EXERCISES COURSE

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Contents

1. Week 2 (3.1)

Problem 1.1. *Calculate* $\displaystyle\int_0^2 \frac{\mathrm{d}x}{\sqrt{(4-x^2)(9-x^2)}}$.

Solution. Let $x = 2\sin\theta$. The integral becomes

$$\int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{9-4\sin^2\theta}} = \frac{1}{3}\int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1-\frac{4}{9}\sin^2\theta}} = \frac{1}{3}F\left(\frac{2}{3}, \frac{\pi}{2}\right).$$

□

Problem 1.2. *Calculate* $\displaystyle\int_0^1 \frac{\mathrm{d}x}{\sqrt{(1+x^2)(1+2x^2)}}$.

Date: June 9, 2023.

Solution. Let $x = \tan\theta$. The integral becomes

$$\begin{aligned}\int_0^{\pi/4} \frac{\sec^2\theta \mathrm{d}\theta}{\sqrt{1+\tan^2\theta}\sqrt{1+2\tan^2\theta}} &= \int_0^{\pi/4} \frac{\mathrm{d}\theta}{\sqrt{\cos^2\theta + 2\sin^2\theta}} \\ &= \int_0^{\pi/4} \frac{\mathrm{d}\theta}{\sqrt{2-\cos^2\theta}} \\ &= \frac{1}{\sqrt{2}}\int_0^{\pi/4} \frac{\mathrm{d}\theta}{\sqrt{1-\frac{1}{2}\cos^2\theta}}.\end{aligned}$$

Let $\phi = \pi/2 - \theta$. The integral becomes

$$\frac{1}{\sqrt{2}}\int_{\pi/4}^{\pi/2} \frac{\mathrm{d}\phi}{\sqrt{1-\frac{1}{2}\sin^2\phi}} = \frac{1}{\sqrt{2}}\left\{F\left(\frac{1}{\sqrt{2}}, \frac{\pi}{2}\right) - F\left(\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)\right\}.$$

□

Problem 1.3. *Calculate* $\displaystyle\int_4^6 \frac{\mathrm{d}x}{\sqrt{(x-1)(x-2)(x-3)}}$.

Solution. Let $u = \sqrt{x-3}$ or $x = 3+u^2$. The integral becomes

$$2\int_1^{\sqrt{3}} \frac{\mathrm{d}u}{\sqrt{(u^2+2)(u^2+1)}}.$$

Let $u = \tan\theta$. The integral becomes

$$\begin{aligned}2\int_{\pi/4}^{\pi/3} \frac{\mathrm{d}\theta}{\sqrt{2\cos^2\theta+\sin^2\theta}} = 2\int_{\pi/4}^{\pi/3} \frac{\mathrm{d}\theta}{\sqrt{2-\sin^2\theta}} &= \sqrt{2}\int_{\pi/4}^{\pi/3} \frac{\mathrm{d}\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}} \\ &= \sqrt{2}\left\{F\left(\frac{1}{\sqrt{2}}, \frac{\pi}{3}\right) - F\left(\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)\right\}.\end{aligned}$$

□

Problem 1.4. *Calculate* $\displaystyle\int \sqrt{\sin x}\,\mathrm{d}x$.

Solution.

$$\begin{aligned}\int \sqrt{\sin x}\,\mathrm{d}x &= \int \sqrt{\cos\left(x-\frac{\pi}{2}\right)}\,\mathrm{d}x \\ &= \int \sqrt{1-2\sin^2\left(\frac{x}{2}-\frac{\pi}{4}\right)}\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= 2\int \sqrt{1-2\sin^2\varphi}\,\mathrm{d}\varphi \\
&= 2E(\sqrt{2},\varphi).
\end{aligned}
$$

□

Problem 1.5. *Calculate* $\displaystyle\int_0^{2\pi} \sqrt{2+\sin 2\theta}\,\mathrm{d}\theta$.

Solution.

$$
\begin{aligned}
\int_0^{2\pi} \sqrt{2+\sin 2\theta}\,\mathrm{d}\theta &= 2\int_0^{\pi} \sqrt{2+\sin 2\theta}\,\mathrm{d}\theta \\
&= 2\int_0^{\pi} \sqrt{2+\cos 2\left(\theta-\frac{\pi}{4}\right)}\,\mathrm{d}\theta \\
&= 2\int_0^{2\pi} \sqrt{3-2\sin^2\left(\theta-\frac{\pi}{4}\right)}\,\mathrm{d}\theta \\
&= 2\sqrt{3}\int_{-\frac{\pi}{4}}^{\frac{3}{4}\pi} \sqrt{1-\frac{2}{3}\sin^2\theta}\,\mathrm{d}\theta \\
&= 2\sqrt{3}\int_0^{\frac{\pi}{4}} \sqrt{1-\frac{2}{3}\sin^2\theta}\,\mathrm{d}\theta + 2\sqrt{3}\int_0^{\frac{3}{4}\pi} \sqrt{1-\frac{2}{3}\sin^2\theta}\,\mathrm{d}\theta \\
&= 2\sqrt{3}\left(E\left(\frac{\sqrt{6}}{3},\frac{\pi}{4}\right)+E\left(\frac{\sqrt{6}}{3},\frac{3\pi}{4}\right)\right).
\end{aligned}
$$

□

Problem 1.6. *Calculate the perimeter of an ellipse.*

Solution. The ellipse $x = a\cos\theta$, $4y = b\sin\theta$, $a > b > 0$, has length

$$
\begin{aligned}
L &= 4\int_0^{\pi/2} \sqrt{\mathrm{d}x^2+\mathrm{d}y^2} = 4\int_0^{\pi/2} \sqrt{a^2\cos^2\theta+b^2\sin^2\theta}\mathrm{d}\theta \\
&= 4\int_0^{\pi/2} \sqrt{a^2-(a^2-b^2)\sin^2\theta}\mathrm{d}\theta = 4a\int_0^{\pi/2} \sqrt{1-e^2\sin^2\theta}\mathrm{d}\theta,
\end{aligned}
$$

where $e^2 = (a^2-b^2)/a^2 = c^2/a^2$ is the square of the eccentricity of the ellipse. The result can be written as

$$
L = 4aE\left(e,\frac{\pi}{2}\right) = 4aE(e).
$$

For the special case of a circle, $a = b = r$, i.e., $e = 0$, and $E(0) = \pi/2$, and we recover the circumference of a circle: $L = 2\pi r$. □

Remark 1.7. *The term elliptic integral was coined by Count Fagnano (1682-1766) in 1750. He discovered that the arclength of the lemniscate can be expressed in terms of an elliptic integral of the first kind.*

Problem 1.8. *Calculate the arclength of a lemniscate.*

Solution. The lemniscate is the curve:

$$\left(x^2+y^2\right)^2 = a^2\left(x^2-y^2\right)^2,$$

or in polar form

$$r^2 = a^2\cos 2\theta.$$

From $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2$,

$$\begin{aligned} L = 4\int_{r=0}^{\pi/4}\mathrm{d}s = 4a\int_0^{\pi/4}\frac{\mathrm{d}\theta}{\sqrt{\cos 2\theta}}, \quad (\cos 2\theta = \cos^2 u) \\ = \int_0^{\pi/2}\frac{\mathrm{d}u}{\sqrt{2-\sin^2 u}} = \frac{1}{\sqrt{2}}\int_0^{\pi/2}\frac{\mathrm{d}u}{\sqrt{1-\frac{1}{2}\sin^2 u}} = \frac{1}{\sqrt{2}}\cdot F\left(\frac{1}{\sqrt{2}},\frac{\pi}{2}\right). \end{aligned}$$

Thus,

$$L = 4a\cdot\frac{1}{\sqrt{2}}F\left(\frac{1}{\sqrt{2}},\frac{\pi}{2}\right) = a\cdot 2\sqrt{2}(1.85407) = 5.244102a.$$

□

Remark 1.9. *(Historical note: The rectification of the lemniscate was first done by Fagnano in 1718. The lemniscatus, L. 'decorated by ribbons', was first studied in astronomy in 1680 by Cassini, known as the ovals of Cassini, but his book was published in 1749, many years after his death. The curves were popularized by the Bernoulli brothers in 1694.) Cassini considered more general forms of the lemniscate for whose points the products of the distances to two foci is a constant:*

$$d_1d_2 = b^2,$$

$$b^4 = r^4 + \frac{a^2}{4} - r^2a^2\cos 2\theta.$$

When $b = \frac{a}{\sqrt{2}}$ centered at the origin, we get the ribbon-shaped curve.

Exercise 1.10 (Leave to the reader)**.** *Calculate the finite-amplitude pendulum.*

Solution. The equation of motion is:

$$ml\ddot{\theta} = -mg\sin\theta.$$

Let $p = \dot{\theta}$, we have

$$p\frac{\mathrm{d}p}{\mathrm{d}\theta} = -\frac{g}{l}\sin\theta,$$

which yields

$$\frac{p^2}{2} = \frac{g}{l}\cos\theta + C.$$

For the initial condition, $t = 0 : \theta = \theta_0$, $\dot{\theta} = 0$, we have

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\sqrt{\frac{2g}{l}}\sqrt{\cos\theta - \cos\theta_0}.$$

The period, T, is given by

$$\frac{T}{4} = -\sqrt{\frac{l}{2g}}\int_{\theta_0}^{0}\frac{\mathrm{d}\theta}{\sqrt{\cos\theta - \cos\theta_0}}$$

$$\text{or,}\qquad T = 4\sqrt{\frac{l}{2g}}\int_0^{\theta_0}\frac{\mathrm{d}\theta}{\sqrt{\cos\theta - \cos\theta_0}} = 2\sqrt{\frac{l}{g}}\int_0^{\theta_0}\frac{\mathrm{d}\theta}{\sqrt{\sin^2(\theta_0/2) - \sin^2(\theta/2)}}$$

$$= 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{\mathrm{d}u}{\sqrt{1-k^2\sin^2 u}},\ \sin\left(\frac{\theta}{2}\right) = \sin\frac{\theta_0}{2}\cdot\sin u,\ k = \sin\left(\frac{\theta_0}{2}\right).$$

Hence

$$T = 4\sqrt{\frac{l}{g}}\cdot F\left(k, \frac{\pi}{2}\right),$$

an elliptic integral. For the special case of smalloscillations, $k = 0$, we get the classical result:

$$T = 2\pi\sqrt{\frac{l}{g}}$$

$\square$

Problem 2.1. *Calculate the area of regions closed by the following curves.*

(1) *The cycloid:* $x = a(t - \sin t)$, $y = a(1 - \cos t)$ $(0 \le t \le 2\pi,\ a > 0)$ *and* x*-axis.*
(2) *The evolvent of sphere* $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$ $(0 \le t \le 2\pi,\ a > 0)$ *and* $x = a$.
(3) *The rhodonea curve* $r = a\sin 3\theta$.
(4) *The folium of Descartes* $r = \dfrac{3a\sin\theta\cos\theta}{\sin^3\theta + \cos^3\theta}$.

