k \left(\frac{1}{x} \right) \cdot \frac{2}{x^3} \cdot e^{-\frac{1}{x^2}} \\ &= \left[P_k\,(y)\,(-y^2) + P_k(y)(2y^3) \right] \Big|_{y=\frac{1}{x}} \cdot e^{-\frac{1}{x^2}}. \end{aligned}$$

Denote $P_{k+1} \left(\frac{1}{x} \right) := \left[P_k\,(y)\,(-y^2) + P_k(y)(2y^3) \right] \Big|_{y=\frac{1}{x}}$, then we obtain (7.1).

Finally, we prove $f^{(n)}(0) = 0$ by induction. It is true for $n = 1, 2$. We suppose that there is $f^{(k)}(0) = 0$, we prove that $f^{(k+1)}(0) = 0$. Indeed,

$$\begin{aligned} f^{(k+1)}(0) &= \lim_{x \to 0} \frac{f^{(k)}(x) - f^{(k)}(0)}{x} = \lim_{x \to 0} \frac{f^{(k)}(x)}{x} \\ &= \lim_{x \to 0} \frac{1}{x} \cdot P_k \left(\frac{1}{x} \right) \cdot e^{-\frac{1}{x^2}} = 0. \end{aligned}$$

The graph of $f(x)$ is as follows:

□

Exercise 7.5. *Prove that there exists a smooth function* $f : \mathbb{R} \longrightarrow [0, 1]$ *such that* $f\big|_{(-\infty,0]} = 0$ *and* $f\big|_{[1,+\infty)} = 1$.

Hint: Firstly, define

$$g(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

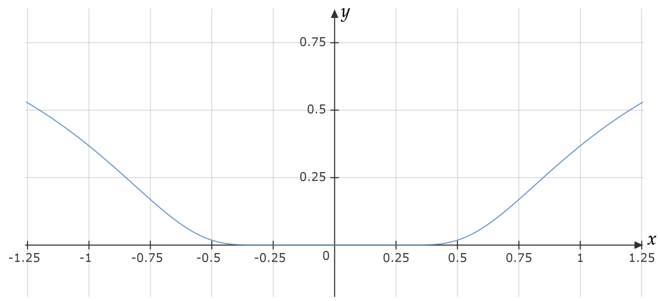

FIGURE 5. Graph of $f(x)$

Next, let

$$f(x) = \frac{g(x)}{g(x) + g(1-x)}.$$

It's easy to verify that f satisfies the condition. $\square$

Problem 7.6. *Suppose that f is a polynomial of degree 7. If $f(x)+1$ is divisible by $(x-1)^4$ and $f(x)-1$ is divisible by $(x+1)^4$. Find f by the method of derivatives.*

Proof. Note that

$$f(x) + 1 = p(x)(x-1)^4 \Rightarrow f'(x) = p'(x)(x-1)^4 + 4p(x)(x-1)^3 \Rightarrow f'(1) = 0,$$
$$f(x) - 1 = q(x)(x+1)^4 \Rightarrow f'(x) = q'(x)(x+1)^4 + 4q(x)(x+1)^3 \Rightarrow f'(-1) = 0.$$

Hence we have

$$f'(x) = a(x-1)^3(x+1)^3.$$

Then

$$f(x) = \frac{a}{7}x^7 - \frac{3a}{5}x^5 + ax^3 - ax + b.$$

Since $f(1) = -1, f(-1) = 1$, we have that

$$-\frac{16}{35}a + b = -1, \quad \frac{16}{35}a + b = 1,$$

i.e.

$$a = \frac{35}{16}, \quad b = 0.$$

Hence

$$f(x) = \frac{1}{16}x\left(5x^6 - 21x^4 + 35x^2 - 35\right).$$

$\square$

Problem 7.7. *Suppose that $f(x) = ax^2 + bx + c$, and $|f(x)| \leq 1$, when $|x| \leq 1$. Prove that $|f'(x)| \leq 4$, when $|x| \leq 1$.*

Proof. By $|f(-1)| = |a - b + c| \leq 1$, $|f(0)| = |c| \leq 1$, $|f(1)| = |a + b + c| \leq 1$, there is

$$\begin{aligned} |2a + b| &= \left| \frac{1}{2}(a - b + c) - 2c + \frac{3}{2}(a + b + c) \right| \\ &\leq \frac{1}{2}|a - b + c| + 2|c| + \frac{3}{2}|a + b + c| \\ &\leq 4. \end{aligned}$$

Similarly, we have

$$| - 2a + b| = \left| -\frac{3}{2}(a - b + c) + 2c - \frac{1}{2}(a + b + c) \right| \leq 4.$$

Since the maximum of linear functions is achieved at endpoints, we know that

$$|f'(x)| = |2ax + b| \leq \max\{|2a + b|, | - 2a + b|\} \leq 4, \quad \forall x \in [-1, 1].$$

□

Problem 7.8. *Prove that for every $n \in \mathbb{N}_+$ there is*

$$\sum_{k=0}^{n} (-1)^k C_n^k k^m = \begin{cases} 0, & 0 \leq m \leq n - 1, \\ (-1)^n n!, & m = n. \end{cases}$$

Proof. Let $S_n^m = \sum_{k=0}^{n} (-1)^k C_n^k k^m$. We show that $S_n^m = 0$ if $0 \leq m \leq n - 1$. Firstly, we have

$$S_n^0 = \sum_{k=0}^{n} (-1)^k C_n^k = (1 - 1)^n = 0.$$

In particular, $S_1^0 = 0$. By

$$kC_n^k = k\frac{n!}{k!(n - k)!} = nC_{n-1}^{k-1},$$

we know that when $1 \leq m \leq n - 1$, there is

$$\begin{aligned} S_n^m &= n \sum_{k=1}^{n} (-1)^k C_{n-1}^{k-1} k^{m-1} \\ &= -n \sum_{k=0}^{n-1} (-1)^k C_{n-1}^{k} (k + 1)^{m-1} \end{aligned}$$

(7.2) $$= -n \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} (-1)^k C_{n-1}^k C_{m-1}^l k^l$$
$$= -n \sum_{l=0}^{m-1} \sum_{k=0}^{n-1} (-1)^k C_{n-1}^k C_{m-1}^l k^l$$
$$= -n \sum_{l=0}^{m-1} C_{m-1}^l S_{n-1}^l.$$

Hence, by induction, we have that $S_n^m = 0$ for $0 \leq m \leq n-1$. What's more, by (7.2), we have

$$S_n^n = -n \sum_{l=0}^{n-1} C_{n-1}^l S_{n-1}^l = -nS_{n-1}^{n-1} = (-1)^n n!.$$

Then the result follows. □

Exercise 7.9. *Given a positive integer n. Find*

$$S = \sum_{k=0}^{n} (-1)^k \binom{n}{k} k^{n+2},$$

and

$$T = \sum_{k=0}^{n} (-1)^k \binom{n}{k} k^{n+3}.$$

Hint: Using (7.2) and induction, we have

$$S = \frac{(-1)^n n(3n+1)(n+2)!}{24},$$

and

$$T = \frac{(-1)^n n^2(n+1)(n+3)!}{48}.$$

□

Problem 7.10. *Suppose that $f(x) = x^n \ln x$, $n \in \mathbb{N}_+$. Calculate $\lim_{n\to\infty} \frac{f^{(n)}(1/n)}{n!}$.*

Proof. Denote that $f_n(x) = x^n \ln x$. Then

$$f_n'(x) = nx^{n-1} \ln x + x^{n-1} = nf_{n-1}(x) + x^{n-1}.$$

Hence, we have

$$f_n^{(n)}(x) = nf_{n-1}^{(n-1)}(x) + (n-1)!,$$

which gives us

$$\frac{f_n^{(n)}(x)}{n!} = \frac{f_{n-1}^{(n-1)}(x)}{(n-1)!} + \frac{1}{n} = \cdots = \ln x + 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Then take $x = \frac{1}{n}$, ther is

$$\lim_{n\to\infty} \frac{f_n^{(n)}(1/n)}{n!} = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n\right) = c,$$

where c is the Euler constant. □

Problem 7.11 (5.1)**.** *Prove the generalized Rolle's theorem, i.e. suppose that $f(x)$ is differentiable on (a,b), and $f(a+0) = f(b-0) = A$. Then there exists $\xi \in (a,b)$, such that $f'(\xi) = 0$, where a can be $-\infty$, b can be $+\infty$, A can be $+\infty$ or $-\infty$.*

Proof. We only prove the case that a, b and A are finite, others are similar and we leave them to the reader. The conclusion is clear if $f(x) = A$, $\forall\, x \in (a,b)$. Hence, without loss of generality, we may assume that there is at least a $x_0 \in (a,b)$ such that $f(x_0) > A$. By the definition of limits, we have there is a small $\delta > 0$ such that

$$f(x) < f(x_0), \quad \forall\, x \in (a, a+\delta) \cup (b-\delta, b).$$

Hence, we know that the maximum of $f(x)$ is achieved on $[a+\delta, b-\delta]$, thus there exists $\xi \in (a,b)$, such that $f'(\xi) = 0$. □

Problem 7.12 (5.5)**.** *Prove that the Chebyshev-Laguerre polynomial*

$$L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x})$$

has n different zero points.

Proof. By Leibniz formula, it's easy to see that $L_n(x)$ is a polynomial of degree n. Hence, it has at most n zero points. The conclusion is clear for $n = 0, 1$, we show it's true for $n \geq 2$ in the following. Denote $g(x) = x^n e^{-x}$, then $L_n(x) = e^x g^{(n)}(x)$. It suffices to find all zero points of $g^{(n)}(x)$. Note that

$$g^{(l)}(x) = \sum_{k=0}^{l} C_l^k (x^n)^{(k)} (e^{-x})^{(l-k)} = \sum_{k=0}^{l} C_l^k n(n-1)\cdots(n-k+1) x^{n-k} (-1)^{l-k} e^{-x}.$$

Hence for $l < n$, there are always $g^{(l)}(0) = 0$ and $\lim_{x\to+\infty} g^{(l)}(x) = 0$. Hence, by Rolle's theorem (Problem 7.11) and induction, we know that there is at least $n-1$ zero points of $g^{(l)}(x)$ between $(0, +\infty)$.By Rolle's theorem again, we have that $g^{(n)}(x)$ has at least n zero points in $(0, +\infty)$. Therefore, we know that $L_n(x)$ has n different zero points. □

Problem 7.13. *Suppose that f is continuous on $[x_1, x_2]$, differentiable on (x_1, x_2). Show that there exists $\xi \in (x_1, x_2)$, such that* $\frac{1}{x_1 - x_2}\begin{vmatrix} x_1 & x_2 \\ f(x_1) & f(x_2) \end{vmatrix} = f(\xi) - \xi f'(\xi)$.

Proof. Note that

$$\frac{1}{x_1 - x_2}\begin{vmatrix} x_1 & x_2 \\ f(x_1) & f(x_2) \end{vmatrix} = \frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = \frac{\frac{f(x_2)}{x_2} - \frac{f(x_1)}{x_1}}{\frac{1}{x_2} - \frac{1}{x_1}}.$$

By the Cauchy mean value theorem, we know that there exists $\xi \in (x_1, x_2)$, such that

$$\frac{\frac{f(x_2)}{x_2} - \frac{f(x_1)}{x_1}}{\frac{1}{x_2} - \frac{1}{x_1}} = \frac{\frac{\xi f'(\xi) - f()\xi}{\xi^2}}{-\frac{1}{\xi^2}} = f(\xi) - \xi f'(\xi).$$

Then the result follows. □

Problem 7.14. *Suppose that $f(x)$ is differentiable on (a, b), $b < +\infty$, and $\lim_{x \to b-0} f(x) = +\infty$. Prove that $\overline{\lim_{x \to b-0}} f'(x) = +\infty$.*

Proof. Prove by contradiction. Assume that $\overline{\lim_{x \to b-0}} f'(x) < +\infty$. Then there exist $M \in \mathbb{R}$, $\delta > 0$, such that $\forall\, x \in (b - \delta, b)$, there is $f'(x) \le M$. Hence by the Lagrange mean value theorem, we know that $\forall\, x, y \in (b - \delta, b)$, $x > y$, there exist $\xi \in (b - \delta, b)$ such that

$$f(x) - f(y) = f'(\xi)(x - y) \le M(x - y).$$

Let $x \to b - 0$, and by $\lim_{x \to b-0} f(x) = +\infty$, we have that

$$+\infty \le M(b - y),$$

contradiction. □

Problem 7.15. *Suppose that $f(x)$ is continuous on $[a, b]$, differentiable on (a, b), and f is not a linear function. Prove that there exists $\xi \in (a, b)$, such that* $f'(\xi) > \frac{f(b) - f(a)}{b - a}$.

Proof. Prove by contradiction. Assume that $f'(x) \le \frac{f(b) - f(a)}{b - a}$, $\forall\, x \in (a, b)$. Define

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Note that $F(a) = F(b) = 0$, and

$$F'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} \le 0.$$

Hence

$$0 = F(a) \geq F(x) \geq F(b) = 0, \quad \forall\, x \in (a,b),$$

i.e. $F(x) = 0$, $\forall\, x \in (a,b)$. Therefore, we have

$$f(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a), \quad \forall\, x \in (a,b),$$

which implies that f is a linear function, contradiction. □

Exercise 7.16. *Suppose that $f(x)$ is continuous on $[0,1]$, differentiable on $(0,1)$, and f is not a constant function. If $f(0) = 0$, prove that there exists $\xi \in (0,1)$ such that $f(\xi)f'(\xi) > 0$.*

Hint: Consider the function $F(x) := f^2(x)$. □

Problem 7.17. *Suppose that $f(x)$ is differentiable on $[0,1]$, $f(0) = 0$, $f(1) = 1$, $k_1, \cdots, k_n$ are positive numbers. Prove that there are $x_1, \cdots, x_n \in [0,1]$, $x_i \neq x_j$, such that*

$$\sum_{i=1}^{n} \frac{k_i}{f'(x_i)} = \sum_{i=1}^{n} k_i.$$

Proof. Denote that $m = \sum_{i=1}^{n} k_i$, $\lambda_i = \frac{k_i}{m}$. Then $0 < \lambda_i < 1$, $\lambda_1 + \cdots + \lambda_n = 1$. Since $f(0) = 0$, $f(1) = 1$ and $f(x)$ is continuous on $[0,1]$, we know that there exists $c_1 \in (0,1)$ such that $f(c_1) = \lambda_1$. Again, we know that there exists $c_2 \in (c_1, 1)$ such that $f(c_2) = \lambda_1 + \lambda_2$ since $\lambda_1 < \lambda_1 + \lambda_2 < 1$. Proceeding like this, we can find

$$0 < c_1 < c_2 < \cdots < c_n = 1,$$

such that

$$f(c_i) = \sum_{k=1}^{i} \lambda_k \quad (i = 1, 2, \cdots, n).$$

By the Lagrange mean value theorem, we have $x_i \in (c_{i-1}, c_i)$ $(c_0 = 0)$, such that

$$f'(x_i) = \frac{f(c_i) - f(c_{i-1})}{c_i - c_{i-1}} = \frac{\lambda_i}{c_i - c_{i-1}},$$

i.e.

$$\frac{\lambda_i}{f'(x_i)} = c_i - c_{i-1} \quad (i = 1, 2, \cdots, n).$$

Hence, we have

$$\sum_{i=1}^{n} \frac{\lambda_i}{f'(x_i)} = \sum_{i=1}^{n} (c_i - c_{i-1}) = c_n - c_0 = 1.$$

Recall that $\lambda_i = \frac{k_i}{m}$, we obtain

$$\sum_{i=1}^{n} \frac{k_i}{f'(x_i)} = \sum_{i=1}^{n} k_i.$$

$\square$

Problem 7.18. *Suppose that $f(x)$ is differentiable on (a, b). Prove that the points in (a, b) are either the continuous point of $f'(x)$, or the discontinuous point of secong kind. i.e. $f'(x)$ has no discontinous points of first kind.*

Proof. Since $f(x)$ is differentiable on (a, b), we know that $\forall\, x_0 \in (a, b)$, there is

$$\begin{aligned} f'(x_0) = f'_+(x_0) &= \lim_{x \to x_0+0} \frac{f(x) - f(x_0)}{x - x_0} \\ &= \lim_{x \to x_0+0} \frac{f'(\xi)(x - x_0)}{x - x_0} \\ &= \lim_{x \to x_0+0} f'(\xi) \quad (x_0 < \xi < x). \end{aligned}$$

Hence, if $\lim_{x \to x_0+0} f'(x)$ exists, there must be

$$f'(x_0) = \lim_{\xi \to x_0+0} f'(\xi) = f'(x_0 + 0).$$

Similarly, if $\lim_{x \to x_0-0} f'(x)$ exists, there must be

$$f'(x_0) = f'(x_0 - 0).$$

Therefore $f'(x)$ is continuous at $x = x_0$ unless at least one of $\lim_{x \to x_0+0} f'(x)$, $\lim_{x \to x_0-0} f'(x)$ does not exist. $\square$

Exercise 7.19 (Darboux Theorem)**.** *Suppose that $f(x)$ is differentiable on $[a, b]$, and $f'(a) < f'(b)$. Then $\forall\, c : f'(a) < c < f'(b)$, there exists $\xi \in (a, b)$, such that $f'(\xi) = c$.*

Hint: Define

$$g(x) = f(x) - cx, \quad \forall\, x \in [a, b].$$

Hence, $g(x)$ is differentiable on $[a, b]$. What's more, there are $g'(a) = f'(a) - c < 0$, $g'(b) = f'(b) - c > 0$. Note that

$$\lim_{x \to a+0} \frac{g(x) - g(a)}{x - a} = g'(a) < 0,$$

which implies that there is $\delta > 0$, such that $\forall\, x \in (a, a + \delta)$, there is $g(x) < g(a)$. Similarly, there is $\delta > 0$, such that $\forall\, x \in (b - \delta, b)$, there is $g(x) < g(b)$. Hence, the minimum of $g(x)$ is achieved on $[a + \delta, b - \delta]$, which yields that there is $\xi \in (a, b)$, such that $g'(\xi) = 0$, i.e. $f'(\xi) = c$. $\square$

Exercise 7.20. *Suppuse that $f(x)$ is continuous on $[a, b]$, and differentiable on (a, c), (c, b). Prove that there exists $\xi \in (a, b)$ such that $|f(b) - f(a)| \leq |f'(\xi)||b - a|$.*

Hint: Using the Lagrange mean value theorem on (a, c), (c, b), respectively, and taking the maximum of intermediate points. □

Exercise 7.21. *Suppuse that $f(x)$ is continuous on $[0, 1]$, differentiable on $(0, 1)$, and $|f'(x)| < 1$. If $f(0) = f(1)$, prove that for any $x_1, x_2 \in (0, 1)$, there is*

$$|f(x_1) - f(x_2)| < \frac{1}{2}.$$

Hint: Consider $|x_1 - x_2| < \frac{1}{2}$ and $|x_1 - x_2| \geq \frac{1}{2}$, respectively. □

Exercise 7.22 (Challenge!). *Suppose that $f(x)$, $f'_+(x) \in C(\mathbb{R})$. Prove that $f(x)$ is differentiable on $\mathbb{R}$.*

Hint: Firstly, prove the two lemmas:

Suppose that $f(x) \in C[a, b]$, $f(a) = f(b)$, $f'_+(x)$ exists on $[a, b)$. Prove that there exist $c, d \in [a, b)$ such that $f'_+(c) \leq 0$, $f'_+(d) \geq 0$.

Suppose that $f(x) \in C[a, b]$, $f'_+(x)$ exists on $[a, b)$. Prove that there exist $c, d \in [a, b)$ such that $f'_+(c) \leq \frac{f(b) - f(a)}{b - a} \leq f'_+(d)$.

Finally, using the continuity of $f'_+(x)$ to prove that $\forall\, x_0 \in \mathbb{R}$, there is

$$f'_-(x_0) = \lim_{x \to x_0 - 0} \frac{f(x) - f(x_0)}{x - x_0} = f'_+(x_0).$$

□

Problem 8.1. *Calculate the following limitations.*

(1) $\lim\limits_{x\to 0+0} x^x$;

(2) $\lim\limits_{x\to 0+0} x^{x^x-1}$;

(3) $\lim\limits_{x\to +\infty} \left(\sqrt[3]{x^3-3x}-\sqrt{x^2-2x}\right)$;

(4) $\lim\limits_{x\to 0+0} \dfrac{x^x-(\sin x)^x}{x^2\ln(1+x)}$.