Solution. (1)

$$\begin{aligned} S = \int_0^{2\pi a} y\,\mathrm{d}x &= \int_0^{2\pi} a(1-\cos t)\cdot a(1-\cos t)\,\mathrm{d}\theta \\ &= a^2\int_0^{2\pi}(\cos^2 t - 2\cos t + 1)\,\mathrm{d}\theta \\ &= 3\pi a^2. \end{aligned}$$

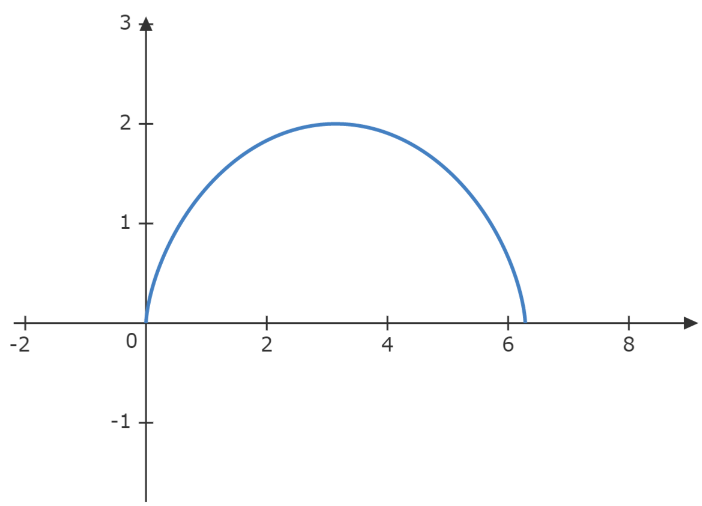

FIGURE 1. Graph of the cycloid for $a = 1$

(2)

$$\begin{aligned} S = \int (x-a)\,\mathrm{d}y &= a^2\int_0^{2\pi}(\cos t + t\sin t - 1)(\cos t - \cos t + t\sin t)\,\mathrm{d}t \\ &= a^2\int_0^{2\pi}\left(t\sin t\cos t + t^2\sin^2 t - t\sin t\right)\,\mathrm{d}t \end{aligned}$$

$$= \frac{a^2}{3}(4\pi^3 + 3\pi)$$

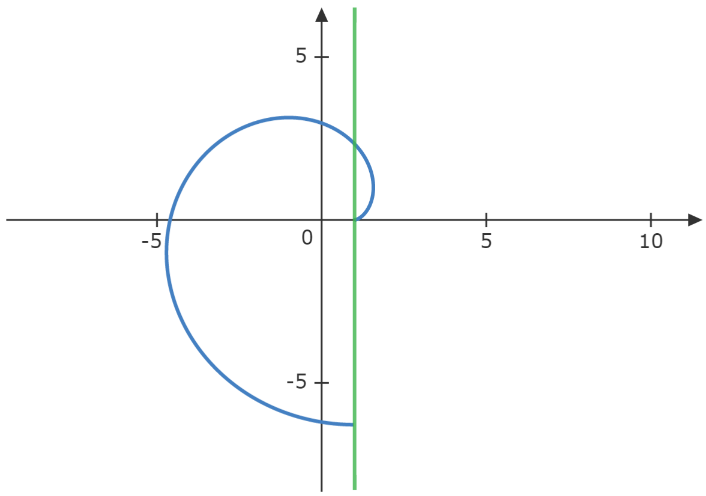

FIGURE 2. Graph of the evolvent for $a = 1$

(3)

$$\begin{aligned} S &= 3 \cdot \frac{1}{2} \int_0^{\frac{\pi}{3}} a^2 \sin^2 3\theta \, \mathrm{d}\theta \\ &= 3 \cdot \frac{a^2}{2} \int_0^{\frac{\pi}{3}} \frac{1 - \cos 6\theta}{2} \, \mathrm{d}\theta \\ &= \frac{\pi}{4} a^2. \end{aligned}$$

(4)

$$\begin{aligned} S &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{9a^2 \sin^2 \cos^2 \theta}{\left(\sin^3 \theta + \cos^3 \theta\right)^2} \, \mathrm{d}\theta \\ &= \frac{9a^2}{2} \int_0^{\frac{\pi}{2}} \frac{\tan^2 \theta}{(\tan^3 \theta + 1)^2} \, \mathrm{d} \tan \theta \\ &= \frac{3a^2}{2} \int_0^{\frac{\pi}{2}} \frac{1}{(\tan^3 \theta + 1)^2} \, \mathrm{d} \tan \theta \\ &= \frac{3a^2}{2} \frac{-1}{\tan^3 \theta + 1} \bigg|_0^{\frac{\pi}{2}} = \frac{3a^2}{2}. \end{aligned}$$

$\square$

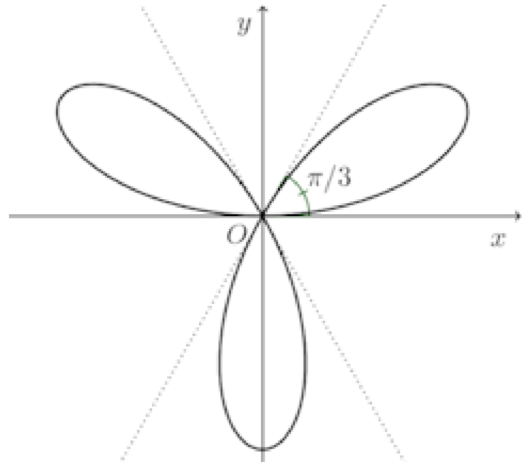

FIGURE 3. Graph of the rhodonea curve for $a = 1$

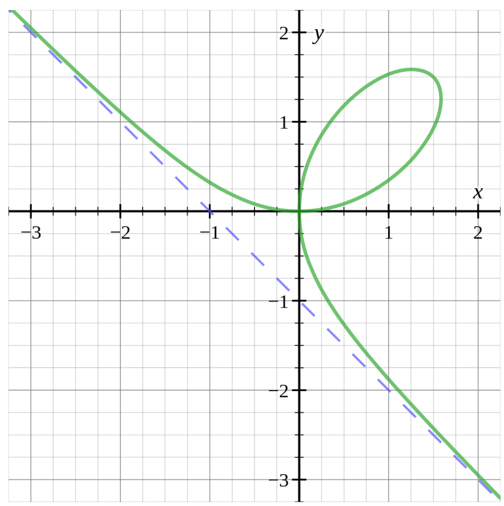

FIGURE 4. Graph of the folium of Descartes for $a = 1$

Problem 2.2 (7.43). *Find volumes of the solid of revolution obtained by rotating the region closed by $y = x^2\,(0 \le x \le h)$, $y = 0$ and $x = h$ around x-axis, y-axis, respectively.*

Solution. Around x-axis:

$$V = \pi \int_0^h y^2 \,\mathrm{d}x = \pi \int_0^h x^4 \,\mathrm{d}x = \frac{\pi}{5} h^5.$$

Around y-axis: **First way**:

$$V = \pi \int_0^h (h^2 - x^2)\,\mathrm{d}y = \pi \int_0^h (h^2 - x^2) 2x\,\mathrm{d}x = \frac{\pi}{2} h^4.$$

Second way:

$$V = 2\pi \int_0^h xy\,\mathrm{d}x = 2\pi \int_0^h x^3\,\mathrm{d}x = \frac{\pi}{2} h^4.$$

□

Problem 2.3 (7.49)**.** *Calculate the arc length of the following curves.*

(2) *Archimedean spiral:* $r = a\theta$ $(0 \le \theta \le 2\pi)$.

(4) *The evolvent of sphere* $x = a(\cos t + t \sin t)$, $y = a(\sin t - t\cos t)$ $(0 \le t \le 2\pi,\ a > 0)$.

Solution. (2)

$$\begin{aligned} L = \int_0^{2\pi} \sqrt{r^2 + r'^2}\,\mathrm{d}\theta &= a \int_0^{2\pi} \sqrt{1+\theta^2}\,\mathrm{d}\theta \\ &= \frac{a}{2}\theta\sqrt{1+\theta^2} + \frac{a}{2}\ln(\theta + \sqrt{1+\theta^2})\Big|_0^{2\pi} \\ &= a\pi\sqrt{1+4\pi^2} + \frac{a}{2}\ln(4\pi + \sqrt{1+4\pi^2}). \end{aligned}$$

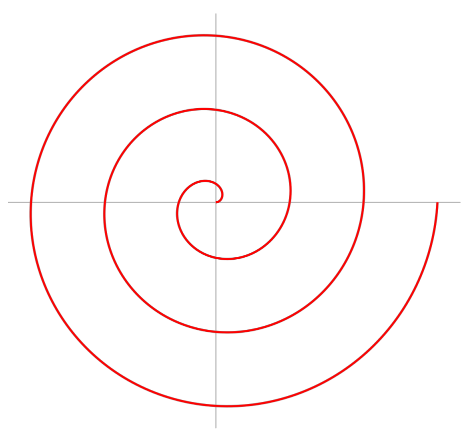

FIGURE 5. Graph of the Archimedean spiral for $a = 1$

(4)

$$L = \int_0^{2\pi} \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = a \int_0^{2\pi} \sqrt{(t\cos t)^2 + (t\sin t)^2}\,\mathrm{d}t$$

$$= a\int_0^{2\pi} t\,\mathrm{d}t = 2\pi^2 a.$$

□

Problem 2.4 (7.51(4)). *Calculate the area of the surface of revolution obtained by rotating the curve* $y = \sin x\,(0 \le x \le \pi)$ *around* x*-axis.*

Solution.

$$\begin{aligned} dS &= 2\pi y\sqrt{y'^2+1}\,\mathrm{d}x \\ &= 2\pi\sin x\sqrt{1+\cos^2 x}\,\mathrm{d}x. \end{aligned}$$

$$\begin{aligned} S &= 2\pi\int_0^{\pi} -\sqrt{1+\cos^2 x}\,\mathrm{d}\cos x \\ &= 4\pi\int_0^1 \sqrt{1+t^2}\,\mathrm{d}t = 2\sqrt{2}\pi + 2\pi\ln(\sqrt{2}+1). \end{aligned}$$

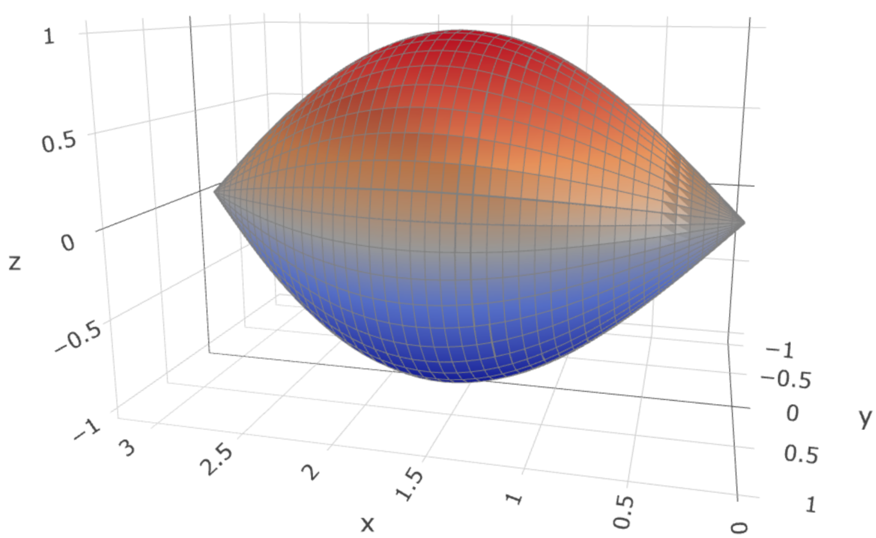

FIGURE 6. $y = \sin x$ rotated around x-axis

□

Problem 2.5. *Find the area of the region determined by* $x^2 + xy + y^2 = 1$.

Solution. **First way:** From the equation, we have

$$y_{1,2}(x) = -\frac{x}{2} \pm \sqrt{1-\frac{3}{4}x^2}, \quad -\frac{2}{\sqrt{3}} \le x \le \frac{2}{\sqrt{3}}.$$

Then

$$S = \int_{-2/\sqrt{3}}^{2/\sqrt{3}} [y_1(x) - y_2(x)] \, \mathrm{d}x = 2 \int_{-2/\sqrt{3}}^{2/\sqrt{3}} \sqrt{1 - \frac{3}{4}x^2} \, \mathrm{d}x$$
$$= \frac{4}{\sqrt{3}} \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta = \frac{2\pi}{\sqrt{3}}.$$

Second way: Let $x = r\cos\theta$, $y = r\sin\theta$, we have

$$r^2 = \frac{1}{1 + \sin\theta\cos\theta}.$$

Then

$$S = \frac{1}{2}\int_0^{2\pi} r^2 \, \mathrm{d}\theta = \frac{1}{2}\int_0^{2\pi} \frac{\mathrm{d}\theta}{1 + \sin\theta\cos\theta} = \int_0^{2\pi} \frac{\mathrm{d}\theta}{2 + \sin\theta} = \frac{2\pi}{\sqrt{3}}.$$

Third way: Note that

$$1 = x^2 + xy + y^2 = \frac{3}{4}x^2 + \left(y + \frac{x}{2}\right)^2.$$

Let

$$x = \frac{2}{\sqrt{3}}\cos t, \; y = \sin t - \frac{1}{\sqrt{3}}\cos t, \quad 0 \le t \le 2\pi.$$

Since

$$\begin{aligned} & x(t)y'(t) - y(t)x'(t) \\ & = \frac{2}{\sqrt{3}}\cos t\left(\cos t + \frac{1}{\sqrt{3}}\sin t\right) - \left(\sin t - \frac{1}{\sqrt{3}}\cos t\right)\left(-\frac{2}{\sqrt{3}}\sin t\right) \\ & = \frac{2}{\sqrt{3}}, \end{aligned}$$

we have

$$S = \frac{1}{2}\int_0^{2\pi} (x \, \mathrm{d}y - y \, \mathrm{d}x) = \frac{1}{2}\int_0^{2\pi} \frac{2}{\sqrt{3}} \, \mathrm{d}t = \frac{2\pi}{\sqrt{3}}.$$

□

Problem 2.6. *Suppose that a curve is given by* $y = \int_0^x \sqrt{\sin t} \, \mathrm{d}t$, $0 \le x \le \pi$. *Calculate its arc length.*

Solution. By the formula of arc length calculation, we have

$$L = \int_0^{\pi} \sqrt{1 + f'^2(x)} \, \mathrm{d}x = \int_0^{\pi} \sqrt{1 + \sin x} \, \mathrm{d}x$$
$$= \int_0^{\pi} \left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) \mathrm{d}x$$

$$
\begin{aligned}
&= 2\left(\int_0^{\pi/2} \sin t \, \mathrm{d}t + \int_0^{\pi/2} \cos t \, \mathrm{d}t\right) \\
&= 4\int_0^{\pi/2} \sin t \, \mathrm{d}t = 4.
\end{aligned}
$$

□

Problem 2.7. *Use Young's inequality to prove when $a, b \geq 1$, there is $ab \leq e^{a-1} + b \ln b$.*

Proof. Choosing $f(x) = e^x - 1$, $g(y) = \ln(y+1)$, we have

$$
\begin{aligned}
(a-1)(b-1) &\leq \int_0^{a-1} (e^x - 1) \, \mathrm{d}x + \int_0^{b-1} \ln(y+1) \, \mathrm{d}y \\
&= e^{a-1} - (a-1) - 1 + b \ln b - (b-1) \\
&= e^{a-1} + b \ln b - a - b + 1,
\end{aligned}
$$

i.e.

$$
ab \leq e^{a-1} + b \ln b.
$$

□

Exercise 2.8 (Leave to the reader)**.** *Use Minkowski's inequality to show that*

$$
\int_0^{\pi} |f(x) - \sin x|^2 \, \mathrm{d}x \leq \frac{3}{4} \quad \textit{and} \quad \int_0^{\pi} |f(x) - \cos x|^2 \, \mathrm{d}x \leq \frac{3}{4}
$$

cannot be simultaneously true, where $f \in R[a, b]$.

Hint:

$$
\begin{aligned}
\sqrt{\pi} &= \left(\int_0^{\pi} |\sin x - \cos x|^2 \, \mathrm{d}x\right)^{\frac{1}{2}} \\
&= \left(\int_0^{\pi} |(f(x) - \cos x) - (f(x) - \sin x)|^2 \, \mathrm{d}x\right)^{\frac{1}{2}} \\
&\leq \left(\int_0^{\pi} |f(x) - \sin x|^2 \, \mathrm{d}x\right)^{\frac{1}{2}} + \left(\int_0^{\pi} |f(x) - \cos x|^2 \, \mathrm{d}x\right)^{\frac{1}{2}} \\
&\leq \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \sqrt{3},
\end{aligned}
$$

contradiction. □

Problem 2.9. *Find the minimum of the area enclosed by the parabola $y^2 = 2x$ and its chord (passing through the focus).*

Solution. Since the focus of parabola $y^2 = 2x$ is $(\frac{1}{2}, 0)$, we can assume that the equation of a chord is $x = ky + \frac{1}{2}$. Then

$$\begin{aligned}
S &= \int_{y_1}^{y_2} \left[\left(ky + \frac{1}{2} \right) - \frac{y^2}{2} \right] \mathrm{d}y \\
&= \frac{k}{2} (y_2 - y_1) \cdot (y_2 + y_1) + \frac{1}{2} (y_2 - y_1) - \frac{1}{6} (y_2 - y_1) \left(y_2^2 + y_1 y_2 + y_1^2 \right),
\end{aligned}$$

where y_1, y_2 are the intersection points of the parabola and its chord. By $y^2 = 2x$, we have

$$y^2 = 2ky + 1.$$

Hence, the Vieta theorem gives us that

$$\begin{aligned}
y_1 + y_2 &= 2k, \\
y_1 \cdot y_2 &= -1.
\end{aligned}$$

Then

$$\begin{aligned}
S &= \frac{1}{6} (y_2 - y_1) \cdot \left[3k (y_2 + y_1) + 3 - \left(y_2^2 + y_1 y_2 + y_1^2 \right) \right] \\
&= \frac{1}{6} \cdot 2 \cdot \sqrt{k^2 + 1} \cdot \left[3k \cdot 2k + 3 - \left(4k^2 + 1 \right) \right] \\
&= \frac{1}{3} \sqrt{k^2 + 1} \cdot \left(2k^2 + 2 \right) \\
&= \frac{2}{3} \cdot \left(k^2 + 1 \right)^{\frac{3}{2}}.
\end{aligned}$$

Hence

$$S_{\min} = \frac{2}{3}.$$

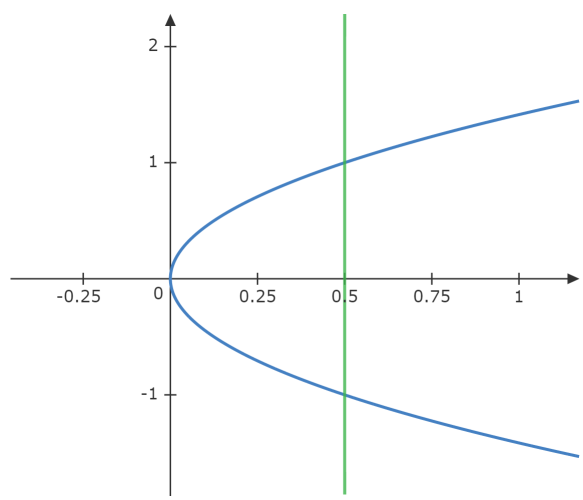

FIGURE 7. Graph of $y^2 = 2x$ and $x = \frac{1}{2}$

$\square$

Problem 2.10 (Wirtinger's inequality)**.** *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous differentiable function with period 2π, and satisfies $\int_0^{2\pi} f(x)\,\mathrm{d}x = 0$. Prove that*

$$\int_0^{2\pi} f'(x)^2\,\mathrm{d}x \geq \int_0^{2\pi} f(x)^2\,\mathrm{d}x.$$

"=" holds iff $f(x) = a\cos x + b\sin x$.

Proof. Firstly, we show there is a $t_0 \in [0, \pi)$ such that $f(t_0) = f(t_0 + \pi)$. Indeed, if $f(0) = f(\pi)$, we can take $t_0 = 0$. If $f(0) \neq f(\pi)$, let $g(x) = f(x) - f(x + \pi)$, then $g(0)g(\pi) < 0$, hence by the intermediate value theorem, we know that there exists a $t_0 \in [0, \pi)$ such that $g(t_0) = 0$, i.e. $f(t_0) = f(t_0 + \pi)$.