Solution. (1) By L'Hospital's rule, we have

$$\begin{aligned}\lim_{x\to 0+0} x^x &= \lim_{x\to 0+0} e^{x\ln x}\\ &= \exp\left\{\lim_{x\to 0+0}\frac{\ln x}{\frac{1}{x}}\right\}\\ &= \exp\left\{\lim_{x\to 0+0}\frac{\frac{1}{x}}{-\frac{1}{x^2}}\right\}\\ &= \exp\left\{\lim_{x\to 0+0} -x\right\}\\ &= 1.\end{aligned}$$

(2) By Taylor's formula, we have

$$\begin{aligned}\lim_{x\to 0+0} x^{x^x-1} &= e^{\lim\limits_{x\to 0+0}(x^x-1)\ln x}\\ &= e^{\lim\limits_{x\to 0+0} x(\ln x)^2} = 1.\end{aligned}$$

(3) By Taylor's formula, we have

$$\begin{aligned}\lim_{x\to +\infty}\left(\sqrt[3]{x^3-3x}-\sqrt{x^2-2x}\right) &= \lim_{x\to +\infty} x\left(\sqrt[3]{1-3x^{-2}}-\sqrt{1-2x^{-1}}\right)\\ &= \lim_{x\to +\infty} x(1-x^{-2}-1+x^{-1})\\ &= 1.\end{aligned}$$

(4) By Taylor's formula, we have

$$\begin{aligned}\lim_{x\to 0+0}\frac{x^x-(\sin x)^x}{x^2\ln(1+x)} &= \lim_{x\to 0+0}\frac{e^{x\ln x}-e^{x\ln\sin x}}{x^3}\\ &= \lim_{x\to 0+0} e^{x\ln x}\frac{1-e^{x\ln\frac{\sin x}{x}}}{x^3}\end{aligned}$$

$$= \lim_{x\to 0+0} \frac{1-e^{-\frac{1}{6}x^3}}{x^3}$$
$$= \frac{1}{6}.$$

□

Exercise 8.2. *Prove that*
$$\left(1+\frac{1}{n}\right)^n = e - \frac{e}{2n} + \frac{11e}{24n^2} + o\left(\frac{1}{n^2}\right), \quad (n\to\infty).$$

Hint: By Taylor's formula, we have
$$\left(1+\frac{1}{n}\right)^n = e^{n\ln\left(1+\frac{1}{n}\right)}$$
$$= e^{1-\frac{1}{2n}+\frac{1}{3n^2}+o\left(\frac{1}{n^2}\right)}$$
$$= e\left(1-\frac{1}{2n}+\frac{1}{3n^2}+\frac{1}{2}\left(-\frac{1}{2n}+\frac{1}{3n^2}\right)^2\right)+o\left(\frac{1}{n^2}\right)$$
$$= e - \frac{e}{2n} + \frac{11e}{24n^2} + o\left(\frac{1}{n^2}\right), \quad (n\to\infty).$$

□

Remark 8.3. *Similarly, for* $(1+x)^{\frac{1}{x}}$*, there is*
$$(1+x)^{\frac{1}{x}} = e - \frac{e}{2}x + \frac{11e}{24}x^2 + o\left(x^2\right), \quad (x\to 0).$$

Problem 8.4. *Suppose that* $f(x)$ *is twice differentiable on* $[0,1]$*,* $f(0)=f(1)=0$*,* $\max_{x\in[0,1]} f(x) = 2$*. Prove that* $\inf_{x\in[0,1]} f''(x) \le -16$*.*

Proof. Since $f(x)$ is continuous on $[0,1]$, $f(0)=f(1)=0$, and $\max_{x\in[0,1]} f(x) = 2$, we know that there is $x_0\in(0,1)$ such that
$$f(x_0) = \max_{x\in[0,1]} f(x).$$
Hence, by Fermat's theorem, there is
$$f'(x_0) = 0.$$
By Taylor's formula at $x = x_0$, we have
$$0 = f(0) = f(x_0) + \frac{1}{2}f''(\xi)(0-x_0)^2 = 2 + \frac{1}{2}f''(\xi)x_0^2,$$

$$0 = f(1) = f(x_0) + \frac{1}{2}f''(\eta)(1-x_0)^2 = 2 + \frac{1}{2}f''(\eta)(1-x_0)^2.$$

Hence, we know

$$\inf_{x\in[0,1]} f''(x) \le \min\{f''(\xi), f''(\eta)\} = \min\left\{-\frac{4}{x_0^2}, -\frac{4}{(1-x_0)^2}\right\}.$$

Note that

$$\min\left\{-\frac{4}{x_0^2}, -\frac{4}{(1-x_0)^2}\right\} = -\frac{4}{(1-x_0)^2} \le -16, \quad x_0 \in \left[\frac{1}{2}, 1\right],$$
$$\min\left\{-\frac{4}{x_0^2}, -\frac{4}{(1-x_0)^2}\right\} = -\frac{4}{x_0^2} \le -16, \quad x_0 \in \left[0, \frac{1}{2}\right].$$

Therefore, we obtain that

$$\inf_{x\in[0,1]} f''(x) \le -16.$$

□

Remark 8.5. *Use the same method, we can prove that if $f(x)$ is twice differentiable on $[0,1]$, $f(0) = f(1) = 0$, $\min_{x\in[0,1]} f(x) = -1$, then $\sup_{x\in[0,1]} f''(x) \ge 8$.*

Problem 8.6 (5.16)**.** *Suppose that $f(x)$ is three times differentiable on $[a,b]$. Prove that there is $\xi \in (a,b)$, such that*

$$f(b) = f(a) + \frac{1}{2}(b-a)(f'(a) + f'(b)) - \frac{1}{12}(b-a)^3 f^{(3)}(\xi).$$

Proof. Firstly, we choose M such that

$$f(b) = f(a) + \frac{1}{2}(b-a)(f'(a) + f'(b)) - \frac{1}{12}(b-a)^3 M.$$

Define

$$F(x) = f(x) - f(a) - \frac{1}{2}(x-a)(f'(x) + f'(a)) + \frac{1}{12}(x-a)^3 M.$$

Hence, there is $F(a) = F(b) = 0$. By Rolle's theorem, we know that there exists $\eta \in (a,b)$ such that $F'(\eta) = 0$. Note that

$$F'(x) = f'(x) - \frac{1}{2}(f'(x) + f'(a)) - \frac{1}{2}(x-a)f''(x) + \frac{1}{4}(x-a)^2 M.$$

Hence, there is $F'(a) = F'(\eta) = 0$. By Rolle's theorem again, we know that there exists $\xi \in (a,\eta)$ such that $F''(\xi) = 0$. Note that

$$F''(x) = -\frac{1}{2}(x-a)f^{(3)}(x) + \frac{1}{2}(x-a)M.$$

Therefore, we have $M = f^{(3)}(\xi)$, i.e.

$$f(b) = f(a) + \frac{1}{2}(b-a)(f'(a) + f'(b)) - \frac{1}{12}(b-a)^3 f^{(3)}(\xi).$$

□

Problem 8.7. *Suppose that $f(x)$ is twice differentiable on $[-1,1]$, $f(0)=f'(0)=0$. Assume that $|f''(x)|\le |f(x)|+|f'(x)|$, $\forall\, x\in[-1,1]$. Prove that there exists $\delta>0$, such that $f(x)=0$, $\forall\, x\in(-\delta,\delta)$.*

Proof. Choose $\delta = 1/4$. Since $|f(x)|+|f'(x)|$ is continuous on $[-1/4,1/4]$, we know that there is $x_0\in[-1/4,1/4]$ such that

$$|f(x_0)|+|f'(x_0)| = \max_{x\in[-1/4,1/4]} |f(x)|+|f'(x)| =: M.$$

By Taylor's formula, we have

$$\begin{aligned} f(x_0) &= f(0)+f'(0)x_0+\frac{f''(\xi)}{2}x_0^2 = \frac{f''(\xi)}{2}x_0^2,\\ f'(x_0) &= f''(\eta)x_0. \end{aligned}$$

Hence, there is

$$\begin{aligned} |f(x_0)|+|f'(x_0)| &= \left|\frac{f''(\xi)}{2}x_0^2\right| + |f''(\eta)x_0|\\ &\le \frac{1}{4}(|f(\xi)|+|f'(\xi)|)+\frac{1}{4}(|f(\eta)|+|f'(\eta)|)\\ &\le \frac{1}{2}M, \end{aligned}$$

which implies that $M=0$, i.e. $f(x)=0$, $\forall\, x\in[-1/4,1/4]$. □

Problem 8.8. *Suppose that $f(x)$ is twice differentiable on $\mathbb{R}$, and $f(x)$ is also a bounded function. Prove that there is $\xi\in\mathbb{R}$ such that $f''(\xi)=0$.*

Proof. Assume that $f''(x)\ne 0$, $\forall\, x\in\mathbb{R}$. By Darboux's theorem (Exercise 7.19), we know that $f''(x)$ does not change sign. Without loss of generality, we may assume that $f''(x)>0$, $\forall\, x\in\mathbb{R}$. Choosing $x_0\in\mathbb{R}$ such that $f'(x_0)\ne 0$. By Taylor' formula, we have

$$f(x) = f(x_0)+f'(x_0)(x-x_0)+\frac{f''(\xi)}{2}(x-x_0)^2 \ge f(x_0)+f'(x_0)(x-x_0),$$

which contradicts with f is bounded on $\mathbb{R}$. □

Problem 8.9. *Suppose that $f\in C^n(\mathbb{R})$, and there exist constants M_0, M_1 such that $|f(x)|\le M_0$, $|f^{(n)}(x)|\le M_1$, $\forall\, x\in\mathbb{R}$. Prove that there is $M>0$ such that $|f^{(j)}(x)|\le M$, $j=1,2,\cdots,n-1$, $\forall\, x\in\mathbb{R}$.*

Proof. By Taylor's formula, we have

$$f(x+m) = f(x) + mf'(x) + \frac{m^2}{2!}f''(x) + \cdots + \frac{m^{n-1}}{(n-1)!}f^{(n-1)}(x) + \frac{m^n}{n!}f^{(n)}(\xi_m),$$

where $x < \xi_m < x+m$, $m = 1, 2, \cdots, n$. This is a linear system of equations about $f'(x)$, $f''(x)$, $\cdots$, $f^{(n-1)}(x)$, and the determinant of its coefficients is

$$\begin{vmatrix} 1 & 1 & \frac{1}{2!} & \cdots & \frac{1}{(n-1)!} \\ 1 & 2 & \frac{2^2}{2!} & \cdots & \frac{2^{n-1}}{(n-1)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & n & \frac{n^2}{2!} & \cdots & \frac{n^{n-1}}{(n-1)!} \end{vmatrix} = \frac{1}{1!2!\cdots(n-1)!} \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2^2 & \cdots & 2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & n & n^2 & \cdots & n^{n-1} \end{vmatrix} = 1.$$

Hence, we know that $f'(x)$, $f''(x)$, $\cdots$, $f^{(n-1)}(x)$ can be written as linear combinations of $f(x+m)$ and $f^{(n)}(\xi_m)$, $m = 1, 2, \cdots, n$. Since $|f(x)| \le M_0$, $|f^{(n)}(x)| \le M_1$, $\forall\, x \in \mathbb{R}$, we have that there exists $M > 0$ such that $|f^{(j)}(x)| \le M$, $j = 1, 2, \cdots, n-1$, $\forall\, x \in \mathbb{R}$. ☐

Problem 8.10. *Suppose that $f(x)$ is bounded on $\mathbb{R}$ and $f'(x)$ is uniformly continuous on $\mathbb{R}$. Prove that $f'(x)$ is also bounded.*

Proof. Prove by contradiction. Without loss of generality, we may assume that $f'(x)$ has no upper bound. Hence, we know that $\forall\, n \in \mathbb{N}$, there exists $x_n \in \mathbb{R}$, such that $f'(x_n) > n$. Since $f'(x)$ is uniformly continuous on $\mathbb{R}$, we have that there exists $\delta > 0$ such that $\forall\, x, y : |x-y| < \delta$, there is

$$|f(x) - f(y)| < 1.$$

Then, there is $f(x) > f(x_n) - 1 > n - 1$, $\forall\, x \in (x_n, x_n + \delta)$. By Taylor's formula, we have

$$2\sup_{x\in\mathbb{R}} |f(x)| \ge |f(x_n + \delta) - f(x_n)| = |f'(\xi_n)\delta| > (n-1)\delta \to +\infty, \quad n \to \infty,$$

which contradicts with $f(x)$ is bounded on $\mathbb{R}$. ☐

Remark 8.11. *Note that $\lim\limits_{x\to+\infty} f'(x)$ may not exist, for example, consider $f(x) = \sin x$. But if $\lim\limits_{x\to+\infty} f'(x)$ exists, there must be $\lim\limits_{x\to+\infty} f'(x) = 0$.*

Problem 8.12. *Suppose that $f \in C^3(\mathbb{R})$, and there exists $\theta \in (0,1)$ such that*

$$f(x+h) = f(x) + hf'(x+\theta h), \quad \forall\, h \in \mathbb{R}. \tag{8.1}$$

Prove that f is a linear function or a quadratic function.

Proof. Differentiating (8.1) respect to h, we have

$$f'(x+h) = f'(x+\theta h) + \theta h f''(x+\theta h). \tag{8.2}$$

Hence, there is

$$\frac{f'(x+h) - f'(x) + f'(x) - f'(x+\theta h)}{h} = \theta f''(x+\theta h).$$

Letting $h \to 0$, we have

$$f''(x) - \theta f''(x) = \theta f''(x),$$

i.e.

$$f''(x) = 2\theta f''(x).$$

If $\theta \neq \frac{1}{2}$, we know that $f''(x) = 0$, $\forall\, x \in \mathbb{R}$, thus $f(x)$ is a linear function. If $\theta = \frac{1}{2}$, (8.2) yields

$$f'(x+h) = f'\left(x + \frac{1}{2}h\right) + \frac{1}{2}hf''\left(x + \frac{1}{2}h\right).$$

Differentiating the above formula respect to h yields

$$f''(x+h) = f''\left(x + \frac{1}{2}h\right) + \frac{1}{4}hf'''\left(x + \frac{1}{2}h\right).$$

Hence

$$\frac{f''(x+h) - f''\left(x + \frac{1}{2}h\right)}{\frac{1}{2}h} = \frac{1}{2}hf'''\left(x + \frac{1}{2}h\right).$$

Letting $h \to 0$, we have

$$f'''(x) = \frac{1}{2}f'''(x),$$

i.e. $f'''(x) = 0$, $\forall\, x \in \mathbb{R}$, thus $f(x)$ is a quadratic function. □

Problem 8.13. *If f is defined on $(0, +\infty)$ and f', f'' exists , with $\lim_{x\to+\infty} f(x)$ exists and f'' bounded, prove that $\lim_{x\to+\infty} f'(x) = 0$.*

Proof. Without loss of generality, we assume that $\lim_{x\to+\infty} f(x) = 0$ (otherwise, replace $f(x)$ by $f(x) - \lim_{x\to+\infty} f(x)$, and those conditions are still satisfied). Then $\forall\, \varepsilon > 0$, $\exists\, a \in \mathbb{R}$ such that $\sup_{x\in(a,\infty)} |f(x)| < \varepsilon$.

Let

$$\sup_{x\in(0,\infty)} |f''(x)| = M_2,$$

(exists finitely in $\mathbb{R}$ as f'' is bounded.) So, for a defined above, there is

$$\sup_{x\in(a,\infty)} |f''(x)| \leq M_2.$$

Taking $h > 0$, by Taylor's theorem we have

$$f'(x) = \frac{1}{2h}[f(x+2h) - f(x)] - hf''(\xi)$$

for some $\xi \in (x, x+2h)$. Hence

$$|f'(x)| \leq \frac{\varepsilon}{h} + hM_2.$$

Then we obtain

$$h^2 M_2 - h\,|f'(x)| + \varepsilon \geq 0, \quad \forall\, x \in (a, \infty),$$

which is a quadratic in h, and since $M_2 > 0$, we have

$$|f'(x)|^2 \leq 4M_2\varepsilon, \quad \forall\, x \in (a, \infty).$$

Hence

$$\lim_{x\to\infty} |f'(x)|^2 = 0 \Longrightarrow \lim_{x\to\infty} |f'(x)| = 0 \Longrightarrow \lim_{x\to\infty} f'(x) = 0.$$

□

Problem 8.14. *Suppose that $f \in C^\infty(a,b)$, and $f^{(n)}(x) \geq 0$ for all $n \in \mathbb{N}_+$. If $|f(x)| \leq M$, prove that for every $x \in (a,b)$, $r > 0$, $x + r \in (a,b)$, there is*

$$f^{(n)}(x) \leq \frac{2Mn!}{r^n}, \quad \forall\, n \in \mathbb{N}_+.$$

Proof. By Taylor's formula, we have

$$\begin{aligned} f(x+r) &= f(x) + f'(x)r + \cdots + \frac{f^{(n)}(x)}{n!}r^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}r^{n+1} \\ &\geq f(x) + \frac{f^{(n)}(x)}{n!}r^n, \end{aligned}$$

which implies

$$f^{(n)}(x) \leq \frac{(f(x+r) - f(x))n!}{r^n} \leq \frac{2Mn!}{r^n}.$$

□

Problem 8.15 (Bernstein Theorem)**.** *Suppose that $f \in C^\infty(a,b)$, and $f^{(n)}(x) \geq 0$ for all $n \in \mathbb{N}_+$. Prove that for every $x_0 \in (a,b)$, there exists $r > 0$, such that $\forall\, x \in [x_0 - r, x_0 + r] \subset (a,b)$, there is*

$$f(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

Proof. Choosing $r > 0$ small enough, such that $[x_0 - 3r, x_0 + 3r] \subset (a, b)$. Denote

$$M = \sup_{x \in [x_0 - 2r, x_0 + 2r]} |f(x)|.$$

By Problem 8.14, we have for every $x \in [x_0 - 2r, x_0 + 2r]$, there is $f^{(n)}(x) \leq \dfrac{2Mn!}{(2r)^n}$. Then, we have

$$\begin{aligned}\left| f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \right| &= \frac{f^{(n+1)}(\xi)}{(n+1)!}|x - x_0|^{n+1} \\ &\leq \frac{f^{(n+1)}(\xi)}{(n+1)!} r^{n+1} \leq \frac{M}{2^n}.\end{aligned}$$

Hence, there is

$$f(x) = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

□

Exercise 8.16 (5.12)**.** *Suppose that $f(x)$ is differentiable on (a, b), and $f'(x)$ is monotonic. Prove that $f'(x)$ is continuous on (a, b).*

Hint: Since $f'(x)$ is monotonic, we know that for any $x_0 \in (a, b)$, $\lim\limits_{x \to x_0 + 0} f'(x)$ and $\lim\limits_{x \to x_0 - 0} f'(x)$ exist. Then by Problem 7.18, we have that $f'(x)$ is continuous on (a, b). □

Exercise 8.17. *Suppose that $f(x)$ is continuous on $[a, b]$, and twice differentiable on (a, b). If $|f''(x)| \geq m > 0$, and $f(a) = f(b) = 0$. Prove that $\max\limits_{x \in [a,b]} |f(x)| \geq \dfrac{m}{8}(b - a)^2$.*

Hint: Denote $|f(x_0)| = \max\limits_{x \in [a,b]} |f(x)|$. It's easy to see that $f'(x_0) = 0$. Then

$$f(a) = f(x_0) + f'(x_0)(a - x_0) + \frac{f''(\xi)}{2}(x_0 - a)^2 = f(x_0) + \frac{f''(\xi)}{2}(x_0 - a)^2,$$
$$f(b) = f(x_0) + f'(x_0)(b - x_0) + \frac{f''(\eta)}{2}(x_0 - b)^2 = f(x_0) + \frac{f''(\eta)}{2}(x_0 - b)^2.$$

Hence

$$|f(x_0)| \geq \frac{m}{2}(x_0 - a)^2 \geq \frac{m}{8}(b - a)^2, \quad x_0 \in \left[\frac{a + b}{2}, b\right],$$
$$|f(x_0)| \geq \frac{m}{2}(x_0 - b)^2 \geq \frac{m}{8}(b - a)^2, \quad x_0 \in \left[a, \frac{a + b}{2}\right].$$

□

Problem 9.1 (Schwarz Theorem). *Define the generalized second order derivative as follows*

$$f^{[2]}(x) = \lim_{h\to 0+0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2}.$$

If $f \in C[a,b]$, and $f^{[2]}(x) = 0$ on (a,b), prove that f is a linear function.

Proof. For any $x \in [a,b]$ and $\forall\, \varepsilon > 0$, define

$$f_\varepsilon(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a) + \varepsilon(x-a)(x-b).$$

Then $f_\varepsilon(a) = f_\varepsilon(b) = 0$. Note that

$$\begin{aligned}
f_\varepsilon^{[2]}(x) = &\lim_{h\to 0+0} \frac{f_\varepsilon(x+h)+f_\varepsilon(x-h)-2f_\varepsilon(x)}{h^2}\\
=&\lim_{h\to 0+0} \frac{1}{h^2}\Bigg(f(x+h) - f(a) - \frac{f(b)-f(a)}{b-a}(x+h-a) + \varepsilon(x+h-a)(x+h-b)\\
&+ f(x-h) - f(a) - \frac{f(b)-f(a)}{b-a}(x-h-a) + \varepsilon(x-h-a)(x-h-b)\\
&- 2\left(f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a) + \varepsilon(x-a)(x-b)\right)\\
=&\lim_{h\to 0+0} \frac{1}{h^2}\left(f(x+h)+f(x-h)-2f(x)+\varepsilon\cdot 2h^2\right)\\
=&f^{[2]}(x) + 2\varepsilon = 2\varepsilon.
\end{aligned}$$

We claim that $f_\varepsilon \le 0$, when $\varepsilon > 0$. Indeed, if there is $x_0 \in (a,b)$ such that $f_\varepsilon(x_0) > 0$, by $f_\varepsilon(a) = f_\varepsilon(b) = 0$ and the continuity of f_ε, we know that the maximum of f_ε is achieved in (a,b), say $f_\varepsilon(x^*) = \max\limits_{x\in[a,b]} f_\varepsilon(x)$. Hence, there is

$$f_\varepsilon(x^*+h) + f_\varepsilon(x^*-h) \le 2f_\varepsilon(x^*).$$

Since $f_\varepsilon^{[2]}(x^*) = 2\varepsilon > 0$, we know that there is $h > 0$ such that

$$f_\varepsilon(x^*+h) + f_\varepsilon(x^*-h) - 2f_\varepsilon(x^*) > 0.$$

Therefore, there is

$$f_\varepsilon(x^*) < \frac{1}{2}(f_\varepsilon(x^*+h) + f_\varepsilon(x^*-h)) \le f_\varepsilon(x^*),$$

contradiction. Similarly, we can prove that $f_\varepsilon \ge 0$, when $\varepsilon < 0$. By the continuity of f_ε respect to ε , we know that

$$f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a) = f_0(x) = 0,$$

i.e. f is a linear function. ◻

Problem 9.2. *If $f'(x_0) > (<)0$, can we say that f is monotonic on a small enough neighborhood of $x = x_0$?*

Solution. No. For example (6.59 in book), let

$$f(x) = \begin{cases} x + 2x^2 \sin \dfrac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

It's easy to see that $f'(0) = 1 > 0$. However, by

$$f'(x) = 1 + 4x \sin \frac{1}{x} - 2 \cos \frac{1}{x}, \quad x \neq 0,$$

we have

$$f'\left(\frac{1}{n\pi}\right) = 1 - 2(-1)^n,$$

which means that $f'(x)$ can not be always positve or negative on any neighboehoods of $x = 0$, hence f is not monotonic on any neighboehoods of $x = 0$. The graph of $f(x)$ is as follows:

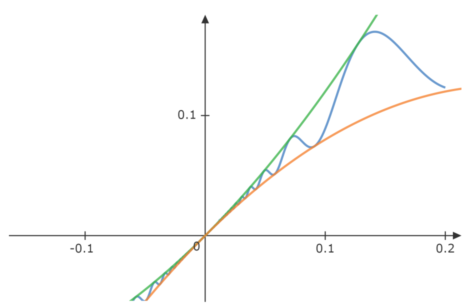

FIGURE 6. Graph of $f(x)$

◻

Problem 9.3 (5.37)**.** *Suppose that $f(x)$ has nth-order derivative on $(a, +\infty)$. If $\lim_{x \to +\infty} f(x) = A$, $\lim_{x \to +\infty} f^{(n)}(x) = B$. Prove that $B = 0$.*

Proof. By L'Hospital's rule, we have

$$\begin{aligned} 0 &= \lim_{x \to +\infty} \frac{f(x)}{x^n} \\ &= \lim_{x \to +\infty} \frac{f'(x)}{nx^{n-1}} \end{aligned}$$

$$
\begin{aligned}
&\vdots \\
&= \lim_{x\to+\infty} \frac{f^{(n)}(x)}{n!} \\
&= \frac{B}{n!},
\end{aligned}
$$

which gives us $B = 0$. □

Problem 9.4 (5.48)**.** *Suppose that $f(x)$ is defined on $[a, b]$ satisfying*

$$
|f(x) - f(y)| \le k|x - y|^{1+\alpha}, \quad \forall\, x, y \in [a, b],\ \alpha > 0. \tag{9.1}
$$

Prove that $f(x)$ is constant.