Next, let $c = f(t_0) = f(t_0 + \pi)$. It's easy to calculate that (Leave to the reader!)

$$f'^2 - (f-c)^2 - \left(f' - (f-c)\cot(t-t_0)\right)^2 = \left((f-c)^2\cot(t-t_0)\right)'.$$

Note that $(f-c)^2\cot(t-t_0)$ is continuous at t_0 and $t_0 + \pi$, we have

$$\int_0^{2\pi} f'(t)^2\ \mathrm{d}t - \int_0^{2\pi} (f(t)-c)^2\ \mathrm{d}t \geq \left((f-c)^2\cot(t-t_0)\right)\Big|_0^{2\pi} = 0.$$

So

$$\int_0^{2\pi} f'(t)^2\ \mathrm{d}t - \int_0^{2\pi} f(t)^2\ \mathrm{d}t \geq 2\pi c^2 \geq 0.$$

"=" holds iff $c = 0$, and $f'(t) = f(t)\cot(t-t_0)$, then $f(t) = A\sin(t-t_0)$. $\square$

Problem 2.11 (Isoperimetric inequality)**.** *Prove the isoperimetric inequality in $\mathbb{R}^2$: $4\pi A \leq L^2$, "=" holds iff the curve is a circle.*

Proof. Suppose that $\gamma(t) = (x(t), y(t)) : [0, 2\pi] \to \mathbb{R}^2$ is a curve parameterized by arc length. Without loss of generality, we may assume that $\int_0^{2\pi} x(t)\mathrm{d}t = 0$. Then, by Problem 2.10, we have

$$\begin{aligned} L^2 - 4\pi A =& 2\pi \int_0^{2\pi} \left(x'^2 + y'^2\right)\mathrm{d}t - 4\pi \int_0^{2\pi} xy'\mathrm{d}t \\ =& 2\pi \int_0^{2\pi} \left(x'^2 - x^2 + (y'-x)^2\right)\mathrm{d}t \\ \geq& 2\pi \int_0^{2\pi} \left(x'^2 - x^2\right)\mathrm{d}t \geq 0. \end{aligned}$$

"=" holds iff $x(t) = a\cos t + b\sin t, y'(t) = x(t)$, thus it is a circle. $\square$

Exercise 2.12. *Suppose a flat plate of uniform density has the shape contained by $y = x^2$, $y = 1$, and $x = 0$, in the first quadrant. Find the center of mass.*

Hint: We compute the moment around the y-axis:

$$M_y = \int_0^1 x\left(1 - x^2\right) dx = \frac{1}{4},$$

and the total mass

$$M = \int_0^1 \left(1 - x^2\right) dx = \frac{2}{3},$$

and finally

$$\bar{x} = \frac{1}{4} \cdot \frac{3}{2} = \frac{3}{8}.$$

Next we do the same thing to find $\bar{y}$.

$$M_x = \int_0^1 y\sqrt{y}dy = \frac{2}{5},$$

and

$$\bar{y} = \frac{2}{5} \cdot \frac{3}{2} = \frac{3}{5},$$

since the total mass M is the same. □

Remark 2.13. *Since the density is constant, the center of mass depends only on the shape of the plate, not the density, or in other words, this is a purely geometric quantity. In such a case the center of mass is called the centroid.*

Exercise 2.14. *Suppose that a water tank is shaped like a right circular cone with the tip at the bottom, and has height 10 meters and radius 2 meters at the top. If the tank is full, how much work is required to pump all the water out over the top?*

Hint: Here we have a large number of atoms of water that must be lifted different distances to get to the top of the tank. Fortunately, we don't really have to deal with individual atoms–we can consider all the atoms at a given depth together. To approximate the work, we can divide the water in the tank into horizontal sections, approximate the volume of water in a section by a thin disk, and compute the amount of work required to lift each disk to the top of the tank. As usual, we take the limit as the sections get thinner and thinner to get the total work.

At depth h the circular cross-section through the tank has radius $r = (10 - h)/5$, by similar triangles, and area $\pi(10 - h)^2/25$. A section of the tank at depth h thus has volume approximately $\pi(10 - h)^2/25\Delta h$ and so contains $\sigma\pi(10 - h)^2/25\Delta h$ kilograms of water, where σ is the density of water in kilograms per cubic meter; $\sigma \approx 1000$. The force due to gravity on this much water is $9.8\sigma\pi(10 - h)^2/25\Delta h$, and final1y, this section of

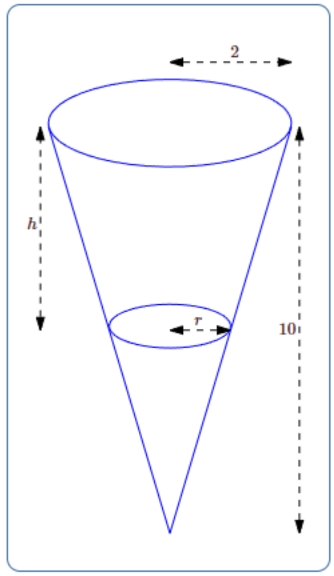

FIGURE 8. A conical water tank

water molst be lifted a distance h, which requires $h 9.8\sigma\pi(10-h)^2/25\Delta h$ Newton-meters of work. The total work is therefore

$$W = \frac{9.8\sigma\pi}{25}\int_0^{10} h(10-h)^2 dh = \frac{980000}{3}\pi \approx 1026254 \text{ Newton-meters.}$$

$\square$

Problem 3.1 (8.1)**.** *Calculate the following improper integrals in infinite intervals.*

(1) $\displaystyle\int_1^{+\infty} \frac{\ln x}{(1+x)^2}\,\mathrm{d}x.$

(3) $\displaystyle\int_1^{+\infty} \frac{\mathrm{d}x}{x\sqrt{x^2+x+1}}.$

(5) $\displaystyle\int_0^{+\infty} \frac{\mathrm{d}x}{(1+x^2)^n}.$

Solution. (1)

$$\begin{aligned}\int_1^{+\infty} \frac{\ln x}{(1+x)^2}\,\mathrm{d}x &= -\int_1^{+\infty} \ln x\,\mathrm{d}\left(\frac{1}{1+x}\right)\\ &= -\left.\frac{\ln x}{x+1}\right|_1^{+\infty} + \int_1^{+\infty} \frac{\mathrm{d}x}{x(x+1)}\\ &= -(0-0) + \ln\left.\left(\frac{x}{x+1}\right)\right|_1^{+\infty}\\ &= \ln 2.\end{aligned}$$

(3)

$$\begin{aligned}&\int_1^0 \frac{\mathrm{d}\left(\frac{1}{t}\right)}{\frac{1}{t}\sqrt{\frac{1}{t^2}+\frac{1}{t}+1}} = \int_0^1 \frac{\mathrm{d}t}{\sqrt{t^2+t+1}}\\ &= \int_0^1 \frac{\mathrm{d}t}{\sqrt{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}}}\\ &= \ln\left.\left[\left(t+\frac{1}{2}\right)+\sqrt{t^2+t+1}\right]\right|_0^1\\ &= \ln\frac{\sqrt{3}+\frac{3}{2}}{1+\frac{1}{2}} = \ln\left(\frac{2+\sqrt{3}}{\sqrt{3}}\right).\end{aligned}$$

(5) **First way:**

$$
\begin{aligned}
I_n &= \int_0^{+\infty} \frac{\mathrm{d}x}{(1+x^2)^n} \\
&= \left.\frac{x}{(1+x^2)^n}\right|_0^{+\infty} + \int_0^{+\infty} \frac{2nx^2}{(1+x^2)^{n+1}}\,\mathrm{d}x \\
&= 2n\left(I_n - I_{n+1}\right), \\
&\Rightarrow 2nI_{n+1} = (2n-1)I_n, \text{ i.e. } I_{n+1} = \frac{2n-1}{2n}I_n. \\
I_1 &= \int_0^{+\infty} \frac{\mathrm{d}x}{1+x^2} = \arctan x|_0^{+\infty} = \frac{\pi}{2} \\
I_n &= \frac{2n-3}{2n-2}\cdot\dots\cdot\frac{1}{2}I_1 = \frac{\pi}{2}\cdot\frac{(2n-3)!!}{(2n-2)!!} \\
&\Rightarrow I_n = \begin{cases} \frac{\pi}{2}, n=1 \\ \frac{(2n-3)!!}{(2n-2)!!}\frac{\pi}{2}, n\geq 2. \end{cases}
\end{aligned}
$$

Second way:

$$
\begin{aligned}
I_n &= \int_0^{+\infty} \frac{\mathrm{d}x}{(1+x^2)^n} = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{\sec^{2n-2}x} \\
&= \int_0^{\frac{\pi}{2}} \frac{\sec^2 x - \tan^2 x}{\sec^{2n-2}x}\,\mathrm{d}x \\
&= I_{n-1} - \int_0^{\frac{\pi}{2}} \frac{\tan^2 x}{\sec^{2n-2}x}\,\mathrm{d}x \\
&= I_{n-1} + \left.\frac{1}{2n-2}\cdot\frac{\tan x}{\sec^{2n-2}x}\right|_0^{\frac{\pi}{2}} - \frac{1}{2n-2}I_{n-1} \\
&= \frac{2n-3}{2n-2}I_{n-1}, \quad n\geq 2.
\end{aligned}
$$

Then

$$
I_1 = \int_0^{+\infty} \frac{\mathrm{d}x}{1+x^2} = \frac{\pi}{2},
$$

and

$$
I_n = \frac{2n-3}{2n-2}\cdot\dots\cdot\frac{1}{2}I_1 = \frac{\pi}{2}\cdot\frac{(2n-3)!!}{(2n-2)!!}, \quad n\geq 2.
$$

□

Problem 3.2 (8.2)**.** *Determine the convergence or divergence of the following improper intergals.*

(1) $\displaystyle\int_0^{+\infty} \frac{\sin x}{1+e^{-x}}\,\mathrm{d}x.$

(6) $\displaystyle\int_1^{+\infty} x\left(1-\cos\frac{1}{x}\right)^{\alpha}\,\mathrm{d}x,\ (\alpha>0).$

(7) $\displaystyle\int_1^{+\infty} \ln\left(\cos\frac{1}{x}+\sin^p\frac{1}{x}\right)\,\mathrm{d}x,\ (p>1).$

Hints. (1)

$$\begin{aligned}
&\int_{2N\pi}^{(2N+1)\pi}\frac{\sin x}{2}\,\mathrm{d}x \le \int_{2N\pi}^{(2N+1)\pi}\frac{\sin x}{1+e^{-x}}\,\mathrm{d}x\\
\Rightarrow\quad & 1\le \int_0^{(2N+1)\pi}\frac{\sin x}{1+e^{-x}}\,\mathrm{d}x-\int_0^{2N\pi}\frac{\sin x}{1+e^{-x}}\,\mathrm{d}x.
\end{aligned}$$

(6) Note that

$$1-\cos\frac{1}{x}=\frac{1}{2}\cdot\frac{1}{x^2}+o\left(\frac{1}{x^2}\right),\ x\to+\infty.$$