Proof. For $\forall\, x_0 \in [a, b]$, by (9.1), we have

$$
\left| \frac{f(x) - f(x_0)}{x - x_0} \right| \le k|x - x_0|^{\alpha}, \quad \forall\, x \ne x_0.
$$

Hence, letting $x \to x_0$, we know

$$
f'(x_0) = 0, \quad \forall\, x_0 \in [a, b],
$$

which means that $f(x)$ is constant. □

Problem 9.5. *Find the extreme value of the following functions.*

(50(3)) $f(x) = \dfrac{(\ln x)^2}{x}$;

(50(4)) $f(x) = |x(x^2 - 1)|$;

(51(2)) $f(x) = 2\tan x - \tan^2 x$, $x \in [0, \pi/2)$;

(51(3)) $f(x) = \sqrt{x} \ln x$, $x \in (0, +\infty)$.

Solution. (50(3)) The derivative of $f(x)$ is

$$
f'(x) = \frac{\ln x(2 - \ln x)}{x^2}.
$$

Hence the local minimum is $f(1) = 0$ and the local maximum is $f(e^2) = 4/e^2$. The graph of $f(x)$ is as follows:

(50(4))The graph of $f(x)$ is as follows: When $x = 0$, $x = \pm 1$, $f(x)$ has local minimun 0. When $x = \pm\frac{\sqrt{3}}{3}$, $f(x)$ has local maximun $\frac{2\sqrt{3}}{9}$.

(51(2)) The graph of $f(x)$ is as follows:

(51(3)) The graph of $f(x)$ is as follows:

□

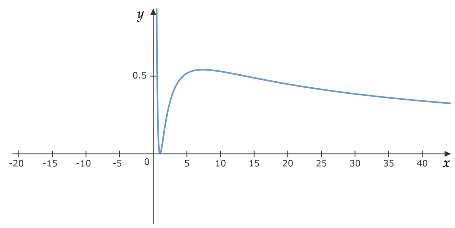

FIGURE 7. Graph of $f(x)$

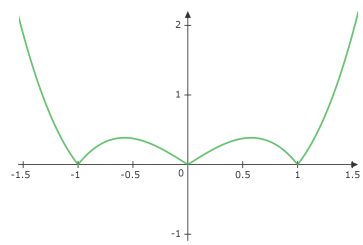

FIGURE 8. Graph of $f(x)$

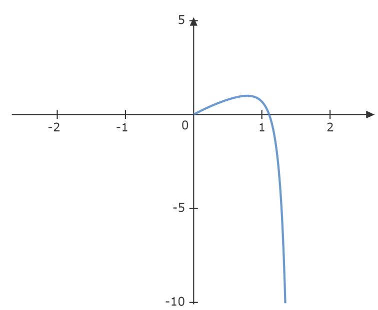

FIGURE 9. Graph of $f(x)$

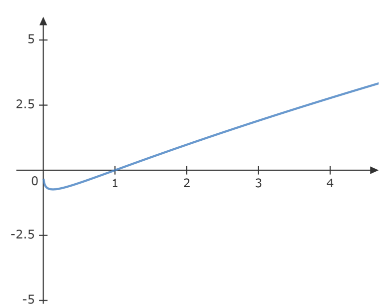

FIGURE 10. Graph of $f(x)$

Problem 9.6 (5.54(2))**.** *Suppose that $f(x)$ and $g(x)$ are differentiable on (a, b), and denote*

$$F(x) = f(x)g'(x) - f'(x)g(x), \quad x \in (a, b).$$

If $F(x) > 0$, $x \in (a, b)$. Prove that there must be zero points of $g(x)$ between the two zero points of $f(x)$.

Proof. Assume that x_1, $x_2 \in (a, b)$ are two zero points of $f(x)$. If $\forall\, x \in (x_1, x_2)$, there is $g(x) \neq 0$. Define

$$G(x) = \frac{f(x)}{g(x)}, \quad \forall\, x \in (x_1, x_2).$$

Hence, we have

$$G'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)} < 0,$$

which gives us that $G(x)$ is monotonic decreasing on $[x_1, x_2]$. Then there is

$$0 = G(x_1) \geq G(x) \geq G(x_2) = 0,$$

i.e. $G(x) \equiv 0$, which contradicts with $F(x) > 0$. Hence we know that there must be zero points of $g(x)$ between x_1 and x_2. □

Problem 9.7. *Suppose that $f(x)$ is twice differentiable on $[a, b]$ satisfying*

$$f''(x) + b(x)f'(x) + c(x)f(x) = 0, \quad \forall\, x \in [a, b], \tag{9.2}$$

where $c(x) < 0$. Prove that

(1) *$f(x)$ can not admit a positive maximum or negative minimum in (a, b);*
(2) *If $f(a) = f(b) = 0$, then $f(x) \equiv 0$.*

Proof. (1) Prove by contradiction. Assume that $f(x)$ has a positive maximum in (a, b), i.e. there is $x_0 \in (a, b)$ such that $f(x_0) = \max\limits_{x \in [a,b]} f(x) > 0$. By the necessary condition for maximum value, we know that $f'(x_0) = 0$, $f''(x_0) \leq 0$. Then combining with $c(x_0) < 0$, we have

$$f''(x_0) + b(x_0)f'(x_0) + c(x_0)f(x_0) \leq c(x_0)f(x_0) < 0,$$

which contradicts with (9.2). Hence, we have that $f(x)$ can not admit a positive maximum in (a, b). Using the same method, we can prove that $f(x)$ can not admit a negative minimum in (a, b).

(2) By (1), we know that $\max\limits_{x \in [a,b]} f(x) \leq 0$ and $\min\limits_{x \in [a,b]} f(x) \geq 0$. Hence

$$0 \leq \min_{x \in [a,b]} f(x) \leq \max_{x \in [a,b]} f(x) \leq 0,$$

i.e. $f(x) \equiv 0$. □

Exercise 9.8 (5.67)**.** *Suppose that the tangent line of the elliptic* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *intersects the* x*-axis and* y*-axis at* A *and* B*, respectively.*

(1) *Find the minimum length of* AB*;*

(2) *Find the minimum area of the triangle formed by* AB *and the coordinate axis.*

Hint: (1) The tangent line at (x_0, y_0) is

$$\frac{x_0x}{a^2}+\frac{y_0y}{b^2}=1.$$

Hence we know that $A\left(\frac{a^2}{x_0},0\right)$, $B\left(\frac{b^2}{y_0},0\right)$. Then

$$\begin{aligned}|AB| &= \sqrt{\frac{a^4}{x_0^2}+\frac{b^4}{y_0^2}} \\ &= \sqrt{\frac{a^2}{\frac{x_0^2}{a^2}}+\frac{b^2}{\frac{y_0^2}{b^2}}} \\ &\geq \sqrt{\frac{(a+b)^2}{\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}}} \\ &= a+b.\end{aligned}$$

(2) By (1), we know

$$S_{\Delta AOB}=\frac{1}{2}\frac{a^2b^2}{|x_0y_0|}\geq\frac{ab}{\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}}=ab.$$

The graph of the elliptic when $a=2$, $b=\sqrt{3}$ is as follows:

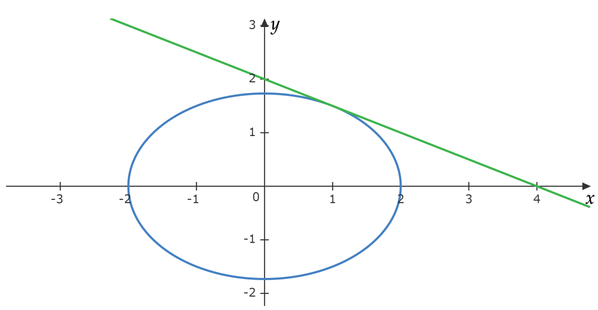

FIGURE 11. Graph of $\frac{x^2}{4}+\frac{y^2}{3}=1$

□

Problem 9.9. *Suppose that $P(x)$ is a polynomial function. Prove that*

(1) *If $P'(x) + P(x) \geq 0$, then $P(x) \geq 0$;*

(2) *If $P(x) - P'(x) \geq 0$, then $P(x) \geq 0$;*

(3) *If $P'''(x) - P''(x) - P'(x) + P(x) \geq 0$, then $P(x) \geq 0$.*

Proof. (1) Let
$$F(x) = P(x)e^x.$$
Then there is
$$F'(x) = (P'(x) + P(x))e^x \geq 0.$$
Since $\lim_{x \to -\infty} F(x) = 0$, we know that $F(x) \geq 0$, $\forall\, x \in \mathbb{R}$. Note that $e^x > 0$, $\forall\, x \in \mathbb{R}$, we have $P(x) \geq 0$.

(2) Let
$$G(x) = P(x)e^{-x}.$$
Then there is
$$G'(x) = (P'(x) - P(x))e^{-x} \leq 0.$$
Since $\lim_{x \to +\infty} G(x) = 0$, we know that $G(x) \geq 0$, $\forall\, x \in \mathbb{R}$. Note that $e^{-x} > 0$, $\forall\, x \in \mathbb{R}$, we have $P(x) \geq 0$.

(3) Denote
$$Q(x) = P''(x) - P(x).$$
We have
$$Q'(x) - Q(x) \geq 0.$$
Hence by (2), we know that $Q(x) \leq 0$, i.e. $P(x) - P''(x) \geq 0$. Note that
$$P(x) - P''(x) = P(x) - P'(x) + P'(x) - P''(x),$$
we have by (1) that $P(x) - P'(x) \geq 0$. Then by (2) again, we have that $P(x) \geq 0$. □

Problem 9.10. *Suppose that $f(x)$ is a bounded convex function on (a, b). Prove that $\lim_{x \to a+0} f(x)$ and $\lim_{x \to b-0} f(x)$ exist.*

Proof. Since $f(x)$ is bounded, we may assume that $|f(x)| \leq M$, $\forall\, x \in (a, b)$. Let $x > x_1 > x_0$ be any points in (a, b). By the convexity of $f(x)$, we have $\frac{f(x) - f(x_0)}{x - x_0}$ is monotonic inreasing respect to x. Since
$$\frac{f(x) - f(x_0)}{x - x_0} \leq \frac{M - f(x_0)}{x_1 - x_0}, \quad \forall\, x > x_1 > x_0,$$

we have

$$\lim_{x \to b-0} \frac{f(x) - f(x_0)}{x - x_0} = A$$

by the monotone bounded convergence theorem. Then

$$\begin{aligned}\lim_{x \to b-0} f(x) &= \lim_{x \to b-0} \left[(x - x_0) \cdot \frac{f(x) - f(x_0)}{x - x_0} + f(x_0)\right] \\ &= A(b - x_0) + f(x_0),\end{aligned}$$

which is $\lim_{x \to b-0} f(x)$ exists. Similarly, we can prove that $\lim_{x \to a+0} f(x)$ exists. □

Problem 9.11. *Suppose that $f(x)$ is convex on (a, b). Prove that $\forall\, [c, d] \subset (a, b)$, $f(x)$ is Lipschitz continuous on $[c, d]$.*

Proof. Since $[c, d] \subset (a, b)$, we can choose $h > 0$ small enough such that

$$[c - h, d + h] \subset (a, b).$$

Indeed, it suffices to choose $0 < h < \min\{c - a, b - d\}$. Then $\forall\, x_1, x_2 \in [c, d]$, if $x_1 < x_2$, we take $x_3 = x_2 + h$. By the convexity of $f(x)$, we have

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2} \leq \frac{M - m}{h},$$

where $M = \sup_{x \in [c-h, d+h]} f(x)$, $m = \inf_{x \in [c-h, d+h]} f(x)$. If $x_2 < x_1$, we take $x_3 = x_2 - h$. By the convexity of $f(x)$, we have

$$\frac{f(x_2) - f(x_3)}{x_2 - x_3} \leq \frac{f(x_1) - f(x_2)}{x_1 - x_2}.$$

Then

$$\frac{f(x_2) - f(x_1)}{x_1 - x_2} \leq \frac{f(x_3) - f(x_2)}{x_2 - x_3} \leq \frac{M - m}{h}.$$

Hence, we have

$$f(x_2) - f(x_1) \leq \frac{M - m}{h} |x_1 - x_2|.$$

Switching x_1 and x_2, we know the above inequality is still valid. Hence, we know

$$|f(x_1) - f(x_2)| \leq \frac{M - m}{h} |x_1 - x_2|, \quad \forall\, x_1, x_2 \in [c, d],$$

i.e. $f(x)$ is Lipschitz continuous on $[c, d]$. □

Problem 9.12. *Prove that the non-differentiable points of a convex function are countable.*

Proof. Let $f(x)$ be a convex function. For $\forall\, x < x_0 < y$, by the convexity of f, we have

$$\frac{f(x)-f(x_0)}{x-x_0} \le \frac{f(y)-f(x_0)}{y-x_0}.$$

It's easy to see that $\dfrac{f(x)-f(x_0)}{x-x_0}$ is monotonic inreasing respect to x. Hence by the monotone bounded convergence theorem, we have

$$f'_-(x_0) \le f'_+(x_0).$$

If $f(x)$ is non-differentiable at $x = x_0$, there is $f'_-(x_0) < f'_+(x_0)$. Then every non-differentiable point x_0 is corresponding to a rational number in $\big(f'_-(x_0), f'_+(x_0)\big)$, and $\big\{\big(f'_-(x_0), f'_+(x_0)\big)\big\}$ is disjoint pairwise, thus the non-differentiable points of $f(x)$ are countable. □

Exercise 9.13. *Suppose that $f(x)$ is a bounded convex function on $\mathbb{R}$. Prove that f is a constant function.*

Hint: Prove by contradiction. Suppose that f is not a constant function. By Problem 9.12, we know that $f'_-(x)$ and $f'_+(x)$ exist. Then there is at least a x_0 such that $f'_-(x_0) \ne 0$ or $f'_+(x_0) \ne 0$. Without loss of generality, we assume that $f'_+(x_0) \ne 0$. Then by the convexity of f, we have

$$f(x) \ge f(x_0) + f'_+(x_0)(x - x_0),$$

which contradicts with f is bounded. □

Exercise 9.14. *Suppose that $f \in C^2[0,1]$ is a nonnegative function. Assume that $f(0) = 0$, $f'(0) = 1$, $f''(0) = -1$, and $\forall\, x \in (0,1]$, $f(x) \ne x$. For any given $x_0 \in (0,1]$, define $x_{n+1} = f(x_n)$. Calculate the limitation $\lim_{n\to\infty} nx_n$.*

Hint: By Taylor's formula, we have

$$f(x) = x - \frac{1}{2}x^2 + o(x^2).$$

Hence, for $x > 0$ small enough, we have $f(x) < x$. By $\forall\, x \in (0,1]$, $f(x) \ne x$ and the continuity of f, we know that $f(x) \le x$. Hence, $\lim_{n\to\infty} x_n$ exists and $\lim_{n\to\infty} x_n = 0$. By Stolz theorem and Taylor's formula, we have

$$\begin{aligned}\lim_{n\to\infty} nx_n &= \lim_{n\to\infty} \frac{n}{\frac{1}{x_n}} = \lim_{n\to\infty} \frac{x_n x_{n+1}}{x_n - x_{n+1}} \\ &= \lim_{n\to\infty} \frac{x_n f(x_n)}{x_n - f(x_n)} = \lim_{n\to\infty} \frac{x_n^2}{\frac{1}{2}x_n^2} = 2.\end{aligned}$$

□

Exercise 9.15. *Suppose that f has nth-order derivatives at $x = x_0$. Prove that*

$$f^{(n)}(x_0) = \lim_{h\to 0} \frac{1}{h^n} \sum_{k=0}^{n} (-1)^{n-k} C_n^k f(x_0 + kh).$$

Hint:

$$\begin{aligned}
&\lim_{h\to 0} \frac{\sum_{k=0}^{n}(-1)^{n-k}\mathrm{C}_n^k f\left(x_0+kh\right)}{h^n} \\
&= \lim_{h\to 0} \frac{\sum_{k=0}^{n}(-1)^{n-k}\mathrm{C}_n^k \left(\sum_{m=0}^{n} \frac{f^{(m)}(x_0)}{m!} k^m h^m + o\left(h^n\right)\right)}{h^n} \\
&= \lim_{h\to 0} \frac{\sum_{k=0}^{n}(-1)^{n-k}C_n^k \sum_{m=0}^{n} \frac{f^{(m)}(x_0)}{m!} k^m h^m}{h^n} \\
&= \lim_{h\to 0} \frac{\sum_{k=0}^{n}\sum_{m=0}^{n}(-1)^{n-k}C_n^k \frac{f^{(m)}(x_0)}{m!} k^m h^m}{h^n} \\
&= \lim_{h\to 0} \frac{\sum_{m=0}^{n}\sum_{k=0}^{n}(-1)^{n-k}C_n^k \frac{f^{(m)}(x_0)}{m!} k^m h^m}{h^n} \\
&= \lim_{h\to 0} \frac{\sum_{m=0}^{n} \frac{f^{(m)}(x_0)}{m!} h^m \sum_{k=0}^{n}(-1)^{n-k}C_n^k k^m}{h^n} \\
&= \lim_{h\to 0} \frac{(-1)^n \cdot (-1)^n n! \frac{f^{(n)}(x_0)}{n!} h^n}{h^n} \\
&= f^{(n)}\left(x_0\right),
\end{aligned}$$

where we used Problem 7.8 in the last equality. □

10. Week 13 (11.28)

Problem 10.1. *Prove that $f(x)$ is a convex function on (a,b) if and only if $\forall\, x_1, x_2 \in (a,b)$, there is $\varphi(\lambda) = f(\lambda x_1 + (1-\lambda)x_2)$ is a convex function on $[0,1]$.*

Proof. "$\Rightarrow$" For $\forall\, \lambda_1, \lambda_2 \in [0,1]$, $\alpha \in [0,1]$, there is

$$\begin{aligned}\varphi(\alpha\lambda_1 + (1-\alpha)\lambda_2) &= f((\alpha\lambda_1 + (1-\alpha)\lambda_2)x_1 + (1-(\alpha\lambda_1 + (1-\alpha)\lambda_2))x_2) \\ &= f(\alpha(\lambda_1 x_1 + (1-\lambda_1)x_2) + (1-\alpha)(\lambda_2 x_1 + (1-\lambda_2)x_2)) \\ &\le \alpha\varphi(\lambda_1) + (1-\alpha)\varphi(\lambda_2),\end{aligned}$$

where we used the convexity of f in the last inequality. Hence $\varphi(\lambda)$ is a convex function on $[0,1]$.

"$\Leftarrow$" It is easy to see that

$$\begin{aligned}f(\lambda x_1 + (1-\lambda)x_2) &= \varphi(\lambda) = \varphi(\lambda \cdot 1 + (1-\lambda)\cdot 0) \\ &\le \lambda\varphi(1) + (1-\lambda)\varphi(0) \\ &= \lambda f(x_1) + (1-\lambda) f(x_2),\end{aligned}$$

i.e. $f(x)$ is a convex function on (a,b). □

Problem 10.2. *Suppose that $f(x)$ is a strict convex function on I. Prove that if $f(x)$ has minimum $f(x_0)$, then $f(x_0)$ is unique, i.e. $\forall\, x \in I\backslash\{x_0\}$, there is $f(x) > f(x_0)$.*

Proof. Prove by contradiction. If there exists $x_1 \in I\backslash\{x_0\}$, such that $f(x_1) \le f(x_0)$. Then $\forall\, \lambda \in (0,1)$, there is

$$f(\lambda x_0 + (1-\lambda)x_1) < \lambda f(x_0) + (1-\lambda) f(x_1) \le f(x_0).$$

Then for any neighborhood of x_0, say $U(x_0,\delta)$ $(0<\delta<|x_1-x_0|)$, in I, we know that if we let $\lambda : 1-\lambda < \dfrac{\delta}{|x_1 - x_0|}$, and take $x = \lambda x_0 + (1-\lambda)x_1$, then there is $x \in U(x_0,\delta)$ and

$$f(x) = f(\lambda x_0 + (1-\lambda)x_1) < f(x_0),$$

which contradicts with $f(x_0)$ is the minimum of $f(x)$. □

Problem 10.3 (Challenge!)**.** *Suppose that $f(x) = \dfrac{1}{1+e^x}$.*

(i) *Prove that $f(x)$ is a convex function on $[0,+\infty)$. Moreover, there is $f(x) + f(y) \le f(0) + f(x+y)$, $\forall\, x, y \ge 0$.*

(ii) *Assume $n \ge 3$, determine the set $E = \left\{ \sum_{k=1}^{n} f(x_k) \,\middle|\, \sum_{k=1}^{n} x_k = 0,\ x_1, \cdots, x_n \in \mathbb{R} \right\}$.*

Proof. (i) Note that

$$f'(x) = -\frac{e^x}{(1+e^x)^2}, \quad f''(x) = \frac{e^x(e^x-1)}{(1+e^x)^3}.$$

When $x \geq 0$, there is $f''(x) \geq 0$, hence $f(x)$ is a convex function on $[0, +\infty)$. For $x, y \geq 0$, we define

$$g(x) = f(x+y) - f(x) - f(y) + f(0).$$

Then

$$g'(x) = f'(x+y) - f'(x) \geq 0,$$

since $f'(x)$ is increasing on $[0, +\infty)$. Hence $g(x) \geq g(0) = 0$, i.e. $f(x) + f(y) \leq f(0) + f(x+y)$.

(ii) By the continuity of f, it's easy to see that E is an interval. Hence, it suffices to find the infimum and supremum of E. Note that $x_1 + \cdots + x_n = 0$.

If $x_1 = x_2 = \cdots = x_n = 0$, then $\sum_{j=1}^{n} f(x_j) = \frac{n}{2}$.

If $x_1, \cdots, x_n$ are not all zero, we assume that the number of negative is k, and the number of nonnegative is m, then $k+m = n$, $1 \leq k \leq n-1$. Without loss of generality, we may assmue that $x_1, \cdots, x_m \geq 0$, $x_{m+1}, \cdots, x_n < 0$. Denote $y_1 = -x_{m+1}, \cdots, y_k = -x_n$, $x = x_1 + \cdots + x_m = y_1 + \cdots + y_k$. By (i), we have

$$f(y_1) + \cdots + f(y_k) \leq (k-1)f(0) + f(x).$$

Note that $mf\left(\frac{x}{m}\right) - f(x)$ is strictly decreasing on $[0, +\infty)$ and $f(x) + f(-x) = 1$, we have

$$\begin{aligned}
&\sum_{j=1}^{n} f(x_j) = \sum_{j=1}^{m} f(x_j) + k - \sum_{j=1}^{k} f(y_j) \\
\geqslant& mf\left(\frac{x}{m}\right) + k - ((k-1)f(0) + f(x)) \\
>& \lim_{u\to+\infty} \left[mf\left(\frac{u}{m}\right) + k - ((k-1)f(0) + f(u)) \right] \\
=& \frac{k+1}{2} \geqslant 1,
\end{aligned}$$

which implies that $\inf E \geq 1$ and $1 \notin E$. On the other hand, for $u > 0$, let $x_1 = x_2 = \cdots = x_{n-1} = \frac{u}{n-1}$, $x_n = -u$, we have

$$\lim_{u\to+\infty} \sum_{j=1}^{n} f(x_j) = \lim_{u\to+\infty} \left((n-1)f\left(\frac{u}{n-1}\right) + 1 - f(u) \right) = 1,$$

thus $\inf E = 1$. Note that $f(-x) = 1 - f(x)$, we have

$$E = \{n - z | z \in E\}.$$

Hence, $\sup E = n-1$, and $n-1 \notin E$. Therefore, we know that $E = (1, n-1)$. □

Exercise 10.4 (Challenge!). *Suppose that $f(x)$ is a concave function on $[a,b]$ satisfying $f(a)=0$, $f(b)>0$ and the right derivative of $f(x)$ at $x=a$ is non-zero. For $n \geq 2$, denote*

$$S_n = \left\{ \sum_{k=1}^{n} k x_k : \sum_{k=1}^{n} k f(x_k) = f(b), x_k \in [a,b] \right\}.$$

(i) *Prove that for $\forall\, \alpha \in (0, f(b))$, there exists a unique $x \in (a,b)$ such that $f(x) = \alpha$.*

(ii) *Find $\lim\limits_{n\to\infty} (\sup S_n - \inf S_n)$.*

Hint: (i) Since $f(x)$ is continuous on $[a,b]$, we know that $\forall\, \alpha \in (0, f(b))$, there exists at least one point $\xi \in (a,b)$ such that $f(\xi)=\alpha$. Next, we prove that it is unique. Assume that there are $\xi, \eta \in (a,b)$ satisfying $\xi < \eta$ and $f(\xi) = f(\eta) = \alpha$. Then the point $(\eta, f(\eta)) = (\eta, \alpha)$ lies below in the segment connecting $(\xi, f(\xi)) = (\xi, \alpha)$ and $(b, f(b))$, contradicts with the concavity of f.

(ii) Denote

$$T_n = \left\{ (x_1, \cdots, x_n) : \sum_{k=1}^{n} k f(x_k) = f(b), x_k \in [a,b] \right\}, \quad n \geq 2.$$

For $\forall\, (x_1, \cdots, x_n) \in T_n$, by the concavity of $f(x)$, we have

$$\frac{2f(b)}{n(n+1)} = \frac{\sum_{k=1}^{n} k f(x_k)}{1+2+\cdots+n} \leq f\left(\frac{x_1 + 2x_2 + \cdots + n x_n}{1+2+\cdots+n} \right).$$

Hence

$$\frac{x_1 + 2x_2 + \cdots + n x_n}{1+2+\cdots+n} \geq f^{-1}\left(\frac{2f(b)}{n(n+1)} \right),$$

i.e.

$$\sum_{k=1}^{n} k x_k \geq \frac{n(n+1)}{2} f^{-1}\left(\frac{2f(b)}{n(n+1)} \right).$$

It is easy to see that "=" holds iff $x_1 = x_2 = \cdots = x_n = f^{-1}\left(\frac{2f(b)}{n(n+1)} \right)$. Note that $\left(f^{-1}\left(\frac{2f(b)}{n(n+1)} \right), \cdots, f^{-1}\left(\frac{2f(b)}{n(n+1)} \right) \right) \in T_n$, hence

$$\inf S_n = \frac{n(n+1)}{2} f^{-1}\left(\frac{2f(b)}{n(n+1)} \right).$$

On the other hand, by the concavity of $f(x)$, we have

$$\frac{f(b)}{b-a}(x-a) \leq f(x),$$

i.e

$$x \leq \frac{b-a}{f(b)} f(x) + a.$$

Hence

$$\sum_{k=1}^{n} k x_k \leq \frac{b-a}{f(b)} \sum_{k=1}^{n} k f(x_k) + \frac{n(n+1)}{2} a = b - a + \frac{n(n+1)}{2} a.$$

Note that "=" holds iff $x_1 = b, x_2 = x_3 = \cdots = x_n = a$, and $(b, a, a, \cdots, a) \in T_n$, then

$$\sup S_n = b - a + \frac{n(n+1)}{2} a.$$

Therefore, we have

$$\begin{aligned}
&\lim_{n\to\infty} (\sup S_n - \inf S_n) = b - a + \lim_{n\to\infty} \frac{n(n+1)}{2} \left(a - f^{-1}\left(\frac{2f(b)}{n(n+1)} \right) \right) \\
&= b - a + f(b) \lim_{n\to\infty} \frac{a - f^{-1}\left(\frac{2f(b)}{n(n+1)}\right)}{\frac{2f(b)}{n(n+1)}} \\
&= b - a + f(b) \lim_{x\to 0+0} \frac{a - f^{-1}(x)}{x} = b - a + f(b) \lim_{t\to a+0} \frac{a-t}{f(t)} \\
&= b - a - \frac{f(b)}{f'(a)}.
\end{aligned}$$

□

Problem 10.5. *Suppose that x, y, $z > 0$. Given $x + y + z = 1$, prove that:*

$$\frac{1}{x^2+y^2+z^2} + \frac{3}{xy+yz+zx} \geq 12.$$

Proof. First note that $(x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx)$, if we denote $t = xy+yz+zx$, we have

$$\frac{1}{x^2+y^2+z^2} + \frac{3}{xy+yz+zx} = \frac{1}{1-2t} + \frac{3}{t}$$

since $x+y+z=1$. By $xy+yz+zx \leq \frac{1}{3}(x+y+z)^2 = \frac{1}{3}$, we know that it suffices to minimize

$$f(t) := \frac{1}{1-2t} + \frac{3}{t}, \quad 0 < t \leq \frac{1}{3}.$$

Differentiating directly yields

$$f'(t) = \frac{2}{(1-2t)^2} - \frac{1}{t^2} = \frac{-10t^2+12t-3}{(1-2t)^2 t^2} < 0, \quad 0 < t \leq \frac{1}{3}.$$

Hence we know $f(t) \geq f(\frac{1}{3}) = \frac{1}{1-\frac{2}{3}} + 9 = 12$. Then we are done! Finally, it is easy to see that " = " holds iff $x = y = z = \frac{1}{3}$. □

Problem 10.6. *If $a, b, c > 0$ and $2abc + 3(ab + ac + bc) = 27$. Prove that*
$$16(a^2 + b^2 + c^2) + 8abc \geq 135.$$

Proof. First by
$$a^2 + b^2 \geq 2ab,\ b^2 + c^2 \geq 2bc,\ c^2 + a^2 \geq 2ca,$$
we get
$$a^2 + b^2 + c^2 \geq ab + bc + ca.$$
Then combining with $2abc + 3(ab + bc + ca) = 27$ yields
$$\begin{aligned} 16(a^2 + b^2 + c^2) + 8abc &= 16(a^2 + b^2 + c^2) + 4(27 - 3(ab + bc + ca)) \\ &= 16(a^2 + b^2 + c^2) - 12(ab + bc + ca) + 108 \\ &\geq 4(a^2 + b^2 + c^2) + 108. \end{aligned}$$
Hence it suffices to prove $a^2+b^2+c^2 \geq \frac{27}{4}$. Assume by contradiction that $a^2+b^2+c^2 < \frac{27}{4}$, and note that $a^2 + b^2 + c^2 \geq 3\sqrt[3]{a^2b^2c^2}$, we have
$$\begin{aligned} 2abc + 3(ab + bc + ca) &\leq 2\left(\frac{a^2 + b^2 + c^2}{3}\right)^{\frac{3}{2}} + 3(a^2 + b^2 + c^2) \\ &< 2\left(\frac{9}{4}\right)^{\frac{3}{2}} + 3 \times \frac{27}{4} \\ &= 2 \times \frac{27}{8} + 3 \times \frac{27}{4} = 27 \end{aligned}$$
contradict with $2abc+3(ab+bc+ca) = 27$. Hence $a^2 + b^2 + c^2 \geq \frac{27}{4}$. Then we are done. Then it is easy to check that " $=$ " holds iff $(a, b, c) = (\frac{3}{2}, \frac{3}{2}, \frac{3}{2})$. □

Remark 10.7. *We can also get $a^2+b^2+c^2 \geq \frac{27}{4}$ by solving the inequality $2\left(\frac{a^2+b^2+c^2}{3}\right)^{\frac{3}{2}} + 3(a^2 + b^2 + c^2) \geq 27$.*

Problem 10.8. *Prove the following inequalities:*

(1) $\displaystyle\sum_{k=1}^{n}\left(x_k + \frac{1}{x_k}\right)^{\alpha} \geq \frac{(n^2+1)^{\alpha}}{n^{\alpha-1}}$, $(\alpha > 1,\ x_1 + \cdots + x_n = 1)$;

(2) $\displaystyle 1 + \left(\sum_{k=1}^{n} p_k x_k\right)^{-1} \leq \prod_{k=1}^{n}\left(\frac{1 + x_k}{x_k}\right)^{p_k}$, $(p_k > 0\, 0 < x_k < 1,\ p_1 + \cdots + p_k = 1)$;

(3) $(\sin x)^{1-\cos 2x} + (\cos x)^{1+\cos 2x} \geq \sqrt{2}$, *where* $x \in (0, \pi/2)$;

(4) $2^n \geq 1 + n\sqrt{2^{n-1}}$, $n \in \mathbb{N}$;

(5) $\left|\sum_{k=1}^{n} \frac{\sin kx}{k}\right| \leq 2\sqrt{2\pi}$, $x \in \mathbb{R}$, $n \in \mathbb{N}$.

Proof. (1) Since $f(x) = (x + 1/x)^{\alpha}$ is convex on $(0, +\infty)$, we have

$$\left(\frac{n^2+1}{n}\right)^{\alpha} = \left[\frac{1}{n}\sum_{k=1}^{n} x_k + \left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)^{-1}\right]^{\alpha} \leq \frac{1}{n}\sum_{k=1}^{n}\left(x_k + \frac{1}{x_k}\right)^{\alpha}.$$

(2) Since $f(x) = \ln(1 + 1/x)$ is convex on $(0, +\infty)$, we have

$$\ln\left(1 + \left(\sum_{k=1}^{n} p_k x_k\right)^{-1}\right) \leq \sum_{k=1}^{n} p_k \cdot \ln\left(1 + \frac{1}{x_k}\right) = \ln\left(\prod_{k=1}^{n}\left(\frac{1+x_k}{x_k}\right)^{p_k}\right).$$

(3) Note that

$$(\sin x)^{1-\cos 2x} + (\cos x)^{1+\cos 2x} = (\sin^2 x)^{sin^2 x} + (\cos^2 x)^{\cos^2 x}.$$

Since $f(x) = x^x$ is convex on $(0, +\infty)$, we have

$$\left(\frac{1}{2}\right)^{\frac{1}{2}} \leq \frac{1}{2}\left((\sin^2 x)^{sin^2 x} + (\cos^2 x)^{\cos^2 x}\right).$$

(4) Let

$$f(x) = 2^x - 1 - x\sqrt{2^{x-1}}, \quad (x \geq 1).$$

Then

$$f'(x) = 2^{\frac{x-1}{2}}\left(2^{\frac{x+1}{2}}\ln 2 - 1 - \frac{x}{2}\ln 2\right).$$

Let

$$g(x) = 2^{\frac{x+1}{2}}\ln 2 - 1 - \frac{x}{2}\ln 2, \quad (x \geq 1).$$

There is

$$g'(x) = 2^{\frac{x+1}{2}}(\ln 2)^2 \cdot \frac{1}{2} - \frac{1}{2}\ln 2 > 0, \quad (x \geq 1).$$

Hence g is increasing on $[1, +\infty)$, then $g(x) \geq g(1) = \frac{3}{2}\ln 2 - 1 > 0$. Therefore, we know that $f'(x) > 0$, i.e. f is also increasing on $[1, +\infty)$. Hence $f(x) \geq f(1) = 0$, i.e. $2^n \geq 1 + n\sqrt{2^{n-1}}$, $n \in \mathbb{N}$.

(5) It suffices to consider $x \in [0, \pi]$ since $f(x) = \left|\sum_{k=1}^{n} \frac{\sin kx}{k}\right|$ is an even function, and it has period, 2π. When $x = 0, \pi$, the inequality is clearly. Now, we assume that $0 < x < \pi$. We know that there must be some $m \in \mathbb{N}$ such that $m \leq \frac{\sqrt{2\pi}}{x} < m + 1$. Hence

$$\left|\sum_{k=1}^{n} \frac{\sin kx}{k}\right| = \sum_{k=1}^{m}\left|\frac{\sin kx}{k}\right| + \left|\sum_{k=m+1}^{n} \frac{\sin kx}{k}\right|.$$

When $m = 0$, the first formula of RHS is 0, when $m \geq n$, the second formula of RHS is 0. Note that

$$|\sin x| \leq |x| \quad \text{and} \quad \sin x > \frac{2}{\pi}x \ (0 < x < \pi/2).$$

We have

$$\sum_{k=1}^{m} \left|\frac{\sin kx}{k}\right| \leq \sum_{k=1}^{m} \frac{kx}{k} < \sqrt{2\pi},$$

and

$$\begin{aligned}\left|\sum_{k=m+1}^{n} \frac{\sin kx}{k}\right| &= \left|\sum_{k=m+1}^{n-1} S_k \left(\frac{1}{k} - \frac{1}{k+1}\right) + S_n \cdot \frac{1}{n} - S_m \cdot \frac{1}{m+1}\right| \\ &< \frac{1}{\left|\sin \frac{x}{2}\right|} \cdot \frac{2}{m+1} < \frac{\pi}{x} \cdot \frac{2x}{\sqrt{2\pi}} = \sqrt{2\pi},\end{aligned}$$

where $S_k = \sin x + \sin 2x + \cdots + \sin kx$ and $|S_k| < 1/\left|\sin \frac{x}{2}\right|$ (Leave to the reader). Then we have

$$\left|\sum_{k=1}^{n} \frac{\sin kx}{k}\right| \leq 2\sqrt{2\pi}.$$

□

Problem 11.1. *Assume that f is twice diffentiable on $\mathbb{R}$, and such*

$$2f(x) + f''(x) = -xf'(x).$$

Prove that $f(x)$and $f'(x)$ are bounded on $\mathbb{R}$.

Proof. Define

$$g = f^2 + \frac{1}{2}(f')^2.$$

By definition, g is non-negative and differentiable; moreover,

$$g'(x) = f'(x) \cdot (2f(x) + f''(x)) = -x \cdot (f'(x))^2, \quad \forall x \in \mathbb{R}.$$

Therefore, g is increasing on $(-\infty, 0]$ and decreasing on $[0, +\infty)$, so $g(\mathbb{R}) \subset [0, g(0)]$. The conclusion follows. □

Problem 11.2. *Define $f \in C^2[a, b]$ satisfying $f''(x) = e^x f(x)$. Show that $f''(x) = e^x f(x)$ with $f(a) = f(b) = 0$ makes $f \equiv 0$ $\forall x \in [a, b]$.*

Proof. Assume that f is not identically zero on the interval, without loss of generality $f(x_0) > 0$ for some $x_0 \in (a, b)$. Then f attains its maximum $M > 0$ at some point $x_1 \in (a, b)$. At the maximum, we necessarily have

$$f'(x_1) = 0\,, \quad f''(x_1) \leq 0\,,$$

which is a contradiction to the assumption that

$$f''(x_1) = e^{x_1} f(x_1) = e^{x_1} M > 0\,.$$

□

Remark 11.3. *Actually, we can prove a general conclusion: if $f \in C^2[a, b]$ satisfying $f''(x) = g(x)f(x)$ where $g(x) \in C^0[a, b]$ satisfying $g(x) > 0$, and $f(a) = f(b) = 0$, we have $f \equiv 0$ $\forall x \in [a, b]$.*

Problem 11.4. *Suppose that there is equation*

$$x(1 - \ln(\varepsilon\sqrt{x})) = 1, \quad (x > 0,\, \varepsilon > 0). \tag{11.1}$$

Then

(i) *For small enough ε, (11.1) has two solutions (denote the small one as x_ε);*
(ii) $\lim_{\varepsilon \to 0+0} x_\varepsilon = 0$;
(iii) $\lim_{\varepsilon \to 0+0} \varepsilon^{-t} x_\varepsilon = +\infty (t > 0)$.

Proof. (i) For $x > 0$, we have $\varepsilon = e/\sqrt{x}e^{1/x} =: f(x)$. Denote $F(x) = \sqrt{x}e^{1/x} = e/f(x)$, there is

$$F'(x) = x^{-\frac{3}{2}} e^{\frac{1}{x}} \frac{x-2}{2}.$$

Hence $F(x)$ is strictly decreasing on $(0, 2]$ and it is strictly increasing on $[2, +\infty)$. Note that

$$\lim_{x \to 0+0} F(x) = +\infty = \lim_{x \to +\infty} F(x), \quad F(2) > 0,$$

we have $\lim_{x \to 0+0} f(x) = 0 = \lim_{x \to +\infty} f(x)$ and $f(x)$ is strictly increasing on $(0, 2]$ and it is strictly decreasing on $[2, +\infty)$, $f(2) = \sqrt{e/2}$. Let

$$f_1(x) = \begin{cases} f(x), & x \in (0,2), \\ 0, & x \in [2, +\infty), \end{cases} \quad f_2(x) = \begin{cases} 0, & x \in (0,2], \\ f(x), & x \in (2, +\infty). \end{cases}$$

Then for $0 < \varepsilon < \sqrt{e/2}$, (11.1) has two solutions: $x = f_1^{-1}(\varepsilon)$, $x = f_2^{-1}(\varepsilon)$. The small one is $x_\varepsilon = f_1^{-1}(\varepsilon)$.

(ii) Since f_1 is strictly increasing on $(0, 2)$ and it is continuous, we know $\lim_{\varepsilon \to 0+0} x_\varepsilon = 0$ by $f_1(0 + 0) = 0$.

(iii) For $t > 0$, we have

$$\varepsilon^{-t} x_\varepsilon = \left(\frac{\mathrm{e}}{\sqrt{x}\mathrm{e}^{1/x}} \right)^{-t} x = \mathrm{e}^{-t} x^{1+t/2} \mathrm{e}^{t/x} \to +\infty, \quad \text{as } x \to 0+0.$$

□

Problem 11.5. *Draw the graph of* $f(x) = |x + 2|e^{-\frac{1}{x}}$.

Solution. Note that $f(x) \geq 0$ and $f(-2) = 0$. Hence 0 is the minimum of $f(x)$ and $x = -2$ is the minimal point. Rewrite $f(x)$ as

$$f(x) = \begin{cases} (x+2)e^{-\frac{1}{x}}, & x \in [-2, 0) \cup (0, +\infty), \\ -(x+2)e^{-\frac{1}{x}}, & x \in (-\infty, -2). \end{cases}$$

When $x \in [-2, 0) \cup (0, +\infty)$, there is

$$f'(x) = \frac{e^{-\frac{1}{x}}(x^2 + x + 2)}{x^2} > 0.$$

Hence $f(x)$ is strictly inreasing on $x \in [-2, 0) \cup (0, +\infty)$. When $x \in (-\infty, -2)$, there is

$$f'(x) = -\frac{e^{-\frac{1}{x}}(x^2 + x + 2)}{x^2} < 0.$$

Hence $f(x)$ is strictly dereasing on $x \in (-\infty, -2)$.

It is easy to see that

$$\lim_{h \to 0+0} \frac{f(-2+h) - f(-2)}{h} = \lim_{h \to 0+0} \frac{he^{-\frac{1}{h-2}}}{h} = e^{\frac{1}{2}},$$

$$\lim_{h\to 0-0}\frac{f(-2+h)-f(-2)}{h}=\lim_{h\to 0-0}\frac{-he^{-\frac{1}{h-2}}}{h}=-e^{\frac{1}{2}},$$

which gives us that $f(x)$ is nondifferentiable at $x=-2$. Note that

$$\lim_{x\to 0+0} f(x)=\lim_{x\to 0+0}\frac{2+x}{e^{\frac{1}{x}}}=0,$$

$$\lim_{x\to 0-0} f(x)=\lim_{x\to 0-0}\frac{2+x}{e^{\frac{1}{x}}}=+\infty.$$

Hence $x=0$ is a vertical asymptote of $f(x)$. Note that

$$\lim_{x\to +\infty}\frac{f(x)}{x}=\lim_{x\to +\infty}\frac{x+2}{x}e^{-\frac{1}{x}}=1,$$

$$\lim_{x\to +\infty}(f(x)-x)=\lim_{x\to +\infty}((x+2)e^{-\frac{1}{x}}-x)=1.$$

Hence $y=x+1$ is a oblique asymptote of $f(x)$.