Then

$$x\left(1-\cos\frac{1}{x}\right)^{\alpha}\sim\frac{1}{2^{\alpha}}\cdot\frac{1}{x^{2\alpha-1}},\ x\to+\infty.$$

(7) Note that

$$\begin{aligned}
\ln\left(\cos\frac{1}{x}+\sin^p\frac{1}{x}\right)&=\ln\left(1+\cos\frac{1}{x}-1+\sin^p\frac{1}{x}\right)\\
&=\cos\frac{1}{x}-1+\sin^p\frac{1}{x}+o\left(\frac{1}{x^2}\right)+o\left(\frac{1}{x^p}\right)\\
&=-\frac{1}{2}\cdot\frac{1}{x^2}+\frac{1}{x^p}+o\left(\frac{1}{x^2}\right)+o\left(\frac{1}{x^p}\right),\ x\to+\infty.
\end{aligned}$$

□

Problem 3.3 (8.3)**.** *Determine the convergence or absolute convergence of the following improper intergals.*

(3) $\displaystyle\int_1^{+\infty}\frac{\sin x}{x^{\alpha}+\sin x}\,\mathrm{d}x.$

(4) $\displaystyle\int_1^{+\infty}\sin\left(\frac{\sin x}{x}\right)\,\mathrm{d}x.$

Hints. (3)

$$\int_1^{+\infty}\frac{\sin x}{x^{\alpha}+\sin x}\,\mathrm{d}x=\int_1^{+\infty}\frac{\sin x}{x^{\alpha}}\,\mathrm{d}x-\int_1^{+\infty}\frac{\sin^2 x}{x^{\alpha}(x^{\alpha}+\sin x)}\,\mathrm{d}x.$$

(4)

$$\sin\left(\frac{\sin x}{x}\right) = \frac{\sin x}{x} + o\left(\frac{1}{x^3}\right), \ x \to +\infty.$$

$\square$

Problem 3.4 (8.9)**.** *Suppose that $f(x) \in C(-\infty, +\infty)$ is a periodic function with period 2π, and $\int_0^{2\pi} f(x)\ \mathrm{d}x = 0$. Prove that for any $\alpha > 0$, the improper intergral $\int_1^{+\infty} x^{-\alpha} f(x)\ \mathrm{d}x$ converges.*

Proof. Since $f(x)$ is a continuous periodic function on $\mathbb{R}$, we know that its intergral exists on any finite closed intervals. Note that $\forall\, X > 0$, there exist $k \in \mathbb{Z}$ and $0 \le r < 2\pi$ such that

$$X = 2k\pi + r.$$

Then

$$\begin{aligned}
\left|\int_1^X f(x)\ \mathrm{d}x\right| &\le \left|\int_0^1 f(x)\ \mathrm{d}x\right| + \left|\int_0^X f(x)\ \mathrm{d}x\right| \\
&= \left|\int_0^1 f(x)\ \mathrm{d}x\right| + \left|\int_0^{2k\pi+r} f(x)\ \mathrm{d}x\right| \\
&= \left|\int_0^1 f(x)\ \mathrm{d}x\right| + \left|\int_0^{r} f(x)\ \mathrm{d}x\right| \\
&\le 2\int_0^{2\pi} |f(x)|\ \mathrm{d}x.
\end{aligned}$$

Hence by the Dirichlet test rule, we know that the improper intergral $\int_1^{+\infty} x^{-\alpha} f(x)\ \mathrm{d}x$ converges. $\square$

Problem 3.5 (8.10)**.** *Suppose that $f(x)$ is uniformly continuous on $[0, +\infty$, and the improper intergral $\int_0^{+\infty} f(x)\ \mathrm{d}x$ converges. Prove that $\lim\limits_{x\to+\infty} f(x) = 0$.*

Proof. Prove by contradiction. Assume that $\lim\limits_{x\to+\infty} f(x) \neq 0$, i.e. there exists $\varepsilon_0 > 0$, such that $\forall\, n \in \mathbb{N}$, there is $x_n > n$ satisfying $|f(x_n)| \ge \varepsilon_0$. Since $f(x)$ is uniformly continuous, we have that for $\varepsilon_0/2$, there exists a $\delta_0 > 0$, such that $\forall\, y : |y - x_n| < \delta_0$,

$$|f(y) - f(x_n)| < \frac{\varepsilon_0}{2}.$$

Then
$$|f(y)| \geq |f(x_n)| - |f(y) - f(x_n)| > \frac{\varepsilon_0}{2},\ \forall\, y \in (x_n - \delta_0, x_n + \delta_0),$$
and $f(y)$, $f(x_n)$ have the same sign in $(x_n - \delta_0, x_n + \delta_0)$. Therefore, we have
$$\left| \int_{x_n - \frac{\delta_0}{2}}^{x_n + \frac{\delta_0}{2}} f(x)\ \mathrm{d}x \right| \geq \frac{\varepsilon_0 \delta_0}{2},$$
contradicts with the convergence of $\displaystyle\int_0^{+\infty} f(x)\ \mathrm{d}x$. Thus, $\lim\limits_{x \to +\infty} f(x) = 0$. □

Problem 3.6 (8.14)**.** *Calculate the following improper integrals with discontinuous integrand.*

(1) $\displaystyle\int_0^1 \frac{\mathrm{d}x}{(2-x)\sqrt{1-x}}$.

(6) $\displaystyle\int_0^{\frac{\pi}{2}} \sqrt{\tan x}\ \mathrm{d}x$.

Solution. (1)
$$\int_0^1 \frac{\mathrm{d}x}{(2-x)\sqrt{1-x}} = -2\int_0^1 \frac{\mathrm{d}\sqrt{1-x}}{2-x} = \frac{\pi}{2}.$$
(6)
$$I = \int_0^{\pi/2} \sqrt{\tan x}\ \mathrm{d}x = \int_0^{\pi/2} \frac{\sin x}{\sqrt{\cos x \sin x}}\ \mathrm{d}x = \int_0^{\pi/2} \frac{\cos x}{\sqrt{\cos x \sin x}}\ \mathrm{d}x.$$
$$\begin{aligned} 2I &= \int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}}\ \mathrm{d}x \\ &= \int_0^{\pi/2} \frac{\sqrt{2}\ \mathrm{d}(\sin x - \cos x)}{\sqrt{1 - (\sin x - \cos x)^2}} \\ &= \sqrt{2}\pi. \end{aligned}$$
i.e. $I = \frac{\pi}{\sqrt{2}}$. □

Problem 3.7 (8.16)**.** *Determine the convergence or divergence of the following improper intergals.*

(5) $\displaystyle\int_0^{+\infty} \frac{\sin\left(x + \frac{1}{x}\right)}{x^p}\ \mathrm{d}x$.

Hints. Note that
$$\frac{\sin\left(x + \frac{1}{x}\right)}{x^p} = \frac{\sin x \cos\frac{1}{x}}{x^p} + \frac{\cos x \sin\frac{1}{x}}{x^p}.$$

$\square$

Problem 3.8. *Suppose that $f(x)$ is integrable on any finite closed interval, and $\lim_{x\to+\infty} f(x) = A$, $\lim_{x\to-\infty} f(x) = B$. Prove that $\forall\, a > 0$, improper inegral*

$$\int_{-\infty}^{+\infty} [f(x+a) - f(x)]\ \mathrm{d}x$$

converges.

Hints. For any M, N, we have

$$\begin{aligned}\int_M^N [f(x+a) - f(x)]\ \mathrm{d}x &= \int_M^N f(x+a)\ \mathrm{d}x - \int_M^N f(x)\ \mathrm{d}x \\ &= \int_{M+a}^{N+a} f(x)\ \mathrm{d}x - \int_M^N f(x)\ \mathrm{d}x \\ &= \int_N^{N+a} f(x)\ \mathrm{d}x - \int_M^{M+a} f(x)\ \mathrm{d}x \\ &\to a(A-B), \quad \text{as } M \to -\infty, N \to +\infty.\end{aligned}$$

$\square$

Problem 3.9. *Suppose that $f(x)$ is integrable on any finite closed interval, and $\int_{-\infty}^{+\infty} f^2\ \mathrm{d}x$ converges. Prove that $\forall\, a > 0$, improper inegral*

$$\int_{-\infty}^{+\infty} |f(x+a)f(x)|\ \mathrm{d}x$$

converges.

Hints. For any M, N, by Cauchy-Schwarz's inequality, we have

$$\begin{aligned}\left[\int_M^N |f(x+a)f(x)|\ \mathrm{d}x\right]^2 &\le \int_M^N [f(x+a)]^2\ \mathrm{d}x \cdot \int_M^N [f(x)]^2\ \mathrm{d}x \\ &= \int_{M+a}^{N+a} [f(x)]^2\ \mathrm{d}x \cdot \int_M^N [f(x)]^2\ \mathrm{d}x.\end{aligned}$$

Then letting $M \to -\infty, N \to +\infty$. $\square$

Problem 3.10. *Prove that $\displaystyle\int_0^{+\infty} \frac{x}{1 + x^6 \sin^2 x}\ \mathrm{d}x$ converges.*

Hints. It suffices to show that

$$F(A) = \int_0^A \frac{x}{1 + x^6 \sin^2 x} \, \mathrm{d}x$$

is bounded on $[0, +\infty)$. Note that

$$\begin{aligned}
\int_0^{n\pi} \frac{x}{1 + x^6 \sin^2 x} \, \mathrm{d}x &= \sum_{k=1}^{n} \int_{(k-1)\pi}^{k\pi} \frac{x}{1 + x^6 \sin^2 x} \, \mathrm{d}x \\
&\le \sum_{k=1}^{n} k\pi \int_{(k-1)\pi}^{k\pi} \frac{\mathrm{d}x}{1 + (k-1)^6 \pi^6 \sin^2 x} \\
&= \sum_{k=1}^{n} k\pi \int_0^{\pi} \frac{\mathrm{d}x}{1 + (k-1)^6 \pi^6 \sin^2 x} \\
&= \sum_{k=1}^{n} 2k\pi \int_0^{\pi/2} \frac{\mathrm{d}x}{1 + (k-1)^6 \pi^6 \sin^2 x} \\
&\le \sum_{k=1}^{n} 2k\pi \int_0^{\pi/2} \frac{\mathrm{d}x}{1 + 4(k-1)^6 \pi^4 x^2} \\
&= \sum_{k=1}^{n} \frac{k}{\pi (k-1)^3} \int_0^{(k-1)^3 \pi^3} \frac{\mathrm{d}x}{1 + x^2} \\
&\sim \sum_{k=1}^{n} \frac{1}{2k^2} \quad (k \to \infty).
\end{aligned}$$