When $x\geq -2$, $f''(x)=\left(\frac{2}{x^4}-\frac{3}{x^3}\right)e^{-\frac{1}{x}}$. Hence we know that when $-2\leq x\leq \frac{2}{3}$, $f(x)$ is convex; $x>\frac{2}{3}$, $f(x)$ is concave. When $x<-2$, $f''(x)=-\left(\frac{2}{x^4}-\frac{3}{x^3}\right)e^{-\frac{1}{x}}<0$. Hence $f(x)$ is concave.

Combining above, we have the graph of f is as follows:

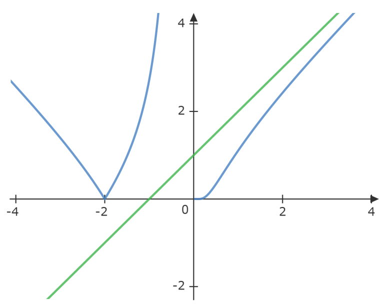

FIGURE 12. Graph of $f(x)$

□

Problem 11.6. *Calculate the following integrals.*

(1) $\int e^{\sqrt{x+1}}\,\mathrm{d}x$;

(2) $\int \frac{\mathrm{d}x}{x^2\sqrt{x^2+x-1}}$;

(3) $\int \frac{x \arctan x}{\left(1+x^2\right)^2} \mathrm{~d}x$;

(4) $\int x \tan^2 x \mathrm{~d}x$;

(5) $\int \frac{\ln(\sin x)}{\sin^2 x} \mathrm{~d}x$;

(6) $\int \frac{\mathrm{d}x}{x\left(1+x^8\right)}$;

(7) $\int \frac{\mathrm{d}x}{\sqrt{e^x-1}}$;

(8) $\int \frac{\sin x}{\sin x-\cos x} \mathrm{~d}x$;

(9) $\int \frac{1+\cos x}{1+\sin x} \mathrm{~d}x$

(10) $\int \frac{e^x(1+\sin x)}{1+\cos x} \mathrm{~d}x$;

(11) $\int \frac{e^{3x}+e^x}{e^{4x}-e^{2x}+1} \mathrm{~d}x$;

(12) $\int \frac{1-\ln x}{(x-\ln x)^2} \mathrm{~d}x$;

(13) $\int \frac{x+\sin x \cos x}{(\cos x-x \sin x)^2} \mathrm{~d}x$;

(14) $\int x^2 e^x \cos 2x \mathrm{~d}x$;

(15) $\int \frac{e^x\left(2-x^2\right)}{(1-x)\sqrt{1-x^2}} \mathrm{~d}x$;

(16) $\int \frac{(1+x)\mathrm{d}x}{x^2 e^x\left(1+x e^x\right)}$;

(17) $\int \frac{x^2 \sin^2 x}{(x+\sin x \cos x)^2} \mathrm{~d}x$;

(18) $\int \frac{x^2}{(x \cos x-\sin x)(x \sin x+\cos x)} \mathrm{d}x$;

(19) $I=\int \frac{\sin^3 x}{\sin^3 x-\cos^3 x} \mathrm{~d}x, \quad J=\int \frac{\cos^3 x}{\sin^3 x-\cos^3 x} \mathrm{~d}x$.

Solution. (1) By changing of variable and integral by parts, we have

$$\begin{aligned}
\int e^{\sqrt{x+1}} \mathrm{~d}x \xlongequal{t=\sqrt{x+1}} & \int 2te^t \mathrm{~d}t \\
= & 2te^t-2\int e^t \mathrm{~d}t \\
= & (2t-2)e^t+C
\end{aligned}$$

$$=(2\sqrt{x+1}-2)e^{\sqrt{x+1}}+C.$$

(2) By changing of variable, we have

$$\begin{aligned}\int \frac{\mathrm{d}x}{x^2\sqrt{x^2+x-1}} &= \int \frac{\mathrm{d}x}{x^3\sqrt{1+\frac{1}{x}-\frac{1}{x^2}}} \overset{t=\frac{1}{x}}{=\!=\!=} -\int \frac{t}{\sqrt{1+t-t^2}}\ \mathrm{d}t \\ &= -\int \frac{t-\frac{1}{2}}{\sqrt{\frac{5}{4}-(t-\frac{1}{2})^2}}\ \mathrm{d}t - \int \frac{\frac{1}{2}}{\sqrt{\frac{5}{4}-(t-\frac{1}{2})^2}}\ \mathrm{d}t \\ &= \sqrt{\frac{5}{4}-(t-\frac{1}{2})^2} - \frac{1}{2}\arcsin\frac{2}{\sqrt{5}}(t-\frac{1}{2})+C \\ &= \frac{\sqrt{x^2+x-1}}{x} - \frac{1}{2}\arcsin\frac{2}{\sqrt{5}}(\frac{1}{x}-\frac{1}{2})+C.\end{aligned}$$

(3) By integral by parts, we have

$$\begin{aligned}\int \frac{x\arctan x}{(1+x^2)^2}\ \mathrm{d}x &= -\frac{1}{2(1+x^2)}\arctan x + \frac{1}{2}\int \frac{1}{(1+x^2)^2}\ \mathrm{d}x \\ &= -\frac{1}{2(1+x^2)}\arctan x + \frac{1}{4}\int \frac{1+x^2+1-x^2}{(1+x^2)^2}\ \mathrm{d}x \\ &= -\frac{1}{2(1+x^2)}\arctan x + \frac{1}{4}\int \frac{1}{1+x^2}\ \mathrm{d}x + \frac{1}{4}\int \frac{1-x^2}{(1+x^2)^2}\ \mathrm{d}x \\ &= -\frac{1}{2(1+x^2)}\arctan x + \frac{1}{4}\arctan x + \frac{x}{4(1+x^2)} + C.\end{aligned}$$

(4) By integral by parts, we have

$$\begin{aligned}\int x\tan^2 x\ \mathrm{d}x &= \int x(\sec^2 x - 1)\ \mathrm{d}x \\ &= \int x\sec^2 x\ \mathrm{d}x - \frac{1}{2}x^2 \\ &= -\frac{1}{2}x^2 + x\tan x - \int \tan x\ \mathrm{d}x \\ &= -\frac{1}{2}x^2 + x\tan x + \ln\cos x + C.\end{aligned}$$

(5) By integral by parts, we have

$$\begin{aligned}\int \frac{\ln(\sin x)}{\sin^2 x}\ \mathrm{d}x &= \int \csc^2 x\ln(\sin x)\ \mathrm{d}x \\ &= -\cot x\ln(\sin x) + \int \cot^2 x\ \mathrm{d}x\end{aligned}$$

$$= -\cot x \ln(\sin x) + \int (\csc^2 x - 1)\,\mathrm{d}x$$
$$= -x - \cot x - \cot x \ln(\sin x) + C.$$

(6) By changing of variable, we have

$$\int \frac{\mathrm{d}x}{x\,(1+x^8)} = \int \frac{x^7 \mathrm{d}x}{x^8\,(1+x^8)}$$
$$\xlongequal{t=x^8} \frac{1}{8}\int \frac{\mathrm{d}t}{t\,(1+t)}$$
$$= \frac{1}{8}\ln\left|\frac{t}{t+1}\right| + C$$
$$= \frac{1}{8}\ln\left(\frac{x^8}{x^8+1}\right) + C.$$

(7) For the integrand $\frac{1}{\sqrt{e^x-1}}$, substitute $u = e^x$ and $du = e^x dx$

$$\int \frac{\mathrm{d}x}{\sqrt{e^x-1}} = \int \frac{1}{\sqrt{u-1}u} du.$$

For the integrand $\frac{1}{\sqrt{u-1}u}$, substitute $s = u - 1$ and $ds = du$, we have

$$\int \frac{1}{\sqrt{u-1}u} du = \int \frac{1}{\sqrt{s}(s+1)} ds.$$

For the integrand $\frac{1}{\sqrt{s}(s+1)}$, substitute $p = \sqrt{s}$ and $dp = \frac{1}{2\sqrt{s}} ds$,we know

$$\int \frac{1}{\sqrt{s}(s+1)} ds = 2\int \frac{1}{p^2+1} dp.$$

The integral of $\frac{1}{p^2+1}$ is $\tan^{-1}(p)$, then

$$2\int \frac{1}{p^2+1} dp = 2\tan^{-1}(p) + C.$$

Substitute back for $p = \sqrt{s}$,

$$\int \frac{1}{\sqrt{s}(s+1)} ds = 2\tan^{-1}(\sqrt{s}) + C.$$

Substitute back for $s = u - 1$,

$$\int \frac{1}{\sqrt{u-1}u} du = 2\tan^{-1}(\sqrt{u-1}) + C.$$

Substitute back for $u = e^x$, we have Answer:

$$\int \frac{\mathrm{d}x}{\sqrt{e^x-1}} = 2\tan^{-1}\left(\sqrt{e^x-1}\right) + C.$$

(8)

$$\begin{aligned}\int \frac{\sin x}{\sin x - \cos x}\ \mathrm{d}x &= \int \frac{\sin(x - \frac{\pi}{4} + \frac{\pi}{4})}{\sqrt{2}\sin(x - \frac{\pi}{4})}\ \mathrm{d}x \\ &= \int \frac{\frac{\sqrt{2}}{2}\sin(x - \frac{\pi}{4}) + \frac{\sqrt{2}}{2}\cos(x - \frac{\pi}{4})}{\sqrt{2}\sin(x - \frac{\pi}{4})}\ \mathrm{d}x \\ &= \frac{1}{2}x + \frac{1}{2}\ln\sin(x - \frac{\pi}{4}) + C.\end{aligned}$$

(9)

$$\begin{aligned}\int \frac{1 + \cos x}{1 + \sin x}\ \mathrm{d}x &= \int \frac{1}{1 + \sin x}\ \mathrm{d}x + \int \frac{\cos x}{1 + \sin x}\ \mathrm{d}x \\ &= \int \frac{1}{1 + \cos(x - \frac{\pi}{2})}\ \mathrm{d}x + \ln(1 + \sin x) \\ &= \int \frac{1}{2\cos^2(\frac{x}{2} - \frac{\pi}{4})} + \ln(1 + \sin x) \\ &= \tan\left(\frac{x}{2} - \frac{\pi}{4}\right) + \ln(1 + \sin x) + C.\end{aligned}$$

(10) By integral by parts, we have

$$\begin{aligned}\int \frac{e^x(1 + \sin x)}{1 + \cos x}\ \mathrm{d}x &= \int \frac{e^x(\sin\frac{x}{2} + \cos\frac{x}{2})^2}{2\cos^2\frac{x}{2}}\ \mathrm{d}x \\ &= \frac{1}{2}\int e^x\left(1 + \tan\frac{x}{2}\right)^2\ \mathrm{d}x \\ &= \frac{1}{2}\int e^x\left(1 + \tan^2\frac{x}{2}\right)\ \mathrm{d}x + \int e^x\tan\frac{x}{2}\ \mathrm{d}x \\ &= \frac{1}{2}\int e^x\sec^2\frac{x}{2}\ \mathrm{d}x + \int e^x\tan\frac{x}{2}\ \mathrm{d}x \\ &= e^x\tan\frac{x}{2} - \int e^x\tan\frac{x}{2}\ \mathrm{d}x + \int e^x\tan\frac{x}{2}\ \mathrm{d}x \\ &= e^x\tan\frac{x}{2} + C.\end{aligned}$$

(11)

$$\begin{aligned}\int \frac{e^{3x} + e^x}{e^{4x} - e^{2x} + 1}\ \mathrm{d}x &= \int \frac{e^x + e^{-x}}{e^{2x} + e^{-2x} - 1}\ \mathrm{d}x \\ &= \int \frac{\mathrm{d}(e^x - e^{-x})}{1 + (e^x - e^{-x})^2} \\ &= \arctan(e^x - e^{-x}) + C.\end{aligned}$$

(12)

$$\begin{aligned}\int \frac{1-\ln x}{(x-\ln x)^2}\ \mathrm{d}x &= \int \frac{1-\ln x}{x^2}\frac{1}{(1-\frac{\ln x}{x})^2}\ \mathrm{d}x \\ &= \int \frac{1}{(1-\frac{\ln x}{x})^2}\ \mathrm{d}\left(\frac{\ln x}{x}\right) \\ &= \frac{x}{x-\ln x}+C.\end{aligned}$$

(13)

$$\begin{aligned}\int \frac{x+\sin x\cos x}{(\cos x - x\sin x)^2}\ \mathrm{d}x &= \int \frac{x\sec^2 x+\tan x}{(1-x\tan x)^2}\ \mathrm{d}x \\ &= \int \frac{\mathrm{d}(x\tan x)}{(1-x\tan x)^2} \\ &= \frac{1}{1-x\tan x}+C.\end{aligned}$$

(14) Firstly, we have by integral by parts that

$$\begin{aligned}\int x^2 e^{(1+2i)x}\ \mathrm{d}x &= \frac{x^2}{1+2i}e^{(1+2i)x} - \frac{2}{1+2i}\int xe^{(1+2i)x}\ \mathrm{d}x \\ &= \frac{x^2}{1+2i}e^{(1+2i)x} - \frac{2x}{(1+2i)^2} + \frac{2}{(1+2i)^2}\int e^{(1+2i)x}\ \mathrm{d}x \\ &= \frac{x^2}{1+2i}e^{(1+2i)x} - \frac{2x}{(1+2i)^2}e^{(1+2i)x} + \frac{2}{(1+2i)^3}e^{(1+2i)x}+C \\ &= \frac{25x^2+30x-22+(-50x^2+40x+4)i}{125}e^{(1+2i)x}+C \\ &= \frac{1}{125}e^x(25x^2+30x-22+(-50x^2+40x+4)i)(\cos 2x+i\sin 2x)+C \\ &= \frac{1}{125}e^x\left((25x^2+30x-22)\cos 2x+(50x^2-40x-4)\sin 2x\right) \\ &\quad+\frac{i}{125}e^x\left((-50x^2+40x+4)\cos 2x+(25x^2+30x-22)\sin 2x\right)+C.\end{aligned}$$

Hence

$$\int x^2 e^x\cos 2x\ \mathrm{d}x = \frac{1}{125}e^x\left((25x^2+30x-22)\cos 2x+(50x^2-40x-4)\sin 2x\right)+C.$$

(15) Note that

$$\mathrm{d}\left(e^x\sqrt{\frac{1+x}{1-x}}\right) = \frac{e^x\left(2-x^2\right)}{(1-x)\sqrt{1-x^2}}\ \mathrm{d}x,$$

we have

$$\int \frac{e^x\left(2-x^2\right)}{(1-x)\sqrt{1-x^2}}\ \mathrm{d}x = e^x\sqrt{\frac{1+x}{1-x}} + C.$$

(16)

$$\begin{aligned}\int \frac{(1+x)\mathrm{d}x}{x^2e^x\left(1+xe^x\right)} &= \int \frac{e^x(1+x)\mathrm{d}x}{x^2e^{2x}\left(1+xe^x\right)}\\ &= \int \frac{\mathrm{d}(xe^x)}{x^2e^{2x}\left(1+xe^x\right)}\\ &= \int \frac{\mathrm{d}(xe^x)}{x^2e^{2x}} - \int \frac{\mathrm{d}(xe^x)}{xe^x} + \int \frac{\mathrm{d}(xe^x)}{1+xe^x}\\ &= -\frac{1}{xe^x} - \ln(xe^x) + \ln(1+xe^x) + C.\end{aligned}$$

(17)

$$\begin{aligned}\int \frac{x^2\sin^2 x}{(x+\sin x\cos x)^2}\ \mathrm{d}x =& \frac{1}{2}\int \frac{x^2(1-\cos 2x)}{(x+\sin x\cos x)^2}\ \mathrm{d}x\\ =& \frac{1}{2}\int \frac{x^2(-1-\cos 2x)}{(x+\sin x\cos x)^2}\ \mathrm{d}x + \int \frac{x^2}{(x+\sin x\cos x)^2}\ \mathrm{d}x\\ =& \frac{1}{2}\cdot\frac{x^2}{x+\sin x\cos x} - \int \frac{x}{x+\sin x\cos x}\ \mathrm{d}x\\ &+ \int \frac{x^2}{(x+\sin x\cos x)^2}\ \mathrm{d}x\\ =& \frac{1}{2}\cdot\frac{x^2}{x+\sin x\cos x} + \int \frac{-x\sin x\cos x}{(x+\sin x\cos x)^2}\ \mathrm{d}x\\ =& \frac{1}{2}\cdot\frac{x^2}{x+\sin x\cos x}\\ &+ \int \frac{x\sin x\cos x}{1+\cos 2x}\cdot\frac{-1-\cos 2x}{(x+\sin x\cos x)^2}\ \mathrm{d}x\\ =& \frac{1}{2}\cdot\frac{x^2}{x+\sin x\cos x} + \frac{x\tan x}{2}\cdot\frac{1}{x+\sin x\cos x}\\ &- \frac{1}{2}\int (x\sec^2 x + \tan x)\cdot\frac{1}{x+\sin x\cos x}\ \mathrm{d}x\\ =& \frac{x^2}{2(x+\sin x\cos x)} + \frac{x\tan x}{2(x+\sin x\cos x)} - \frac{1}{2}\int \sec^2 x\ \mathrm{d}x\\ =& \frac{x^2}{2(x+\sin x\cos x)} + \frac{x\tan x}{2(x+\sin x\cos x)} - \frac{1}{2}\tan x + C\\ =& \frac{x^2-\sin^2 x}{2(x+\sin x\cos x)} + C.\end{aligned}$$

(18)

$$\int \frac{x^2}{(x\cos x - \sin x)(x\sin x + \cos x)}\mathrm{d}x = \int \frac{x\cos x}{x\sin x + \cos x}\mathrm{d}x - \int \frac{-x\sin x}{x\cos x - \sin x}\mathrm{d}x$$
$$= \ln|x\sin x + \cos| - \ln|x\cos - \sin x| + C.$$

(19) It is easy to see that $I - J = x + C$. Note that

$$\begin{aligned}
I + J =& \int \frac{\sin^3 x + \cos^3 x}{\sin^3 x - \cos^3 x}\mathrm{d}x \\
=& \int \frac{(\sin x + \cos x)(\sin^2 x - \sin x\cos x + \cos^2 x)}{(\sin x - \cos x)(\sin^2 x + \sin x\cos x + \cos^2 x)}\mathrm{d}x \\
=& \int \frac{(\sin^2 x - \cos^2 x)(1 - \frac{1}{2}\sin 2x)}{(\sin x - \cos x)^2(1 + \frac{1}{2}\sin 2x)}\mathrm{d}x \\
=& \int \frac{-\cos 2x(1 - \frac{1}{2}\sin 2x)}{(1 - \sin 2x)(1 + \frac{1}{2}\sin 2x)}\mathrm{d}x \\
=& \frac{1}{2}\int \frac{-\cos 2x}{1 + \frac{1}{2}\sin 2x}\mathrm{d}x - \frac{1}{2}\int \frac{\cos 2x}{(1 - \sin 2x)(1 + \frac{1}{2}\sin 2x)}\mathrm{d}x \\
=& -\frac{1}{2}\ln(1 + \frac{1}{2}\sin 2x) + \frac{1}{6}\ln(1 - \sin 2x) - \frac{1}{6}\ln(1 + \frac{1}{2}\sin 2x) + C \\
=& \frac{1}{6}\ln(1 - \sin 2x) - \frac{2}{3}\ln(1 + \frac{1}{2}\sin 2x) + C.
\end{aligned}$$

□

Problem 11.7. *Suppose that $F(x)$ ia a primitive function of $f(x)$ on $(0, +\infty)$, and $F(1) = \frac{\sqrt{2}\pi}{4}$. If there is*

$$f(x)F(x) = \frac{\arctan\sqrt{x}}{\sqrt{x}(1+x)}, \quad x \in (0, +\infty),$$

find $f(x)$.

Proof. A direct integrating yields

$$\frac{1}{2}F(x)^2 = \int \frac{\arctan\sqrt{x}}{\sqrt{x}(1+x)}\mathrm{d}x = (\arctan\sqrt{x})^2 + C.$$

By $F(1) = \frac{\sqrt{2}\pi}{4}$, we know that $C = 0$. Hence $F(x) = \sqrt{2}\arctan\sqrt{x}$, which gives us

$$f(x) = \frac{\sqrt{2}}{2\sqrt{x}(1+x)}.$$

□

Exercise 11.8. *Use an elementary way to show that for positive integer n,*

$$\int \frac{\sin(nx)\sin x}{1-\cos x}\,dx = x + \frac{\sin(nx)}{n} + 2\sum_{k=1}^{n-1}\frac{\sin(kx)}{k}$$

Hint: Let

$$F(x) = x + \frac{\sin(nx)}{n} + 2\sum_{k=1}^{n-1}\frac{\sin(kx)}{k}.$$

Then

$$F'(x) = 1 + \cos(nx) + 2\sum_{k=1}^{n-1}\cos(kx).$$

Therefore, we only need to prove that

$$\frac{\sin(nx)\sin x}{1-\cos x} = 1 + \cos(nx) + 2\sum_{k=1}^{n-1}\cos(kx),$$

or equivalently that

$$\sin(nx)\sin(x) = (1-\cos(x))\left(1 + \cos(nx) + 2\sum_{k=1}^{n-1}\cos(kx)\right). \tag{11.2}$$

Note that

$$z = \cos(x) + i\sin(x)$$

and calculate

$$\begin{aligned}
1 + 2z + 2z^2 + .. + 2z^{n-1} + z^n &= 2(1 + z + z^2 + .. + z^{n-1} + z^n) - 1 - z^n \\
&= 2\frac{1-z^{n+1}}{1-z} - 1 - z^n \\
&= \frac{2 - z^{n+1} - z^n}{1-z} - 1 = \frac{(2 - z^{n+1} - z^n)(1-\bar{z})}{(1-\cos(x))^2 + \sin(x)^2} - 1 \\
&= \frac{2 - z^{n+1} - z^n - 2\bar{z} + z^n + z^{n-1}}{2 - 2\cos(x)} - 1 \\
&= \frac{2 - z^{n+1} - 2\bar{z} + z^{n-1}}{2 - 2\cos(x)} - 1.
\end{aligned}$$

By taking the real parts we get:

$$\begin{aligned}\left(1+\cos(nx)+2\sum_{k=1}^{n-1}\cos(kx)\right) &= \frac{2-\cos((n+1)x)-2\cos(x)+\cos((n-1)x)-2+2\cos(x)}{2-2\cos(x)} \\ &= \frac{-\cos((n+1)x)+\cos((n-1)x)}{2-2\cos(x)} \\ &= \frac{2\sin(nx)\sin(x)}{2-2\cos(x)},\end{aligned}$$

which is exactly what we want. □

Remark 11.9. *We can also try to prove* (11.2) *by writing*

$$(1-\cos(x))\left(\sum_{k=1}^{n-1}\cos(kx)\right) = 2\sin^2(\frac{x}{2})\left(\sum_{k=1}^{n-1}\cos(kx)\right),$$

and use the fact that

$$\sum_{k=1}^{n-1}\cos(kx)\sin(\frac{x}{2})$$

is telescopic.

12. Week 15 (12.12)

Problem 12.1 (7.3)**.** *Use the definition of integral to calculate limitations.*

(1) $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right)$;

(2) $\lim_{n\to\infty}\frac{\sqrt[n]{n(n+1)\cdots(2n-1)}}{n}$;

(3) $\lim_{n\to\infty}\frac{2\pi}{n}\sum_{k=1}^{n}\left(2+\sin\frac{2k\pi}{n}\right)$.

Solution. (1) By definition, we have

$$\begin{aligned}
&\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right)\\
=&\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+\frac{i}{n}}=\int_0^1\frac{1}{1+x}\,\mathrm{d}x\\
=&\ln(1+x)\Big|_0^1=\ln 2.
\end{aligned}$$

(2) Take logarithm. By definition, we have

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\ln\left(1+\frac{i-1}{n}\right)=\int_0^1\ln(1+x)\,\mathrm{d}x\\
=&((1+x)\ln(1+x)-x)\Big|_0^1=2\ln 2-1.
\end{aligned}$$

Hence

$$\lim_{n\to\infty}\frac{\sqrt[n]{n(n+1)\cdots(2n-1)}}{n}=e^{2\ln 2-1}=\frac{4}{e}.$$

(3) By definition, we have

$$\begin{aligned}
&\lim_{n\to\infty}\frac{2\pi}{n}\sum_{k=1}^{n}\left(2+\sin\frac{2k\pi}{n}\right)\\
=&\int_0^{2\pi}(2+\sin x)\,\mathrm{d}x\\
=&(2x-\cos x)\Big|_0^{2\pi}=4\pi.
\end{aligned}$$

□

Problem 12.2 (7.5)**.** *Suppose that $f\in R[a,b]$, $g(x)$ is defined on (a,b) and $g(x)$ is different from $f(x)$ at only a finite number of points on (a,b). Prove that $g(x)$ is* *integrable on (a, b), and there is*

$$\int_a^b g(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x.$$

Proof. Denote $I = \displaystyle\int_a^b f(x)\,\mathrm{d}x$. By definition, we know that $\forall\, \varepsilon > 0$, there exists $\delta' > 0$ such that for any $\Delta' : a = x_0 < x_1 < \cdots < x_n = b$ with $\lambda(\Delta') < \delta'$ and any $\xi_i \in [x_{i-1}, x_i]\,(i = 1, 2, \cdots, n)$, there is

$$\left|\sum_{i=1}^n f(\xi_i)\Delta x_i - I\right| < \frac{\varepsilon}{2},$$

where $\lambda(\Delta') = \max\limits_{1\le i\le n}\{x_i - x_{i-1}\}$. Since $f \in R[a, b]$ and $g(x)$ is different from $f(x)$ at only a finite number of points on (a, b), we know that $f(x)$ and $g(x)$ are both bounded. Assume that $\exists\, M > 0$ such that $|f(x)| \le M$ and $|g(x)| \le M$. Then, if we choose $\delta < \min\{\delta', \frac{\varepsilon}{4kM}\}$, where k is the number of points where $g(x)$ is different from $f(x)$, we have for any $\Delta : a = x_0 < x_1 < \cdots < x_n = b$ with $\lambda(\Delta) < \delta$ and any $\xi_i \in [x_{i-1}, x_i]\,(i = 1, 2, \cdots, n)$ that

$$\begin{aligned}\left|\sum_{i=1}^n g(\xi_i)\Delta x_i - I\right| &\le \left|\sum_{i=1}^n f(\xi_i)\Delta x_i - I\right| + 2kM\lambda(\Delta)\\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Hence $g(x)$ is integrable on (a, b), and

$$\int_a^b g(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x.$$

□

Problem 12.3 (7.6)**.** *Suppose that $f(x)$ is defined on $[a, b]$. Prove that $f(x) \in R[a, b]$ if and only if there exists $I \in \mathbb{R}$, for $\forall\, \varepsilon > 0$, $\exists\, \Delta : a = x_0 < x_1 < \cdots < x_n = b$ and for any $\xi_i \in [x_{i-1}, x_i]\,(i = 1, 2, \cdots, n)$, there is*

$$\left|\sum_{i=1}^n f(\xi_i)\Delta x_i - I\right| < \varepsilon.$$

Proof. "⇒" By the definition of intagral, it's trival.

"⇐" Denote $w_i = \sup\limits_{x\in[x_{i-1},x_i]} f(x) - \inf\limits_{x\in[x_{i-1},x_i]} f(x)$. It suffices to prove that for some partition $\Delta : a = x_0 < x_1 < \cdots < x_n = b$, there is

$$\sum_{i=1}^n w_i\Delta x_i < \varepsilon.$$

By assumption, we know that for $\forall\,\varepsilon > 0$, $\exists\,\Delta : a = x_0 < x_1 < \cdots < x_n = b$ and for any $\xi_i \in [x_{i-1}, x_i]$ $(i = 1, 2, \cdots, n)$, there is

$$\left| \sum_{i=1}^{n} f(\xi_i)\Delta x_i - I \right| < \frac{\varepsilon}{4}.$$

By the definition of superemum and infimum, we have that there exist $\xi_i, \eta_i \in [x_{i-1}, x_i]$ such that

$$f(\xi_i) + \frac{\varepsilon}{4(b-a)} > \sup_{x\in[x_{i-1},x_i]} f(x) \quad \text{and} \quad f(\eta_i) - \frac{\varepsilon}{4(b-a)} < \inf_{x\in[x_{i-1},x_i]} f(x).$$

Hence, there is

$$\begin{aligned}
\sum_{i=1}^{n} w_i \Delta x_i &= \sum_{i=1}^{n} \left(\sup_{x\in[x_{i-1},x_i]} f(x) - \inf_{x\in[x_{i-1},x_i]} f(x) \right) \Delta x_i \\
&\le \left| \sum_{i=1}^{n} (f(\xi_i) - f(\eta_i))\Delta x_i \right| + \frac{\varepsilon}{2} \\
&\le \left| \sum_{i=1}^{n} f(\xi_i)\Delta x_i - I \right| + \left| \sum_{i=1}^{n} f(\eta_i)\Delta x_i - I \right| + \frac{\varepsilon}{2} \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

which gives us that $f(x)$ is integrable on $[a, b]$. □

Problem 12.4 (7.10)**.** *Suppose that $f(x)$ is bounded on $[a, b]$. Prove that $f \in R[a, b]$ if and only if for $\forall\,\varepsilon > 0$, there exist continous functions $g(x)$ and $h(x)$ on $[a, b]$ satisfying*

(1) $g(x) \le f(x) \le h(x)$, $\forall\, x \in [a, b]$;

(2) $\displaystyle\int_a^b [h(x) - g(x)]\,\mathrm{d}x < \varepsilon$.

Proof. "$\Leftarrow$" For $\forall\,\varepsilon > 0$, by the definition of upper integral and lower integral, we know that there exists a partition $\Delta : a = x_0 < x_1 < \cdots < x_n = b$, such that

$$\sum_{i=1}^{n} \sup_{x\in[x_{i-1},x_i]} h(x)\Delta x_i < \int_a^b h(x)\,\mathrm{d}x + \frac{\varepsilon}{4},$$

and

$$\sum_{i=1}^{n} \inf_{x\in[x_{i-1},x_i]} g(x)\Delta x_i > \int_a^b g(x)\,\mathrm{d}x - \frac{\varepsilon}{4},$$

where $\Delta x_i = x_i - x_{i-1}$. Then $\displaystyle\int_a^b [h(x) - g(x)]\,\mathrm{d}x < \varepsilon/2$ implies

$$\sum_{i=1}^{n}\left[\sup_{x\in[x_{i-1},x_i]} h(x) - \inf_{x\in[x_{i-1},x_i]} g(x)\right]\Delta x_i < \int_a^b (h(x) - g(x))\,\mathrm{d}x + \frac{\varepsilon}{2} < \varepsilon.$$

Since $g(x) \le f(x) \le h(x)$, $\forall\, x \in [a, b]$, we have that

$$w_i = \sup_{x\in[x_{i-1},x_i]} f(x) - \inf_{x\in[x_{i-1},x_i]} f(x) \le \sup_{x\in[x_{i-1},x_i]} h(x) - \inf_{x\in[x_{i-1},x_i]} g(x),$$

i.e.

$$\sum_{i=1}^{n} w_i \Delta x_i \le \sum_{i=1}^{n}\left[\sup_{x\in[x_{i-1},x_i]} h(x) - \inf_{x\in[x_{i-1},x_i]} g(x)\right]\Delta x_i < \varepsilon,$$

which gives us that $f \in R[a, b]$.

"$\Rightarrow$" Since $f \in R[a, b]$, we have that for $\forall\,\varepsilon > 0$, there exists a partition $\Delta : a = x_0 < x_1 < \cdots < x_n = b$, such that

$$\sum_{i=1}^{n}(M_i - m_i)\Delta x_i < \frac{\varepsilon}{2},$$

where

$$M_i = \sup_{x\in[x_{i-1},x_i]} f(x), \quad m_i = \inf_{x\in[x_{i-1},x_i]} f(x), \quad \Delta x_i = x_i - x_{i-1}.$$

Firstly, we define

$$\varphi(x) = \begin{cases} m_i, & x \in [x_{i-1}, x_i),\ i = 1, 2, \cdots, n-1; \\ m_n, & x \in [x_{n-1}, x_n], \end{cases}$$

and

$$\psi(x) = \begin{cases} M_i, & x \in [x_{i-1}, x_i),\ i = 1, 2, \cdots, n-1; \\ M_n, & x \in [x_{n-1}, x_n]. \end{cases}$$

It is clear that $\varphi(x) \le f(x) \le \psi(x)$, and $\displaystyle\int_a^b [\psi(x) - \varphi(x)]\,\mathrm{d}x < \varepsilon/2$.

Next, we assume that $m \le f(x) \le M$ (since $f(x)$ is bounded), $r = \min\limits_{1\le i\le n} \Delta x_i$. Choose $0 < \eta < \min\left\{\dfrac{r}{2}, \dfrac{\varepsilon}{4n(M - m + 1)}\right\}$. Define

$$h(x) = \begin{cases} M_1, & x \in [a, x_1 - \eta), \\ M_i, & x \in [x_i + \eta, x_{i+1} - \eta],\ i = 1, \cdots, n-2; \\ d_i + \frac{M_i - d_i}{\eta}(x_i - x), & x \in [x_i - \eta, x_i],\ i = 1, \cdots, n-1; \\ d_i + \frac{M_{i+1} - d_i}{\eta}(x - x_i), & x \in [x_i, x_i + \eta],\ i = 1, \cdots, n-1; \\ M_n, & x \in [x_{n-1} + \eta, x_n]; \end{cases}$$

and

$$g(x) = \begin{cases} m_1, & x \in [a, x_1 - \eta), \\ m_i, & x \in [x_i + \eta, x_{i+1} - \eta], i = 1, \cdots, n-2; \\ c_i + \frac{m_i - c_i}{\eta}(x_i - x), & x \in [x_i - \eta, x_i], i = 1, \cdots, n-1; \\ c_i + \frac{m_{i+1} - c_i}{\eta}(x - x_i), & x \in [x_i, x_i + \eta], i = 1, \cdots, n-1; \\ m_n, & x \in [x_{n-1} + \eta, x_n]. \end{cases}$$

where $d_i = \max\{M_i, M_{i+1}\}, c_i = \min\{m_i, m_{i+1}\}$.

By the construction of $g(x)$ and $h(x)$, we know that $g(x) \le f(x) \le h(x)$, and

$$\begin{aligned} \int_a^b (h(x) - g(x))\mathrm{d}x &= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} [h(x) - g(x)]\mathrm{d}x \\ &= \int_a^{x_1 - \eta} [h(x) - g(x)]\mathrm{d}x + \sum_{i=1}^{n-1} \int_{x_i - \eta}^{x_i + \eta} [h(x) - g(x)]\mathrm{d}x \\ &\quad + \sum_{i=1}^{n-1} \int_{x_i + \eta}^{x_{i+1} - \eta} [h(x) - g(x)]\mathrm{d}x + \int_{x_{n-1} + \eta}^{b} [h(x) - g(x)]\mathrm{d}x \\ &\le \int_a^b [\psi(x) - \phi(x)]\mathrm{d}x + 2(n-1)(M - m)\eta \\ &< \varepsilon. \end{aligned}$$

□

Problem 12.5 (7.16)**.** *Suppose that $f(x)$ is defined on $\mathbb{R}$, and $f(x)$ is integrable on every finite closed interval. Prove that for any closed interval $[a, b]$, there is*

$$\lim_{h \to 0} \int_a^b |f(x+h) - f(x)| \,\mathrm{d}x = 0. \tag{12.1}$$

Proof. Since $f(x)$ is integrable on $[a-1, b+1]$, we know that for $\forall \varepsilon > 0$, there is $g(x) \in C[a-1, b+1]$ such that

$$\int_{a-1}^{b+1} |f(x) - g(x)| \,\mathrm{d}x < \frac{\varepsilon}{3}.$$

(Leave to the reader.) Since $g(x)$ is continuous on $[a-1, b+1]$, we have by the Cantor theorem that $g(x)$ is uniformly continuous on $[a-1, b+1]$. Then there exists $\delta : 0 < \delta < 1$, such that $\forall h \in (-\delta, \delta)$, there is

$$|g(x+h) - g(x)| < \frac{\varepsilon}{3(b-a)}, \quad \forall x \in [a, b].$$

Therefore, we have for $\forall\, h \in (-\delta, \delta)$ that

$$\begin{aligned}\int_a^b |f(x+h) - f(x)|\, \mathrm{d}x \leq& \int_a^b |f(x+h) - g(x+h)|\, \mathrm{d}x + \int_a^b |g(x+h) - g(x)|\, \mathrm{d}x \\ &+ \int_a^b |g(x) - f(x)|\, \mathrm{d}x \\ =& \int_{a+h}^{b+h} |f(x) - g(x)|\, \mathrm{d}x + \int_a^b |g(x+h) - g(x)|\, \mathrm{d}x \\ &+ \int_a^b |g(x) - f(x)|\, \mathrm{d}x \\ <& \frac{\varepsilon}{3} + \frac{\varepsilon}{3(b-a)}(b-a) + \frac{\varepsilon}{3} \\ =& \varepsilon,\end{aligned}$$

i.e.

$$\lim_{h \to 0} \int_a^b |f(x+h) - f(x)|\, \mathrm{d}x = 0.$$

□

Remark 12.6. (12.1) *is called the absolute continuity of integrals.*

Problem 12.7 (7.18)**.** *Suppose that $f(x)$, $g(x) \in R[a,b]$. Prove the Cauchy-Schwarz inequality:*

$$\left|\int_a^b f(x)g(x)\mathrm{d}x\right| \leq \left[\int_a^b f^2(x)\mathrm{d}x\right]^{\frac{1}{2}} \left[\int_a^b g^2(x)\mathrm{d}x\right]^{\frac{1}{2}}. \tag{12.2}$$

Proof. If $\int_a^b f(x)g(x)\mathrm{d}x = 0$, (12.2) is clearly. Now, we assume that $\int_a^b f(x)g(x)\mathrm{d}x \neq 0$. It's obvious that $\int_a^b f^2(x)\mathrm{d}x \neq 0$ and $\int_a^b g^2(x)\mathrm{d}x \neq 0$ (Leave to the reader). Note that

$$0 \leq \int_a^b (f(x) - tg(x))^2\, \mathrm{d}x = \int_a^b f^2(x)\mathrm{d}x - 2t\int_a^b f(x)g(x)\mathrm{d}x + t^2 \int_a^b g^2(x)\mathrm{d}x, \quad \forall\, t \in \mathbb{R}.$$

Then there is

$$4\left[\int_a^b f(x)g(x)\mathrm{d}x\right]^2 - 4\left[\int_a^b f^2(x)\mathrm{d}x\right]\left[\int_a^b g^2(x)\mathrm{d}x\right] \leq 0,$$

i.e.

$$\left|\int_a^b f(x)g(x)\mathrm{d}x\right| \leq \left[\int_a^b f^2(x)\mathrm{d}x\right]^{\frac{1}{2}} \left[\int_a^b g^2(x)\mathrm{d}x\right]^{\frac{1}{2}}.$$

□

Problem 12.8 (7.19)**.** *Prove the following limitations.*

(1) $\lim_{n\to\infty}\int_{-1}^{1}(1-x^2)^n\mathrm{d}x = 0$;

(2) *Suppose that* $f(x)\in C[-1,1]$, *then*

$$\lim_{n\to\infty}\frac{\int_{-1}^{1} f(x)(1-x^2)^n\mathrm{d}x}{\int_{-1}^{1}(1-x^2)^n\mathrm{d}x} = f(0).$$

Proof. (1) For any $\delta>0$, we have

$$\begin{aligned}\int_{-1}^{1}(1-x^2)^n\mathrm{d}x &= \int_{|x|\le\delta}(1-x^2)^n\mathrm{d}x + \int_{\delta<|x|\le 1}(1-x^2)^n\mathrm{d}x\\ &\le 2\delta + 2(1-\delta^2)^n\\ &\to 2\delta \quad \text{as } n\to\infty.\end{aligned}$$

Since δ is arbitrary, we know that $\lim_{n\to\infty}\int_{-1}^{1}(1-x^2)^n\mathrm{d}x = 0$.

(2) Since $f(x)\in C[-1,1]$, we have that $\forall\,\varepsilon>0$, there exists $\delta>0$, such that $\forall\,x:\ |x|<\delta$, there is

$$|f(x)-f(0)|\le\varepsilon.$$

Then

$$\begin{aligned}\left|\frac{\int_{-1}^{1} f(x)(1-x^2)^n\mathrm{d}x}{\int_{-1}^{1}(1-x^2)^n\mathrm{d}x} - f(0)\right| &= \left|\frac{\int_{-1}^{1}(f(x)-f(0))(1-x^2)^n\mathrm{d}x}{\int_{-1}^{1}(1-x^2)^n\mathrm{d}x}\right|\\ &\le \frac{\int_{|x|<\delta}|f(x)-f(0)|\,(1-x^2)^n\mathrm{d}x}{\int_{-1}^{1}(1-x^2)^n\mathrm{d}x}\\ &\quad+\frac{\int_{\delta\le|x|<1}|f(x)-f(0)|\,(1-x^2)^n\mathrm{d}x}{\int_{-1}^{1}(1-x^2)^n\mathrm{d}x}\\ &\le \varepsilon + 2M\cdot\frac{\int_{\delta\le|x|<1}(1-x^2)^n\mathrm{d}x}{\int_{-1}^{1}(1-x^2)^n\mathrm{d}x}\\ &\le \varepsilon + 2M\cdot\frac{\int_{\delta\le|x|<1}(1-x^2)^n\mathrm{d}x}{\int_{|x|\le\frac{\delta}{2}}(1-x^2)^n\mathrm{d}x}\\ &\le \varepsilon + 2M\frac{(1-\delta^2)^n}{(1-(\delta/2)^2)^n\delta}\\ &\to \varepsilon \quad \text{as } n\to\infty,\end{aligned}$$

where $M = \max_{x\in[-1,1]} f(x)$. Since ε is arbitrary, we have

$$\lim_{n\to\infty} \frac{\int_{-1}^{1} f(x)(1-x^2)^n \mathrm{d}x}{\int_{-1}^{1}(1-x^2)^n \mathrm{d}x} = f(0).$$

□

Exercise 12.9 (7.21(1))**.** *Suppose that $f(x)$ has continuous derivative on $[a,b]$. Prove that for any $x \in [a,b]$, there is*

$$|f(x)| \leq \left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| + \int_a^b |f'(x)|\mathrm{d}x.$$

Hint: By the mean value theorem for definite integrals, we know that there is $\xi \in (a,b)$ such that

$$f(\xi) = \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x.$$

By the Newton-Leibniz formula, we have

$$|f(x) - f(\xi)| = \left|\int_\xi^x f'(t)\mathrm{d}t\right| \leq \int_a^b |f'(x)|\mathrm{d}x.$$

Hence

$$|f(x)| \leq |f(\xi)| + |f(x) - f(\xi)| \leq \left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| + \int_a^b |f'(x)|\mathrm{d}x.$$

□

Problem 12.10 (7.22)**.** *Suppose that $f(x) \in C(\mathbb{R})$ and $f'(0)$ exists. Assume that $\forall\, x \in \mathbb{R}$, there is*

$$\int_0^x f(t)\mathrm{d}x = \frac{1}{2}xf(x). \tag{12.3}$$

Prove that $f(x) \equiv cx$, where $c = f'(0)$.