Hence $F(n\pi)$ is bounded, which gives us that $F(A)$ converges. □

Problem 3.11. *Suppose that $f(x)$ is continuous on $[0, +\infty)$, and $\lim_{x \to +\infty} f(x)$ exists. For $0 < a < b$, calculate*

$$\int_0^{+\infty} \frac{f(ax) - f(bx)}{x} \, \mathrm{d}x.$$

Hints. For $0 < r < R < +\infty$, we have

$$\begin{aligned}
\int_r^R \frac{f(ax) - f(bx)}{x} \, \mathrm{d}x &= \int_r^R \frac{f(ax)}{x} \, \mathrm{d}x - \int_r^R \frac{f(bx)}{x} \, \mathrm{d}x \\
&= \int_{ar}^{aR} \frac{f(x)}{x} \, \mathrm{d}x - \int_{br}^{bR} \frac{f(x)}{x} \, \mathrm{d}x \\
&= \int_{ar}^{br} \frac{f(x)}{x} \, \mathrm{d}x - \int_{aR}^{bR} \frac{f(x)}{x} \, \mathrm{d}x.
\end{aligned}$$

Then by the first mean value theorem for definite integrals, we have

$$\int_{ar}^{br} \frac{f(x)}{x}\,\mathrm{d}x = f(\xi)\int_{ar}^{br} \frac{\mathrm{d}x}{x} = f(\xi)\ln\frac{b}{a} \quad (ar < \xi < br),$$

$$\int_{aR}^{bR} \frac{f(x)}{x}\,\mathrm{d}x = f(\eta)\int_{ar}^{br} \frac{\mathrm{d}x}{x} = f(\eta)\ln\frac{b}{a} \quad (aR < \eta < bR).$$

Then letting $r \to 0$, and $N \to +\infty$ yield

$$\int_0^{+\infty} \frac{f(ax) - f(bx)}{x}\,\mathrm{d}x = [f(0) - f(+\infty)]\ln\frac{b}{a}.$$

□

Problem 3.12. *Prove that for any $\alpha \in \mathbb{R}$, there ia*

$$\int_0^{+\infty} \frac{\mathrm{d}x}{(1+x^2)(1+x^\alpha)} = \frac{\pi}{4}$$

converges.

Hints. Note that by changing of variable, we have

$$\int_0^{1} \frac{\mathrm{d}x}{(1+x^2)(1+x^\alpha)} = \int_1^{+\infty} \frac{x^\alpha\,\mathrm{d}x}{(1+x^2)(1+x^\alpha)}.$$

□

Problem 3.13. *Suppose that $f(x)$ is integrable on any finite closed interval, and $\forall\, p \geq 1$, $\int_{-\infty}^{+\infty} |f|^p\,\mathrm{d}x$ converges. Prove that*

$$\lim_{h\to 0}\int_{-\infty}^{+\infty} |f(x+h) - f(x)|^p\,\mathrm{d}x = 0.$$

Hints. Note that

$$\int_{-\infty}^{+\infty} |f(x+h) - f(x)|^p\,\mathrm{d}x = \int_{-\infty}^{-R} |f(x+h) - f(x)|^p\,\mathrm{d}x + \int_{R}^{+\infty} |f(x+h) - f(x)|^p\,\mathrm{d}x$$
$$+ \int_{-R}^{R} |f(x+h) - f(x)|^p\,\mathrm{d}x.$$

Since $\int_{-\infty}^{+\infty} |f|^p\,\mathrm{d}x$ converges, we know that for $\forall\,\varepsilon > 0$, there exists $R > 0$ such that

$$\int_{-\infty}^{-R} |f(x+h) - f(x)|^p\,\mathrm{d}x < \frac{\varepsilon}{3},$$

and

$$\int_R^{+\infty} |f(x+h) - f(x)|^p \, \mathrm{d}x < \frac{\varepsilon}{3}.$$

Since

$$\lim_{h \to 0} \int_{-R}^{R} |f(x+h) - f(x)|^p \, \mathrm{d}x = 0,$$

we have

$$\int_{-\infty}^{+\infty} |f(x+h) - f(x)|^p \, \mathrm{d}x < \varepsilon.$$

□

Problem 4.1 (8.17). *Determine the conditional convergence or absolute convergence of the following improper intergals.*

(1) $\displaystyle\int_0^{+\infty} |\ln x|^p \frac{\sin x}{x^q}\, \mathrm{d}x$.

Hints. (1) When $p \geq 0$, the flaw points are 0 and $+\infty$. Then

$$\int_0^{+\infty} |\ln x|^p \frac{\sin x}{x^q}\, \mathrm{d}x = \int_0^{1} |\ln x|^p \frac{\sin x}{x^q}\, \mathrm{d}x + \int_1^{+\infty} |\ln x|^p \frac{\sin x}{x^q}\, \mathrm{d}x.$$

Note that

$$\frac{\sin x}{x^q} \sim \frac{1}{x^{q-1}}, \text{ as } x \to 0.$$

Hence when $1 < q < 2$, $\displaystyle\int_0^{+\infty} |\ln x|^p \frac{\sin x}{x^q}\, \mathrm{d}x$ converges absolutely; when $0 < q < 1$, $\displaystyle\int_0^{+\infty} |\ln x|^p \frac{\sin x}{x^q}\, \mathrm{d}x$ converges conditionally.

When $p < 0$, we know that 1 is also the flaw point, and there is

$$|\ln x|^p \sim \frac{1}{|x-1|^{-p}}, \text{ as } x \to 1.$$

Hence for $-1 < p < 0$, we write

$$\int_0^{+\infty} |\ln x|^p \frac{\sin x}{x^q}\, \mathrm{d}x = \int_0^{1/2} |\ln x|^p \frac{\sin x}{x^q}\, \mathrm{d}x + \int_{1/2}^{3/2} |\ln x|^p \frac{\sin x}{x^q}\, \mathrm{d}x + \int_{3/2}^{+\infty} |\ln x|^p \frac{\sin x}{x^q}\, \mathrm{d}x.$$

Hence when $1 < q < 2$, $\displaystyle\int_0^{+\infty} |\ln x|^p \frac{\sin x}{x^q}\, \mathrm{d}x$ converges absolutely; when $0 \leq q < 1$, $\displaystyle\int_0^{+\infty} |\ln x|^p \frac{\sin x}{x^q}\, \mathrm{d}x$ converges conditionally. □

Problem 4.2 (8.21). *Suppose that $f(x)$ is monotonic on $(0, 1]$, and $\displaystyle\int_0^1 f(x)\, \mathrm{d}x$ converges. Prove that*

$$\int_0^1 f(x)\, \mathrm{d}x = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right).$$

If we remove the condition that $f(x)$ is monotonic on $(0, 1]$, is the conclusion still true?

Proof. Without loss of generality, we may assume that $f(x)$ is monotonic decreasing on $(0,1]$. Hence there is

$$\int_{\frac{1}{n}}^{1} f(x)\ \mathrm{d}x + \frac{f(1)}{n} \leq \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) \leq \int_{0}^{1} f(x)\ \mathrm{d}x,$$

which gives us that

$$\int_{0}^{1} f(x)\ \mathrm{d}x = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right).$$

If $f(x)$ is not monotonic decreasing on $(0,1]$, then the conclusion may be wrong. For example, we can consider the function

$$f(x) = \begin{cases} n^2, & x = \dfrac{1}{n}, \\ 0, & others. \end{cases}$$

□

Problem 4.3 (8.22)**.** *Suppose that* $f(x)$ *is monotonic on* $[0,+\infty)$, *and* $\int_{0}^{+\infty} f(x)\ \mathrm{d}x$ *converges. Prove that*

$$\lim_{\lambda\to\infty} \int_{0}^{+\infty} f(x)\sin\lambda x\ \mathrm{d}x = 0.$$

***Hint*.** Consider

$$\int_{0}^{+\infty} f(x)\sin\lambda x\ \mathrm{d}x = \int_{0}^{R} f(x)\sin\lambda x\ \mathrm{d}x + \int_{R}^{+\infty} f(x)\sin\lambda x\ \mathrm{d}x,$$

and use Exercise 12.14 in the notes of *Mathematical analysis I*. □

Problem 4.4 (8.24)**.** *Suppose that* $f(x)$ *is monotonic on* $[0,+\infty)$, *and* $g(x) \not\equiv 0$ *is periodic and continuous on* $\mathbb{R}$ *with period* $T > 0$. *Prove that* $\int_{0}^{+\infty} f(x)\ \mathrm{d}x$ *converges if and only if* $\int_{0}^{+\infty} f(x)|g(x)|\ \mathrm{d}x$ *converges.*

Proof. "$\Longrightarrow$" Since $f(x)$ is monotonic on $[0,+\infty)$, we may assume that $f(x) \geq 0$. Since $g(x$ is periodic and continuous on $\mathbb{R}$, we know that there exists a $M > 0$ such that $|g(x)| \leq M$, $\forall\, x \in \mathbb{R}$. Then, we have

$$\int_{0}^{+\infty} f(x)|g(x)|\ \mathrm{d}x \leq M\int_{0}^{+\infty} f(x)\ \mathrm{d}x < \infty.$$

"$\Longleftarrow$" Since $g(x) \not\equiv 0$ is a continuous and periodic function with period $T > 0$, we know there exist a $A > 0$ and $[a, b] \subset (0, T)$ such that for any $x \in [a, b]$, there is $|g(x)| > A$. Then we know

$$\int_0^T |g(x)|\,\mathrm{d}x \geq \int_a^b |g(x)|\,\mathrm{d}x \geq A(b-a).$$

Without loss of generality, we may assume that $f(x)$ is monotonic decreasing and non-negative. Then

$$\int_{(k-1)T}^{kT} f(x)|g(x)|\ \mathrm{d}x \geq f(kT)\int_{(k-1)T}^{kT} |g(x)|\ \mathrm{d}x \geq A(b-a)f(kT), \quad k \geq 1.$$

Note that

$$\int_{kT}^{(k+1)T} f(x)\ \mathrm{d}x \leq f(kT)T \leq \frac{T}{A(b-a)}\int_{(k-1)T}^{kT} f(x)|g(x)|\ \mathrm{d}x.$$

Then

$$\begin{aligned}
\int_T^{(n+1)T} f(x)\ \mathrm{d}x &= \sum_{k=1}^{n}\int_{kT}^{(k+1)T} f(x)\ \mathrm{d}x \\
&\leq \frac{T}{A(b-a)}\sum_{k=1}^{n}\int_{(k-1)T}^{kT} f(x)|g(x)|\ \mathrm{d}x \\
&= \frac{T}{A(b-a)}\int_0^{nT} f(x)|g(x)|\ \mathrm{d}x \\
&\to \int_0^{+\infty} f(x)|g(x)|\ \mathrm{d}x, \quad \text{as } n \to \infty,
\end{aligned}$$

which implies that $\displaystyle\int_0^{+\infty} f(x)\ \mathrm{d}x$ converges. $\square$

Problem 4.5. *Suppose that $f(x)$ is differentiable, $\displaystyle\int_0^{+\infty} f(x)\ \mathrm{d}x$ and $\displaystyle\int_0^{+\infty} f'(x)\ \mathrm{d}x$ converge. Prove that $\displaystyle\lim_{x\to+\infty} f(x) = 0$.*

Hint. Firstly, using $\displaystyle\int_0^{+\infty} f'(x)\ \mathrm{d}x$ converges to prove $\displaystyle\lim_{x\to+\infty} f(x) = \alpha$ exists (by Cauchy's convergence test and Heine's theorem). Then, using $\displaystyle\int_0^{+\infty} f(x)\ \mathrm{d}x$ to show $\alpha = 0$ (by contradiction). $\square$

Exercise 4.6. *Suppose that* $F(x) = \displaystyle\int_0^x \left(\frac{1}{t} - \left[\frac{1}{t}\right]\right)\,\mathrm{d}t$. *Prove that* $F'(0) = \dfrac{1}{2}$.