Proof. Denote $F(x) = \displaystyle\int_0^x f(t)\mathrm{d}t$. Since $f(x) \in C(\mathbb{R})$, we have that $F(x) \in C^1(\mathbb{R})$. Then by (12.3), we have $f(0) = 0$. Define

$$g(x) = \begin{cases} \dfrac{F(x)}{x^2}, & x \neq 0, \\ \dfrac{1}{2}f'(0), & x = 0. \end{cases}$$

Then by (12.3), we have

$$g'(x) = \frac{x^2 f(x) - 2xF(x)}{x^4} = 0, \quad \forall\, x \neq 0.$$

By definition, we know that

$$g'(0) = \lim_{x\to 0} \frac{g(x) - g(0)}{x} = \lim_{x\to 0} \frac{g'(\xi_x)}{x} = 0.$$

Hence $F(x) = \frac{f'(0)}{2} x^2$, $\forall\, x \in \mathbb{R}$. Then $f(x) = f'(0)x$, $\forall\, x \in \mathbb{R}$. □

Problem 12.11 (7.23)**.** *Suppose that $P_n(x)$ is a polynomial with degree $n \geq 1$, and $[a, b]$ is an closed interval. Prove that*

$$\int_a^b |P_n'(x)| \mathrm{d}x \leq 2n \max_{a\leq x\leq b} \{|P_n(x)|\}.$$

Proof. Since $P_n'(x)$ is a polynomial with degree $n-1$, we may assume that there are $a \leq x_1 \leq \cdots \leq x_k \leq b$ with $k \leq n-1$, where $x_1, \cdots, x_{n-1}$ are zero points of $P_n'(x)$. Hence, we have

$$\begin{aligned}
\int_a^b |P_n'(x)| \mathrm{d}x &= \left| \int_a^{x_1} P_n'(x) \mathrm{d}x \right| + \cdots + \left| \int_{x_k}^b P_n'(x) \mathrm{d}x \right| \\
&= |P_n(x_1) - P_n(a)| + \cdots + |P_n(b) - P_n(x_k)| \\
&\leq 2(k+1) \max_{a\leq x\leq b} \{|P_n(x)|\} \\
&\leq 2n \max_{a\leq x\leq b} \{|P_n(x)|\}.
\end{aligned}$$

□

Problem 12.12 (7.28)**.** *Suppose that $f(x)$ is a periodic function with period 2π and $f(x) \in R[0, 2\pi]$. Prove that*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(x) \mathrm{d}x = \frac{1}{2\pi} \int_0^{2\pi} f(x) \mathrm{d}x.$$

Proof. For any $T > 0$, we know that there exists $n \in \mathbb{N}$ such that $T = 2n\pi + r$, where $0 \leq r < 2\pi$. Then

$$\begin{aligned}
\int_0^T f(x) \mathrm{d}x &= \int_0^{2n\pi + r} f(x) \mathrm{d}x \\
&= \int_0^{2\pi} f(x) \mathrm{d}x + \cdots + \int_{2(n-1)\pi}^{2n\pi} f(x) \mathrm{d}x + \int_{2n\pi}^{2n\pi + r} f(x) \mathrm{d}x \\
&= n \int_0^{2\pi} f(x) \mathrm{d}x + \int_0^r f(x) \mathrm{d}x.
\end{aligned}$$

Hence, we have

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(x)\mathrm{d}x=\lim_{T\to\infty}\frac{n}{2n\pi+r}\int_0^{2\pi} f(x)\mathrm{d}x+\lim_{T\to\infty}\frac{1}{T}\int_0^r f(x)\mathrm{d}x=\frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x.$$

□

Problem 12.13 (7.35)**.** *Suppose that $f(x)\in C[a,b]$ is nonnegative. Denote that $M=\sup_{a\le x\le b} f(x)$. Prove that*

$$\lim_{n\to\infty}\left[\int_a^b f^n(x)\mathrm{d}x\right]^{\frac{1}{n}}=M.$$

Proof. Firstly, it is easy to see that

$$\left[\int_a^b f^n(x)\mathrm{d}x\right]^{\frac{1}{n}}\le M(b-a)^{\frac{1}{n}}.$$

Then

$$\overline{\lim_{n\to\infty}}\left[\int_a^b f^n(x)\mathrm{d}x\right]^{\frac{1}{n}}\le M.$$

Secondly, since $f(x)$ is continuous, we know that there is $x_0\in[a,b]$ such that $f(x_0)=M$. Then $\forall\,\varepsilon>0$, there exists $\delta>0$ such that $\forall\,x:|x-x_0|<\delta$, there is $|f(x)-f(x_0)|<\varepsilon$, thus $f(x)>M-\varepsilon$. Hence, we have

$$\left[\int_a^b f^n(x)\mathrm{d}x\right]^{\frac{1}{n}}\ge\left[\int_{x_0-\delta}^{x_0+\delta} f^n(x)\mathrm{d}x\right]^{\frac{1}{n}}\ge(M-\varepsilon)(2\delta)^{\frac{1}{n}}.$$

Therefore, there is

$$\underline{\lim_{n\to\infty}}\left[\int_a^b f^n(x)\mathrm{d}x\right]^{\frac{1}{n}}\ge M-\varepsilon.$$

Since ε is arbitrary, we have

$$\underline{\lim_{n\to\infty}}\left[\int_a^b f^n(x)\mathrm{d}x\right]^{\frac{1}{n}}\ge M\ge\overline{\lim_{n\to\infty}}\left[\int_a^b f^n(x)\mathrm{d}x\right]^{\frac{1}{n}},$$

which implies

$$\lim_{n\to\infty}\left[\int_a^b f^n(x)\mathrm{d}x\right]^{\frac{1}{n}}=M.$$

□

Exercise 12.14 (7.37). *Suppose that $f(x)$ is monotonic on (a, b), $g(x)$ is a periodic function with period $T > 0$ on $\mathbb{R}$, and $\int_0^T g(x) = 0$. Prove that*

$$\lim_{\lambda\to\infty} \int_a^b f(x)g(\lambda x)\mathrm{d}x = 0.$$

Hint: By the second mean value theorem for definite integrals, we have that there exists $\xi \in (a, b)$ such that

$$\int_a^b f(x)g(\lambda x)\mathrm{d}x = f(a)\int_a^\xi g(\lambda x)\mathrm{d}x + f(b)\int_\xi^b g(\lambda x)\mathrm{d}x.$$

Note that

$$\int_a^\xi g(\lambda x)\mathrm{d}x = \frac{1}{\lambda}\int_{a\lambda}^{\xi\lambda} g(x)\mathrm{d}x = \frac{\xi}{\xi\lambda}\int_0^{\xi\lambda} g(x)\mathrm{d}x - \frac{a}{a\lambda}\int_0^{a\lambda} g(x)\mathrm{d}x.$$

Hence by Problem 12.12 and $\int_0^T g(x) = 0$, we have

$$\lim_{\lambda\to\infty} \int_a^\xi g(\lambda x)\mathrm{d}x = 0.$$

Similarly, we have

$$\lim_{\lambda\to\infty} \int_\xi^b g(\lambda x)\mathrm{d}x = 0.$$

Then we obtain that

$$\lim_{\lambda\to\infty} \int_a^b f(x)g(\lambda x)\mathrm{d}x = 0.$$

□

Remark 12.15. *In fact, we can remove the monotonicity of $f(x)$. It is the Riemann-Lebesgue lemma, which can be proved by real analysis' method, we leave it to the reader.*

Exercise 12.16. *Suppose that $f \in C[-1, 1]$, Prove that*

$$\lim_{h\to 0^+} \int_{-1}^1 \frac{h}{h^2 + x^2} f(x)\,dx = \pi f(0).$$

Hint: Note that

$$\begin{aligned}
&\int_{-1}^{1} \frac{h}{h^2+x^2} f(x)\,dx - 2\arctan\frac{1}{h} f(0)\\
&= \int_{-1}^{1} \frac{h}{h^2+x^2}(f(x)-f(0))\,dx\\
&= \int_{|x|<\delta} \frac{h}{h^2+x^2}(f(x)-f(0))\,dx + \int_{1\ge|x|\ge\delta} \frac{h}{h^2+x^2}(f(x)-f(0))\,dx,
\end{aligned}$$

where $\delta>0$ is such that $|f(x)-f(0)|<\varepsilon$ whence $|x|<\delta$ (Since $f(x)$ is continous). Then, we have

$$\begin{aligned}
&\left|\int_{-1}^{1} \frac{h}{h^2+x^2} f(x)\,dx - 2\arctan\frac{1}{h} f(0)\right|\\
&\le \int_{|x|<\delta} \frac{h}{h^2+x^2}|f(x)-f(0)|\,dx + \int_{1\ge|x|\ge\delta} \frac{h}{h^2+x^2}|f(x)-f(0))|\,dx\\
&\le \varepsilon\int_{-1}^{1} \frac{h}{h^2+x^2}\,dx + 2\sup|f|\int_{1\ge|x|\ge\delta} \frac{h}{h^2+x^2}\,dx\\
&= 2\varepsilon\arctan\frac{1}{h} + 4\sup|f|\left(\arctan\frac{1}{h}-\arctan\frac{\delta}{h}\right)\\
&\le \pi\varepsilon,\ \ as \quad h\to 0.
\end{aligned}$$

Since ε is arbitrary, we know

$$\lim_{h\to 0}\int_{-1}^{1} \frac{h}{h^2+x^2} f(x)\,dx = \lim_{h\to 0} 2\arctan\frac{1}{h} f(0) = \pi f(0).$$

□

Exercise 12.17. *Suppose that* $f\in C[-1,1]$, *given* $\int_{-1}^{1} f(x)x^n dx = 0$ *for* $n=0,1,2,\dots$ *then* $f(x)=0,\ \forall x\in[-1,1]$.

Hint: Since

$$\lim_{n\to\infty}\sup|f(x)-p_n(x)|=0,$$

we know for any $\epsilon>0$, there is $N\in\mathbb{N}$ such that for $\forall n>N$

$$|f-p_n|<\frac{\epsilon}{2M},$$

where $M:=\max_{[-1,1]}|f(x)|$.

Hence

$$\left|\int_{-1}^{1} f(x)\left(f(x)-p_n(x)\right)\,dx\right| < \frac{\epsilon}{2M}\int_{-1}^{1}|f(x)|\,dx \le \epsilon, \quad \forall n>N.$$

□

Exercise 12.18 (Challenge!). *Assume that $f(x) \in C[0,+\infty)$, and for all $a \geq 0$, we have*

$$\lim_{x\to\infty} (f(x+a) - f(x)) = 0. \tag{12.4}$$

Prove that there exists $g(x) \in C[0,+\infty)$ and $h(x) \in C^1[0,+\infty)$ such that $f(x) = g(x) + h(x)$, and such that they satisfy

$$\lim_{x\to\infty} g(x) = 0, \ \lim_{x\to\infty} h'(x) = 0.$$

Hint: By Exercise 4.23, we first know that $f(x)$ is uniformly continuous. Let's chose $a = 1$ and set $h(x) = \int_x^{x+1} f(t)\,dt$. We begin by writing

$$h(x) - f(x) = \int_0^1 (f(x+t) - f(x))\ dt.$$

The integrand converges to 0 pointwise (from condition (12.4)), but this is not quite sufficient! We'll have to be a bit more careful and also use the uniform continuity of f. Let $\epsilon > 0$. Because f is uniformly continuous, there exists an integer $n > 0$ such that for all $x, y \geq 0$ with $|x-y| \leq \frac{1}{n}$, we have $|f(x) - f(y)| \leq \epsilon$.

Now we use condition (12.4) to get that for $1 \leq k \leq n$, there exists x_k such that for all $x \geq x_k$,

$$\left| f\left(x + \frac{k}{n}\right) - f(x) \right| \leq \varepsilon.$$

We set $x_0 = \max\limits_{1\leq k\leq n}(x_k)$. Now, for $x \geq x_0$ we have

$$\begin{aligned}
|h(x) - f(x)| &= \left| \int_0^1 (f(x+t) - f(x))\ dt \right| \\
&\leq \int_0^1 |f(x+t) - f(x)|\ dt \\
&\leq \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \underbrace{\left| f(x+t) - f\left(x + \frac{k}{n}\right) \right|}_{\leq\varepsilon \quad \text{(from continuity)}} + \underbrace{\left| f\left(x+\frac{k}{n}\right) - f(x) \right|}_{\leq\varepsilon \quad \text{(from (12.4))}}\ dt \\
&\leq 2\varepsilon.
\end{aligned}$$

□

Problem 13.1. *Calculate the following limitations.*

(1) $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\left(\dfrac{k}{n^2}\right)^{1+\frac{k}{n^2}}$;

(2) $\lim\limits_{n\to\infty}\dfrac{1}{n^k}\int_0^1 \ln^k(1+e^{nx})\,\mathrm{d}x$;

(3) $\lim\limits_{x\to 0}\dfrac{[1+\ln(1+x)]^{\frac{1}{\tan x}}-e(1-x)}{x^2}$.

Solution. (1) Firstly, we have

$$\left(\frac{k}{n^2}\right)^{1+\frac{k}{n^2}} \le \frac{k}{n^2}, \quad k=1,2,\cdots,n,$$

since $\dfrac{k}{n^2}\le 1$. Then there is

$$\sum_{k=1}^{n}\left(\frac{k}{n^2}\right)^{1+\frac{k}{n^2}} \le \sum_{k=1}^{n}\frac{k}{n^2}=\frac{n(n+1)}{2n^2}=\frac{n+1}{2n}\to\frac{1}{2}, \quad \text{as } n\to\infty.$$

On the other hand, we have

$$\left(\frac{k}{n^2}\right)^{1+\frac{k}{n^2}} \ge \left(\frac{k}{n^2}\right)^{1+\frac{1}{n}}, \quad k=1,2,\cdots,n.$$

Since $u_n(x)=x^{1+\frac{1}{n}}$ is increasing on $(0,1)$, we know that

$$\begin{aligned}\sum_{k=1}^{n}\left(\frac{k}{n^2}\right)^{1+\frac{1}{n}} &= \frac{1}{\sqrt[n]{n}}\cdot\frac{1}{n}\sum_{k=1}^{n}u_n\left(\frac{k}{n}\right)\\ &\ge \frac{1}{\sqrt[n]{n}}\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}}u_n(x)\,\mathrm{d}x\\ &= \frac{1}{\sqrt[n]{n}}\int_0^1 u_n(x)\,\mathrm{d}x\\ &= \frac{n^{1-\frac{1}{n}}}{2n+1}\to\frac{1}{2}, \quad \text{as } n\to\infty.\end{aligned}$$

Hence

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{k}{n^2}\right)^{1+\frac{k}{n^2}}=\frac{1}{2}.$$

(2) By changing of variables and the Stolz theorem, we have

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n^k}\int_0^1 \ln^k(1+e^{nx})\,\mathrm{d}x &\xlongequal{y=nx} \lim_{n\to\infty}\frac{1}{n^{k+1}}\int_0^n \ln^k(1+e^{y})\,\mathrm{d}y\\
&\xlongequal{Stolz} \lim_{n\to\infty}\frac{\int_n^{n+1}\ln^k(1+e^y)\,\mathrm{d}y}{(n+1)^{k+1}-n^{k+1}}\\
&= \lim_{n\to\infty}\frac{\ln^k(1+e^{\theta_n})}{(k+1)n^k}\\
&\xlongequal{(*)} \frac{1}{k+1},
\end{aligned}$$

where we used $\lim_{n\to\infty}\frac{\ln(1+e^{\theta_n})}{n}=1$ in $(*)$ since

$$\frac{\ln(1+e^n)}{n}\le\frac{\ln(1+e^{\theta_n})}{n}\le\frac{\ln(1+e^{n+1})}{n}.$$

(3) By Taylor's formula, we have

$$\begin{aligned}
\lim_{x\to0}\frac{[1+\ln(1+x)]^{\frac{1}{\tan x}}-e(1-x)}{x^2} &= \lim_{x\to0}\frac{e^{\frac{1}{\tan x}\ln(1+\ln(1+x))}-e(1-x)}{x^2}\\
&= \lim_{x\to0}\frac{e^{\frac{1}{\tan x}\ln(1+x-\frac12x^2+\frac13x^3)}-e(1-x)}{x^2}\\
&= \lim_{x\to0}\frac{e^{\frac{(x-\frac12x^2+\frac13x^3)-\frac12(x-\frac12x^2)^2+\frac13x^3}{x+\frac13x^3}}-e(1-x)}{x^2}\\
&= \lim_{x\to0}\frac{e^{\frac{x-x^2+\frac76x^3}{x+\frac13x^3}}-e(1-x)}{x^2}\\
&= \lim_{x\to0}\frac{e^{1-x+\frac56x^2}-e(1-x)}{x^2}\\
&= \lim_{x\to0}\frac{e(1-x+\frac56x^2+\frac12x^2)-e(1-x)}{x^2}\\
&= \lim_{x\to0}\frac{\frac{4e}{3}x^2}{x^2}\\
&= \frac{4e}{3}.
\end{aligned}$$

□

Problem 13.2. *Calculate the following integrals.*

(1) $\displaystyle\int_{-1}^{1}\frac{1}{\sqrt{1+x}+\sqrt{1-x}+2}\,\mathrm{d}x;$

(2) $\displaystyle\int_1^2 \frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}}\,\mathrm{d}x;$

(3) $\displaystyle\int x\sin(\ln x)\,\mathrm{d}x,$ *where* $x>0$;

(4) $\displaystyle\int \frac{1}{x+\sqrt{x^2-x+1}}\,\mathrm{d}x.$

Solution. (1) By changing of variables, we have

$$\begin{aligned}
\int_{-1}^{1}\frac{1}{\sqrt{1+x}+\sqrt{1-x}+2}\,\mathrm{d}x &= 2\int_0^1\frac{1}{\sqrt{1+x}+\sqrt{1-x}+2}\,\mathrm{d}x\\
&\overset{x=\sin 4t}{=\!=\!=} 8\int_0^{\frac{\pi}{8}}\frac{\cos 4t}{\sqrt{1+\sin 4t}+\sqrt{1-\sin 4t}+2}\,\mathrm{d}t\\
&= 8\int_0^{\frac{\pi}{8}}\frac{\cos 4t}{\sqrt{1+2\sin 2t\cos 2t}+\sqrt{1-2\sin 2t\cos 2t}+2}\,\mathrm{d}t\\
&= 8\int_0^{\frac{\pi}{8}}\frac{\cos 4t}{\sin 2t+\cos 2t+\cos 2t-\sin 2t+2}\,\mathrm{d}t\\
&= 4\int_0^{\frac{\pi}{8}}\frac{\cos 4t}{\cos 2t+1}\,\mathrm{d}t\\
&= 2\int_0^{\frac{\pi}{8}}\frac{\cos 4t}{\cos^2 t}\,\mathrm{d}t\\
&= 2\int_0^{\frac{\pi}{8}}\frac{2\cos^2 2t-1}{\cos^2 t}\,\mathrm{d}t\\
&= 2\int_0^{\frac{\pi}{8}}\frac{2(2\cos^2 t-1)^2-1}{\cos^2 t}\,\mathrm{d}t\\
&= 2\int_0^{\frac{\pi}{8}}\frac{8\cos^4 t-8\cos^2 t+1}{\cos^2 t}\,\mathrm{d}t\\
&= 2\int_0^{\frac{\pi}{8}}8\cos^2 t\,\mathrm{d}t-2\int_0^{\frac{\pi}{8}}8\,\mathrm{d}t+2\int_0^{\frac{\pi}{8}}\frac{1}{\cos^2 t}\,\mathrm{d}t\\
&= 16\int_0^{\frac{\pi}{8}}\frac{1+\cos 2t}{2}\,\mathrm{d}t-2\pi+2\tan t\Big|_0^{\frac{\pi}{8}}\\
&= \pi+2\sqrt{2}-2\pi+2\tan\frac{\pi}{8}\\
&= 4\sqrt{2}-2-\pi,
\end{aligned}$$

since

$$1=\frac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$$

gives us that $\tan\frac{\pi}{8} = \sqrt{2} - 1$.

(2) By changing of variables, we have

$$\begin{aligned}
\int_1^2 \frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}}\,\mathrm{d}x &\xlongequal{x=\frac{1}{t^2}} \frac{1}{2}\int_{\frac{1}{4}}^1 \frac{1-t}{\sqrt{t^2-2t+2}}\,\mathrm{d}t \\
&= \frac{1}{2}\int_{\frac{1}{4}}^1 \frac{1-t}{\sqrt{(t-1)^2+1}}\,\mathrm{d}t \\
&= -\frac{1}{2}\sqrt{(t-1)^2+1}\Big|_{\frac{1}{4}}^1 \\
&= \frac{1}{8}.
\end{aligned}$$

(3) By integral by parts, we have

$$\begin{aligned}
\int x\sin(\ln x)\,\mathrm{d}x &= \frac{1}{2}x^2\sin(\ln x) - \frac{1}{2}\int x\cos(\ln x)\,\mathrm{d}x \\
&= \frac{1}{2}x^2\sin(\ln x) - \frac{1}{2}\left(\frac{1}{2}x^2\cos(\ln x) + \frac{1}{2}\int x\sin(\ln x)\,\mathrm{d}x\right) \\
&= \frac{1}{2}x^2\sin(\ln x) - \frac{1}{4}x^2\cos(\ln x) - \frac{1}{4}\int x\sin(\ln x)\,\mathrm{d}x,
\end{aligned}$$

which yields

$$\int x\sin(\ln x)\,\mathrm{d}x = \frac{2}{5}x^2\sin(\ln x) - \frac{1}{5}x^2\cos(\ln x) + C.$$

(4) By changing of variables, we have

$$\begin{aligned}
\int \frac{1}{x+\sqrt{x^2-x+1}}\,\mathrm{d}x &\xlongequal{t=x+\sqrt{x^2-x+1}} 2\int \frac{t^2-t+1}{t(2t-1)^2}\,\mathrm{d}t \\
&= \int \frac{2}{t}\,\mathrm{d}t - \int \frac{3}{2t-1}\,\mathrm{d}t + \int \frac{3}{(2t-1)^2}\,\mathrm{d}t \\
&= \ln t - \frac{3}{2}\ln(2t-1) - \frac{3}{2}\frac{1}{2t-1} + C \\
&= \ln(x+\sqrt{x^2-x+1}) \\
&\quad - \frac{3}{2}\ln(2(x+\sqrt{x^2-x+1})-1) \\
&\quad - \frac{3}{2(2(x+\sqrt{x^2-x+1})-1)} + C.
\end{aligned}$$

□

Problem 13.3. *Suppose that a curve L can be given by $y = y(x) \in C^4(\mathbb{R})$ in the xy-coordinate system. Rotate the xy-coordinate system against the clockwise $\pi/4$ to get* *the new coordinate system, say (t, s). Assume that L can be given by $s = s(t) \in C^4(\mathbb{R})$ in the st-coordinate system. If $y'(x) > -1$ and $y''(x) \neq 0$, prove that $s''(t) \neq 0$ and there is*

$$\left[s''(t)^{-\frac{2}{3}}\right]''(t) = \left[y''(x)^{-\frac{2}{3}}\right]''(x),$$

where $(x, y(x))$ and $(t, s(t))$ are the same point in the curve.

Proof. Note that

$$\begin{cases} t = x\cos\dfrac{\pi}{4} + y\sin\dfrac{\pi}{4}, \\ s = -x\sin\dfrac{\pi}{4} + y\cos\dfrac{\pi}{4}. \end{cases}$$

By $y = y(x)$, we know that L can be given by

$$\begin{cases} t = \dfrac{\sqrt{2}}{2}x + \dfrac{\sqrt{2}}{2}y(x), \\ s = -\dfrac{\sqrt{2}}{2}x + \dfrac{\sqrt{2}}{2}y(x). \end{cases}$$

Hence, there is

$$\begin{cases} \mathrm{d}t = \dfrac{\sqrt{2}}{2}(1 + y'(x))\mathrm{d}x, \\ \mathrm{d}s = \dfrac{\sqrt{2}}{2}(-1 + y'(x))\mathrm{d}x. \end{cases}$$

Then we have

$$s'(t) = \frac{-1 + y'(x)}{1 + y'(x)} \quad \text{and} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\sqrt{2}}{1 + y'(x)}.$$

Taking derivative yields

$$s''(t) = \frac{y''(x)\frac{\mathrm{d}x}{\mathrm{d}t}(1 + y'(x)) - y''(x)\frac{\mathrm{d}x}{\mathrm{d}t}(-1 + y'(x))}{(1 + y'(x))^2} = \frac{2\sqrt{2}y''(x)}{(1 + y'(x))^3}.$$

Since $y''(x) \neq 0$, it is clear that $s''(t) \neq 0$. What's more, since

$$s''(t)^{-\frac{2}{3}} = y''(x)^{-\frac{2}{3}}\frac{(1 + y'(x))^2}{2},$$

we have

$$\begin{aligned} \left[s''(t)^{-\frac{2}{3}}\right]'(t) &= \left[y''(x)^{-\frac{2}{3}}\right]'\frac{\mathrm{d}x}{\mathrm{d}t}\frac{(1 + y'(x))^2}{2} + y''(x)^{-\frac{2}{3}}(1 + y'(x))y''(x)\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= \left[y''(x)^{-\frac{2}{3}}\right]'\frac{1 + y'(x)}{\sqrt{2}} + \sqrt{2}y''(x)^{\frac{1}{3}}. \end{aligned}$$

Then

$$\left[s''(t)^{-\frac{2}{3}}\right]''(t) = \left[y''(x)^{-\frac{2}{3}}\right]''(x) + \left[y''(x)^{-\frac{2}{3}}\right]'\frac{y''(x)}{\sqrt{2}}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\sqrt{2}}{3}y''(x)^{-\frac{2}{3}}y'''(x)\frac{\mathrm{d}x}{\mathrm{d}t}$$

$$
\begin{aligned}
&= \left[y''(x)^{-\frac{2}{3}}\right]''(x) + \left[y''(x)^{-\frac{2}{3}}\right]' y''(x)\frac{1}{1+y'(x)} + \frac{2}{3}y''(x)^{-\frac{2}{3}}y'''(x)\frac{1}{1+y'(x)} \\
&= \left[y''(x)^{-\frac{2}{3}}\right]''(x) - \frac{2}{3}y''(x)^{-\frac{2}{3}}y'''(x)\frac{1}{1+y'(x)} + \frac{2}{3}y''(x)^{-\frac{2}{3}}y'''(x)\frac{1}{1+y'(x)} \\
&= \left[y''(x)^{-\frac{2}{3}}\right]''(x).
\end{aligned}
$$

□

Problem 13.4. *Suppose that $f \in C^\infty(\mathbb{R})$ and for any $k \in \mathbb{N}$, there is*

$$\sup_{x\in\mathbb{R}} \big||x|^k |f(x)| + |f^{(k)}(x)|\big| < +\infty.$$

Prove that for any $k, l \in \mathbb{N}$, there is

$$\sup_{x\in\mathbb{R}} \big||x|^k |f^{(l)}(x)|\big| < +\infty.$$

Proof. We prove the conclusion by induction. For $l = 0$, it's clear that $\sup_{x\in\mathbb{R}} \big||x|^k |f(x)|\big| < +\infty$ for any $k \in \mathbb{N}$. Assume that for any $0 \le l \le n$ and $k \in \mathbb{N}$, there is

$$\sup_{x\in\mathbb{R}} \big||x|^k |f^{(l)}(x)|\big| < +\infty.$$

We will show that $\sup_{x\in\mathbb{R}} \big||x|^k |f^{(n+1)}(x)|\big| < +\infty$ for any $k \in \mathbb{N}$. Indeed, by Taylor's formula, we have for any $x > 0$ that

$$f(x+h) = f(x) + f'(x)h + \cdots + \frac{f^{(n+1)}(x)}{(n+1)!}h^{n+1} + \frac{f^{(n+2)}(\xi)}{(n+2)!}h^{n+2}.$$

Taking $h = |x|^{-k}$, we have

$$
\begin{aligned}
\big||x|^k |f^{(n+1)}(x)|\big| \le (n+1)!\bigg(&|x|^{(n+2)k}|f(x+|x|^{-k})| + |x|^{(n+2)k}|f(x)| \\
&+ |x|^{(n+1)k}|f'(x)| + \cdots + \frac{1}{n!}|x|^{2k}|f^{(n)}(x)| + \frac{|f^{(n+2)}(\xi)|}{(n+2)!}\bigg).
\end{aligned}
$$

By $\sup_{x\in\mathbb{R}} \big||x|^k |f(x)| + |f^{(k)}(x)|\big| < +\infty$, $\sup_{x>0} \dfrac{x}{x+|x|^{-k}} < +\infty$ and the assumption, we know that

$$\sup_{x>0} \big||x|^k f^{(n+1)}(x)|\big| < +\infty \quad \text{for any } k \in \mathbb{N}.$$

For any $x < 0$, we just need to take $h = -|x|^{-k}$. For $x = 0$, it's clear. Hence we know that for any $k, l \in \mathbb{N}$, there is

$$\sup_{x\in\mathbb{R}} \big||x|^k f^{(l)}(x)|\big| < +\infty.$$

□

Problem 13.5. *Suppose that $f(x)$ is twice differentiable on $[-2,2]$, $|f(x)| \leq 1$ and $[f(0)]^2 + [f'(0)]^2 = 4$. Prove that there exists $\xi \in (-2,2)$ such that $f''(\xi) + f(\xi) = 0$.*

Proof. Let

$$F(x) = f(x)^2 + f'(x)^2, \quad \forall x \in [-2,2].$$

Then $F(0) = 4$. By the Lagrange Mean Value Theorem, we know that there exists $x_1 \in (-2,0)$ such that

$$f'(x_1) = \frac{f(0) - f(-2)}{2}.$$

Since $|f(x)| \leq 1$, we have that $|f'(x_1)| \leq 1$. Similarly, we know that there exists $x_2 \in (0,2)$ such that $|f'(x_2)| \leq 1$. Then $F(x_1) \leq 2$ and $F(x_2) \leq 2$. Note that $x_1 < 0 < x_2$ and $F(0) = 4 > 2$, we know that there must be at least a maximum point in (x_1, x_2). Hence, there exists $\xi \in (-2,2)$ such that $F'(\xi) = 0$, i.e. $f(\xi) + f''(\xi) = 0$ since $f'(\xi) \neq 0$. To prove $f'(\xi) \neq 0$, it suffices to note that $F(\xi) \geq 4$ and $f(\xi)^2 \leq 1$. Then we are done. □

Problem 13.6. *Suppose that $f(x)$ is nonnegative convex function on $[-1,1]$, satisfying $f(0) = 0$ and $f(-1) = f(1) = 1$. Define $S(h) = \{x | f(x) \leq h\}$, $\forall h \in [0,1]$.*

(1) *If there exists $\varepsilon > 0$ such that $\forall x \in [-1,1]$, there is $f\left(\frac{x}{2}\right) \leq \frac{1-\varepsilon}{2} f(x)$. Prove that there exist $\alpha > 0$ and $C > 0$ such that $f(x) \leq C|x|^{1+\alpha}$, $\forall x \in [-1,1]$.*

(2) *If there exists $\varepsilon \in (0, 1/2)$ such that $\forall h \in [0,1]$, there is $l\left(\frac{h}{2}\right) \leq (1-\varepsilon) l(h)$, where $l(h)$ is the length of $S(h)$. Prove that there exist $\beta > 0$ and $C > 0$ such that $f(x) \geq C|x|^{1+\beta}$, $\forall x \in [-1,1]$.*

Proof. (1) By $f\left(\frac{x}{2}\right) \leq \frac{1-\varepsilon}{2} f(x)$, we have

$$f\left(\frac{x}{2^k}\right) \leq \left(\frac{1-\varepsilon}{2}\right)^k f(x), \quad \forall x \in [-1,1],\, k \geq 0.$$

Since $f(x)$ is convex and $f(-1) = f(1) = 1$, we know that $f(x) \leq 1$, $\forall x \in [-1,1]$. Choosing $\alpha > 0$ such that $2^{-\alpha} = 1 - \varepsilon$, i.e. $\alpha = -\frac{\ln(1-\varepsilon)}{\ln 2}$. Then there is

$$f\left(\frac{x}{2^k}\right) \leq \left(\frac{1-\varepsilon}{2}\right)^k = \left(\frac{1}{2^k}\right)^{1+\alpha}, \quad \forall x \in [-1,1],\, k \geq 0.$$

Hence, for $\forall x' \in [-1,1]$, we know that there exists $k = k(x')$ such that

$$\frac{1}{2^{k+1}} < |x'| \leq \frac{1}{2^k}.$$

Then taking $x = 2^k x' \in [-1,1]$, we have

$$f(x') = f\left(\frac{x}{2^k}\right) \le \left(\frac{1}{2^k}\right)^{1+\alpha} = \left(\frac{1}{2^{k+1}}\right)^{1+\alpha} 2^{1+\alpha} \le 2^{1+\alpha}|x'|^{1+\alpha}, \quad \forall\, x' \in [-1,1].$$

i.e. there exist $\alpha = -\dfrac{\ln(1-\varepsilon)}{\ln 2}$ and $C = 2^{1+\alpha}$ such that

$$f(x) \le C|x|^{1+\alpha}, \quad \forall\, x \in [-1,1].$$

(2) Similar to (1), we have

$$l(h) \le 2^{1+\alpha}h^{\alpha}, \quad \forall\, h \in [0,1],$$

where $\alpha = -\dfrac{\ln(1-\varepsilon)}{\ln 2}$. Next, we prove that $\forall\, x \in [-1,1]$, there is

$$f(x) \ge 2^{-\left(\frac{2}{\alpha}+1\right)}|x|^{\frac{1}{\alpha}}.$$

We prove the claim by contradiction. Assume that there exists $x_0 \in [-1,1]$ such that

$$f(x_0) < 2^{-\left(\frac{2}{\alpha}+1\right)}|x_0|^{\frac{1}{\alpha}}.$$

Without loss of generality, we may assume that $x_0 > 0$, and it's similar for $x_0 < 0$. Since $f(x)$ is a convex function, we know that $\forall\, x \in [0, x_0]$, there is

$$f(x) \le \lambda f(x_0) + (1-\lambda)f(0) \le f(x_0) < 2^{-\left(\frac{2}{\alpha}+1\right)}|x_0|^{\frac{1}{\alpha}}.$$

Hence $[0, x_0] \subset S(h_0)$, where $h_0 = 2^{-\left(\frac{2}{\alpha}+1\right)}|x_0|^{\frac{1}{\alpha}} < 1$. Then

$$|x_0| \le l(h_0) \le 2^{1+\alpha}\left(2^{-\left(\frac{2}{\alpha}+1\right)}|x_0|^{\frac{1}{\alpha}}\right)^{\alpha} = 2^{1+\alpha}\cdot 2^{-(2-\alpha)}|x_0| = \frac{1}{2}|x_0|,$$

contradiction. Hence, $\forall\, x \in [-1,1]$, there is

$$f(x) \ge 2^{-\left(\frac{2}{\alpha}+1\right)}|x|^{\frac{1}{\alpha}}.$$

Therefore, we can take $C = 2^{-\left(\frac{2}{\alpha}+1\right)}$ and $\beta = \dfrac{1}{\alpha} - 1$. Since $\varepsilon \in (0, 1/2)$ and $\alpha = -\dfrac{\ln(1-\varepsilon)}{\ln 2}$, we know that $\alpha \in (0,1)$, then $\beta > 0$. □

Problem 13.7. *Suppose that $f(x) \in C^1(\mathbb{R})$ satisfying $\sup\limits_{x\in\mathbb{R}}|f(x)| \le A \in (0,+\infty)$ and $\sup\limits_{x\in\mathbb{R},y>x}\left|\dfrac{f'(y)-f'(x)}{y-x}\right| \le B \in (0,+\infty)$. Prove that $\forall\, x \in \mathbb{R}$, there is $|f'(x)| \le \sqrt{2AB}$.*

Proof. By the Newton-Leibniz formula, we have

$$f(x+h) = f(x) + f'(x)h + \int_x^{x+h} \left(f'(t) - f'(x)\right)\, \mathrm{d}t,$$

$$f(x-h) = f(x) - f'(x)h + \int_{x-h}^{x} (f'(t) - f'(x)) \, \mathrm{d}t.$$

Hence there are

$$|f(x+h) - f(x) - f'(x)h| \le B \int_{x}^{x+h} (t-x) \, \mathrm{d}t = \frac{B}{2}h^2,$$

$$|f(x-h) - f(x) + f'(x)h| \le B \int_{x-h}^{x} (x-t) \, \mathrm{d}t = \frac{B}{2}h^2.$$

Then

$$|2hf'(x) + f(x-h) - f(x+h)| \le Bh^2,$$

which yields

$$|f'(x)| \le \frac{1}{2h}\left(Bh^2 + |f(x+h) - f(x-h)|\right) \le \frac{A}{h} + \frac{Bh}{2}.$$

Choosing $h = \sqrt{\dfrac{2A}{B}}$, we have

$$|f'(x)| \le \sqrt{2AB}.$$

□

Problem 13.8. *Suppose $f(x) \in C[0,1]$ is positive, and $\int_0^1 f(x) \, \mathrm{d}x = A$, $\int_0^1 f^2(x) \, \mathrm{d}x = B$.*

(1) *Prove that for any $n \in \mathbb{N}_+$, there exists a partition $\Delta : 0 = x_0 < \cdots < x_n = 1$ such that $\int_{x_{k-1}}^{x_k} f(x) \, \mathrm{d}x = \frac{A}{n}$, $k = 1, 2, \cdots, n$.*

(2) *Find $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(x_k)$.*

Proof. (1) Since $f(x)$ is continuous and positive, we know that $\int_0^x f(t) \, \mathrm{d}t$ is continuous and increasing. By the intermediate value theorem, we have that there exist $0 = x_0 < \cdots < x_n = 1$ such that $\int_0^{x_k} f(t) \, \mathrm{d}t = \frac{kA}{n}$, hence $\int_{x_{k-1}}^{x_k} f(x) \, \mathrm{d}x = \frac{A}{n}$, $k = 1, 2, \cdots, n$.

(2) By the mean value theorems for definite integrals, we know that there exists $\xi_k \in (x_{k-1}, x_k)$, such that

$$\int_{x_{k-1}}^{x_k} f(x) \, \mathrm{d}x = f(\xi_k)(x_k - x_{k-1}) = f(\xi_k)\Delta x_k = \frac{A}{n}, \quad k = 1, 2, \cdots, n.$$

Since $f(x)$ is continuous on $[0,1]$, we know that $f(x)$ is uniformly continuous. Then for $\forall \varepsilon > 0$, there exits $\delta > 0$ such that $\forall x, x' : |x - x'| < \delta$, there is $|f(x) - f(x')| < \varepsilon$.

Hence for n large enough, we have $\Delta x_k < \delta$, which gives us that $|f(x_k) - f(\xi_k)| < \varepsilon$. Then

$$\begin{aligned}\left|\frac{1}{n}\sum_{k=1}^{n} f(x_k) - \frac{B}{A}\right| &= \left|\frac{1}{n}\sum_{k=1}^{n} f(x_k) - \frac{1}{A}\int_0^1 f^2(x)\,\mathrm{d}x\right| \\ &\le \left|\frac{1}{n}\sum_{k=1}^{n} f(x_k) - \frac{1}{A}\sum_{k=1}^{n} f^2(x_k)\Delta x_k\right| \\ &\quad + \left|\frac{1}{A}\sum_{k=1}^{n} f^2(x_k)\Delta x_k - \frac{1}{A}\int_0^1 f^2(x)\,\mathrm{d}x\right| \\ &\xlongequal{\frac{1}{n}=\frac{1}{A}f(\xi_k)\Delta x_k} \left|\frac{1}{A}\sum_{k=1}^{n} f(x_k)f(\xi_k)\Delta x_k - \frac{1}{A}\sum_{k=1}^{n} f^2(x_k)\Delta x_k\right| \\ &\qquad + \left|\frac{1}{A}\sum_{k=1}^{n} f^2(x_k)\Delta x_k - \frac{1}{A}\int_0^1 f^2(x)\,\mathrm{d}x\right| \\ &\le \varepsilon\left|\frac{1}{A}\sum_{k=1}^{n} f(x_k)\Delta x_k\right| + \left|\frac{1}{A}\sum_{k=1}^{n} f^2(x_k)\Delta x_k - \frac{1}{A}\int_0^1 f^2(x)\,\mathrm{d}x\right| \\ &\to \varepsilon, \quad \text{as } n \to \infty.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f(x_k) = \frac{B}{A}.$$

□

Problem 13.9. *Prove that for any $n \in \mathbb{N}_+$, there is* $\left|\int_1^2 \sin\left(nx - \frac{1}{x}\right)\,\mathrm{d}x\right| < \frac{2}{n}$.

Proof. Let

$$t = x - \frac{1}{nx}.$$

It's clear that

$$\frac{\mathrm{d}t}{\mathrm{d}x} = 1 + \frac{1}{nx^2} > 0.$$

Hence we know that there exists inverse function of $t = t(x)$, i.e. $x = x(t)$. What's more, we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \left(1 + \frac{1}{nx^2}\right)^{-1}.$$

By changing of variables, we have

$$\int_1^2 \sin\left(nx - \frac{1}{x}\right)\,\mathrm{d}x = \int_{1-\frac{1}{n}}^{2-\frac{1}{2n}} \sin(nt)x'(t)\,\mathrm{d}t.$$

Note that

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\left(1+\frac{1}{nx^2}\right)^{-2}\frac{-2}{nx^3}\frac{\mathrm{d}x}{\mathrm{d}t} = \left(1+\frac{1}{nx^2}\right)^{-3}\frac{2}{nx^3} > 0,$$

which gives us that $x'(t)$ is monotonic increasing. Then by the second mean value theorem for definite integrals, we know that there exists ξ such that

$$\begin{aligned}
\left|\int_1^2 \sin\left(nx - \frac{1}{x}\right)\,\mathrm{d}x\right| &= \left|\int_{1-\frac{1}{n}}^{2-\frac{1}{2n}} \sin(nt)x'(t)\,\mathrm{d}t\right| \\
&= \left|x'\left(2-\frac{1}{2n}\right)\int_{\xi}^{2-\frac{1}{2n}} \sin(nt)\,\mathrm{d}t\right| \\
&= \left(1+\frac{1}{4n}\right)^{-1}\frac{1}{n}\left|\cos\left(2-\frac{1}{2n}\right) - \cos\xi\right| \\
&\le \left(1+\frac{1}{4n}\right)^{-1}\frac{2}{n} \\
&< \frac{2}{n}.
\end{aligned}$$

□

Problem 13.10. *Suppose that $f(x)$ is a nonnegative monotonic increasing function on $[0, \frac{\pi}{2}]$. Prove that when $x \in [0, \frac{\pi}{2}]$, there is $(1-\cos x)\int_0^x f(t)\,\mathrm{d}t \le x\int_0^x f(t)\sin t\,\mathrm{d}t$.*

Proof. Let

$$g(x) = \frac{1-\cos x}{x},$$

and

$$h(x) = \int_0^x f(t)\sin t\,\mathrm{d}t - g(x)\int_0^x f(t)\,\mathrm{d}t.$$

Then

$$\begin{aligned}
h'(x) &= f(x)\sin x - g(x)f(x) - g'(x)\int_0^x f(t)\,\mathrm{d}t \\
&= f(x)\sin x - f(x)\frac{1-\cos x}{x} - \frac{x\sin x - 1 + \cos x}{x^2}\int_0^x f(t)\,\mathrm{d}t \\
&= \frac{x\sin x - 1 + \cos x}{x^2}\left(xf(x) - \int_0^x f(t)\,\mathrm{d}t\right).
\end{aligned}$$

It's easy to see that $x\sin x - 1 + \cos x \geq 0$ on $[0, \frac{\pi}{2}]$ (Leave to the reader). Since $f(x)$ is nonnegative and monotonic increasing, we have

$$\int_0^x f(t)\,\mathrm{d}t \leq xf(x),$$

which implies

$$h'(x) \geq 0$$

on $[0, \frac{\pi}{2}]$. Note that $h(0) = 0$, we have $h(x) \geq h(0) = 0$, $\forall\, x \in [0, \frac{\pi}{2}]$. Hence

$$(1 - \cos x)\int_0^x f(t)\,\mathrm{d}t \leq x\int_0^x f(t)\sin t\,\mathrm{d}t, \quad \forall\, x \in \left[0, \frac{\pi}{2}\right].$$

□

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