Hints: By changing of variable, $t = \frac{x}{y}$, we have

$$F(x) = \int_0^x \left(\frac{1}{t} - \left[\frac{1}{t}\right]\right)\,\mathrm{d}t = x\int_1^{+\infty} \left(\frac{y}{x} - \left[\frac{y}{x}\right]\right)\frac{1}{y^2}\,\mathrm{d}y.$$

Hence

$$\frac{F(x)}{x} = \int_1^{+\infty} \left(\frac{y}{x} - \left[\frac{y}{x}\right]\right)\frac{1}{y^2}\,\mathrm{d}y = \int_1^{+\infty} \frac{\varphi\left(\frac{y}{x}\right)}{y^2}\,\mathrm{d}y,$$

where $\varphi(t) = t - [t]$ is a periodic function with period $T = 1$. Hence, by Riemann's theorem, we know

$$\begin{aligned} F'(0) &= \lim_{x\to 0}\frac{F(x)}{x} = \frac{1}{T}\int_0^T \varphi(y)\,\mathrm{d}y \cdot \int_1^{+\infty}\frac{1}{y^2}\,\mathrm{d}y \\ &= \int_0^1 y\,\mathrm{d}y = \frac{1}{2}. \end{aligned}$$

□

5. Week 6 (3.29)

Problem 5.1 (9.10)**.** *Discuss the convergence and divergence of the following series.*

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{[\sqrt{n}]}}{n^p}$, $(p > 0)$.

Problem 5.2 (9.16)**.** *Suppose that* $\displaystyle\sum_{n=1}^{\infty} a_n$ *converges, and* $\displaystyle\sum_{n=1}^{\infty}(b_n - b_{n+1})$ *converges absolutely. Prove that* $\displaystyle\sum_{n=1}^{\infty} a_n b_n$ *converges.*

Problem 5.3 (9.21)**.** *Suppose that the sequence* $\{a_n\}$ *satisfies* $\displaystyle\lim_{n\to\infty} na_n$ *exists and series* $\displaystyle\sum_{n=1}^{\infty} n(a_n - a_{n+1})$ *converges. Prove that series* $\displaystyle\sum_{n=1}^{\infty} a_n$ *converges.*

Problem 5.4 (9.23)**.** *Suppose that non-constant function* $f(x)$ *is nonnegative and continuous on* $[0,1]$*, and* $f(x) \le 1$, $x \in [0,1]$*. For* $\forall\, n \in \mathbb{N}$*, defining* $a_n = \left[\displaystyle\int_0^1 f(x)\,\mathrm{d}x\right]^{\frac{1}{n}}$*. Prove that series* $\displaystyle\sum_{n=1}^{\infty}(1 - a_n)$ *diverges.*

Problem 5.5. *Suppose that* $a_n > 0$ *and* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_n}$ *converges. Prove that*

$$\sum_{n=1}^{\infty} \frac{n^2}{(a_1 + a_2 + \cdots + a_n)^2} a_n$$

converges, too.

Problem 5.6. *Suppose that* $x_n = 1 + \dfrac{1}{\sqrt{2}} + \cdots + \dfrac{1}{\sqrt{n}} - 2\sqrt{n}$*. Prove that* $\displaystyle\lim_{n\to\infty} x_n$ *exists.*

Problem 5.7. *Suppose that* $a_n > 0$ *and* $S = \sum_{n=1}^{\infty} a_n$ *converges. Prove that*

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} ka_k}{n} = 0,$$

and

$$\sum_{n=1}^{\infty} \frac{a_1 + 2a_2 + \cdots + na_n}{n(n+1)} = S.$$

Problem 5.8. *Suppose that* $a_n > 0$, $\sum_{n=1}^{\infty} a_n$ *converges and* $\{a_n - a_{n+1}\}$ *is decreasing. Prove that* $\{a_n\}$ *is decreasing and*

$$\lim_{n\to\infty} \frac{1}{a_{n+1}} - \frac{1}{a_n} = +\infty.$$

Problem 5.9. *Suppose that* $\{a_n\}$ *is positive and decreasing. If series* $\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}}$ *converges, prove that* $\sum_{n=1}^{\infty} a_n^2$ *also converges.*

Problem 5.10.

(1) *If for evrey sequence* $\{b_n\}$ *satisfying* $\lim_{n\to\infty} b_n = 0$, *there is* $\sum_{n=1}^{\infty} a_n b_n$ *converges. Prove that* $\sum_{n=1}^{\infty} a_n$ *converges absolutely.*

(2) *If for every convergent series* $\sum_{n=1}^{\infty} b_n$, *we have that series* $\sum_{n=1}^{\infty} a_n b_n$ *converges. Prove that* $\sum_{n=1}^{\infty} |a_n - a_{n+1}|$ *must be convergent.*

Problem 6.1. *Suppose that $\{x_n\}$ is positive, monotonic decreasing and* $\sum_{n=1}^{\infty} x_n = +\infty$. *Prove that*

$$\sum_{n=1}^{\infty} x_n e^{-\frac{x_n}{x_{n+1}}} = +\infty.$$

Proof. **Firsr way:** Clearly, $\{x_n\}$ converges. If $x_n \to a > 0$, then $x_n/x_{n+1} \to 1$, and hence $\mathrm{e}^{-x_n/x_{n+1}}$ lower bounded, say by $b > 0$, and hence

$$\sum x_n \mathrm{e}^{-x_n/x_{n+1}} \geq b \sum x_n = \infty.$$

If $x_n \to 0$, set

$$S_k = \{n \in \mathbb{N} : 2^{-k} < x_n \leq 2^{-k+1}\}$$

Then $\sum_{k=0}^{\infty} \sum_{n \in S_k} x_n = \infty$, and

$$\sum_{k=0}^{\infty} 2^{-k+1}|S_k| \geq \sum_{k=0}^{\infty} \sum_{n \in S_k} x_n \geq \sum_{k=0}^{\infty} 2^{-k}|S_k|.$$

Hence $\sum_{k=0}^{\infty} 2^{-k}|S_k| = \frac{1}{2}\sum_{k=0}^{\infty} 2^{-k+1}|S_k| = \infty$. Also, if $n, n+1 \in S_k$, then $\mathrm{e}^{-x_n/x_{n+1}} \geq \mathrm{e}^{-2}$. Hence, if $S_k = \{j, j+1, \ldots, \ell\}$, then

$$\sum_{n \in S_k} x_n \mathrm{e}^{-x_n/x_{n+1}} > \sum_{n=j}^{\ell-1} x_n \mathrm{e}^{-x_n/x_{n+1}} \geq \sum_{n=j}^{\ell-1} x_n \mathrm{e}^{-2} \geq 2^{-k}(|S_k| - 1)\mathrm{e}^{-2}.$$

Thus

$$\sum_{k=0}^{\infty} \sum_{n \in S_k} x_n \mathrm{e}^{-x_n/x_{n+1}} \geq \sum_{k=0}^{\infty} 2^{-k}(|S_k| - 1)\mathrm{e}^{-2} = \infty$$

Second way: Divide $\mathbb{N}$ into two complementary sets of indices

$$A = \left\{n \in \mathbb{N} \mid x_{n+1} \leq \frac{1}{2}x_n\right\}, B = \left\{n \in \mathbb{N} \mid x_{n+1} > \frac{1}{2}x_n\right\}.$$

If $k < l$ are two elements of A then $x_l \leq \frac{1}{2}x_k$ (here we need the fact the the given sequence is decreasing). It follows that the k'th element of A is $\leq 2^{-k}x_1$. A can be empty, finite, or infinite, but in any case is

$$\sum_{n \in A} x_n < \infty$$

and therefore

$$\sum_{n \in B} x_n = \infty.$$

Then

$$\sum_{n=1}^{\infty} x_n e^{-\frac{x_n}{x_{n+1}}} \geq \sum_{n\in B} x_n e^{-\frac{x_n}{x_{n+1}}} \geq e^{-2} \sum_{n\in B} x_n = \infty\,.$$

□

Remark 6.2. *The proof shows that under the given conditions,*

$$\sum_{n=1}^{\infty} x_n f\left(\frac{x_n}{x_{n+1}}\right) = \infty$$

holds for any positive, monotonic decreasing function $f : (0,\infty) \to (0,\infty)$.

Problem 6.3. *Let* f *be an increasing function on* $[0,1]$ *such that* $0 \leq f(x) \leq 1$ *and* $\int_0^1 (f(x) - x)dx = 0$. *Show that*

$$\int_0^1 |f(x) - x|dx \leq \frac{1}{2}.$$

Proof. Here we present a bit different, calculus-themed approach. In this answer, we will assume that $f : [0,1] \to [0,1]$ is monotone-increasing. We also write $I(f) = \int_0^1 |f(x) - x|\,\mathrm{d}x$ for brevity.

Step 1 - Proof under extra assumptions. Assume further that f is piecewise-smooth, $f(0) = 0$, and $f(1) = 1$. Then by the formula $\int |x|\,\mathrm{d}x = \frac{1}{2}x|x| + \mathsf{C}$, we have

$$\int_0^1 |f(x) - x|(f'(x) - 1)\,\mathrm{d}x = \left[\frac{1}{2}|f(x) - x|(f(x) - x)\right]_0^1 = 0.$$

In particular,

$$I(f) = \frac{1}{2}\int_0^1 |f(x) - x|(f'(x) + 1)\,\mathrm{d}x.$$

Now pick $\alpha \in [0,1]$ so that $f(\alpha) + \alpha = 1$. (This is possible since $x \mapsto f(x) + x$ increases from 0 to 2. Then by triangle inequality,

$$\int_0^\alpha |f(x) - x|(f'(x) + 1)\,\mathrm{d}x \leq \int_0^\alpha (f(x) + x)(f'(x) + 1)\,\mathrm{d}x = \frac{1}{2}.$$

Similarly, by writing $|f(x) - x| = |(1 - f(x)) - (1 - x)| \leq (1 - f(x)) + (1 - x)$, we get

$$\int_\alpha^1 |f(x) - x|(f'(x) + 1)\,\mathrm{d}x \leq \int_\alpha^1 (2 - f(x) - x)(f'(x) + 1)\,\mathrm{d}x = \frac{1}{2}.$$

Therefore $\int_0^1 |f(x) - x|(f'(x) + 1)\,\mathrm{d}x \leq 1$, which in turn implies $I(f) \leq \frac{1}{2}$ as required.

Step 2 - General case. For the general case, let f_n be the linear interpolation of the points

$$(0,0), \quad (\tfrac{1}{n}, f(\tfrac{1}{n})), \quad \cdots, \quad (\tfrac{n-1}{n}, f(\tfrac{n-1}{n})).$$

Then by monotonicity,

$$\begin{aligned}|I(f_n) - I(f)| &\leq \int_0^1 |f_n(x) - f(x)|\,\mathrm{d}x = \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} |f_n(x) - f(x)|\,\mathrm{d}x \\ &\leq \frac{1}{n}\left([f(\tfrac{1}{n}) - 0] + \sum_{k=2}^{n-1} [f(\tfrac{k}{n}) - f(\tfrac{k-1}{n})] + [1 - f(\tfrac{n-1}{n})] \right) \\ &= \frac{1}{n},\end{aligned}$$

hence $I(f_n) \to I(f)$ as $n \to \infty$ and the desired inequality $I(f) \leq \frac{1}{2}$ follows from the previous step. □

Remark 6.4. *Let $\gamma(t) = (f(t) + t, f(t) - t)$. Then $\int_0^1 |f(t) - t|(f'(t) + 1)\,\mathrm{d}t = \int_\gamma |y|\,\mathrm{d}x$ computes the area between the path γ and the horizontal axis. Note that γ is essentially the $-45°$-rotation of the graph $y = f(x)$ up to scaling.*

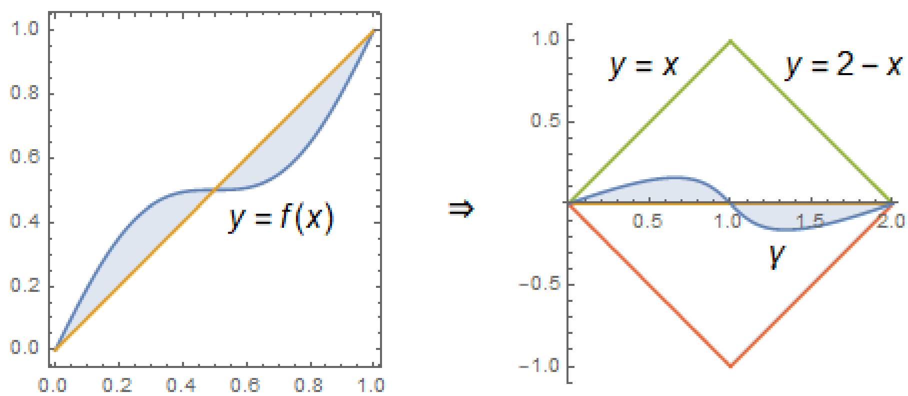

Figure 9. Graph

Then the above bounds immediately follow from the fact that the graph of γ defines a function on $[0, 2]$ which is squeezed between lines $y = \pm x$ and $y = \pm(2 - x)$.

Problem 6.5. *Let f be an increasing function on $[0, 1]$ such that $0 \leq f(x) \leq 1$ and $\int_0^1 (f(x) - x)dx = 0$. Show that*

$$\int_0^1 |f(x) - x|dx \leq \frac{1}{4}.$$

Proof. **Step 1. (Proof under extra assumptions)** Assume that

- $f : [0, 1] \to [0, 1]$ is continuous and non-decreasing;
- $\displaystyle\int_0^1 (f(x) - x)\,\mathrm{d}x = 0$;

- $f(0) = 0$, and $f(1) = 1$.

Then the set $U_+ = \{x \in [0,1] : f(x) > x\}$ is open, hence it is written as the union of at most countably many disjoint open intervals (a_i, b_i), $i = 1, 2, \dots$ Also, the continuity of f forces that $f(b_i) = b_i$. So,

$$\begin{aligned} I_+ := \int_{U_+} |f(x) - x| \, \mathrm{d}x &= \sum_i \int_{(a_i,b_i)} (f(x) - x) \, \mathrm{d}x \\ &\leq \sum_i \int_{(a_i,b_i)} (b_i - x) \, \mathrm{d}x = \sum_i \frac{(b_i - a_i)^2}{2} \\ &\leq \frac{1}{2}\Big(\sum_i (b_i - a_i)\Big)^2 = \frac{1}{2}|U_+|^2. \end{aligned}$$

A similar argument shows that, for $U_- = \{x \in [0,1] : f(x) < x\}$ we have

$$I_- := \int_{U_-} |f(x) - x| \, \mathrm{d}x \leq \frac{1}{2}|(U_-|^2.$$

Moreover, from $\int_0^1 (f(x) - x) \, \mathrm{d}x = 0$ we get $I_+ = I_-$. Therefore, together with the observations$\int_0^1 |f(x) - x| \, \mathrm{d}x = 2I_+ = 2I_-$ and $|U_+| + |U_-| \leq 1$, we conclude that

$$\int_0^1 |f(x) - x| \, \mathrm{d}x \leq \min\{|U_+|, |U_-|\}^2 \leq \frac{1}{4}.$$

These inequalities have a nice interpretation in terms of areas:

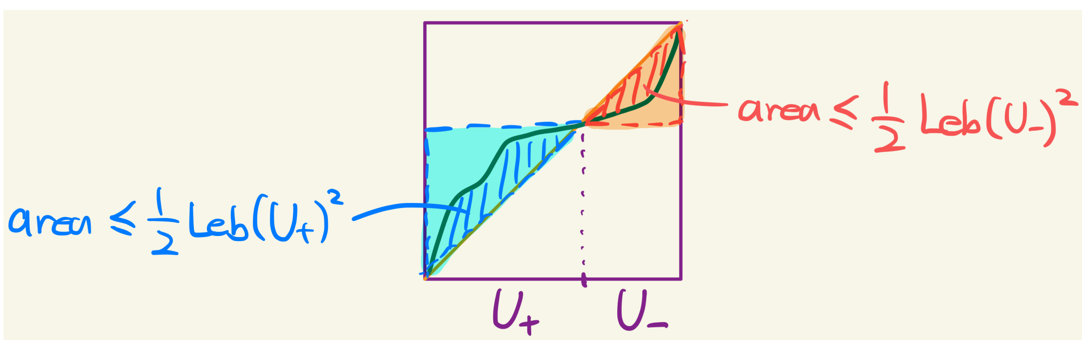

Figure 10. Graph

Step 2. (General case by approximation) Now suppose $f : [0,1] \to [0,1]$ is non-decreasing and satisfies $\int_0^1 (f(x) - x) \, \mathrm{d}x = 0$. Then it is not hard to find a sequence $f_n(x)$ satisfying the conditions in Step 1 and $f_n(x) \to f(x)$ for almost every x. So, by the dominated convergence theorem,

$$\int_0^1 |f(x) - x| \, \mathrm{d}x = \lim_{n\to\infty} \int_0^1 |f_n(x) - x| \, \mathrm{d}x \leq \frac{1}{4}.$$

$\square$

Actually, there is an extension of the above two problems. But the proof needs some tools of *real analysis*. We still give proof here in case someone is interested in. Note that in this case, the bound of RHS can not be improved to 1/4.

Exercise 6.6 (Hard!)**.** *f, g are monotonically increasing in $[0,1]$ and $0 \le f, g \le 1$. $\int_0^1 (f-g)\mathrm{d}x = 0$. Prove that*

$$\int_0^1 |f-g|\mathrm{d}x \le \frac{1}{2}.$$

Hint: Let $f = \mathbf{1}_{[\frac{1}{2},1]}$, $g = \frac{1}{2}$, then $\int_0^1 |f-g|\mathrm{d}x = \frac{1}{2}$. Except for swapping f, g, this is the only case to reach the maximum.

We can decompose $f-g = (f-g)^+ - (f-g)^-$ where $h^+ = \max(0,h), h^- = -\min(h,0)$. So $\int (f-g)^+ = \int (f-g)^-$.

f, g is monotone means $f-g$ has bounded variation. In particular,

$$V(f-g) \le V(f) + V(g) \le 2.$$

So $\sup(f-g)^+ + \sup(f-g)^- \le 1$. To see this we can assume $f(0) = g(0) = 0, f(1) = g(1) = 1$ since we don't assume f, g to be continuous. And for simplicity assume supermum can be taken, say $f(a)-g(a) = \max(f-g)$, $f(b)-g(b) = \min(f-g)$.

Assume $a < b$ otherwise swap f, g. Then

$$\begin{aligned}
V(f-g) &= V_0^a(f-g) + V_a^b(f-g) + V_b^1(f-g) \\
&\ge (\max - 0) + (\max - \min) + (0 - \min) \\
&= 2(\max - \min) \\
&= 2(\sup(f-g)^+ + \sup(f-g)^-).
\end{aligned}$$

Then use

$$\int |f-g| = \int_{(f>g)} (f-g)^+ + \int_{(f<g)} (f-g)^- = 2\int_{(f>g)} (f-g)^+ =: 2I.$$

And we have

$$I \le m(f>g) \cdot \sup(f-g) \text{ and } I \le m(f<g) \cdot \sup(g-f)$$

with $m(f>g) + m(f<g) \le 1$, $\sup(f-g) + \sup(g-f) \le 1$.

If

$$I > \frac{1}{4},\ m(f>g)\sup(f-g) \ge I,$$

then

$$\begin{aligned}
m(f<g)\sup(g-f) &\le (1 - m(f>g))(1 - \max(f-g)) \\
&\le 1 + m(f>g)\sup(f-g) - (m(f>g) + \sup(f-g)) \\
&\le 1 + I - 2\sqrt{I} = (1-\sqrt{I})^2 < I,
\end{aligned}$$

contradiction. □

Problem 7.1. *Suppose that* $f \in C(-\infty, +\infty)$*, let*

$$f_n(x) = \sum_{k=0}^{n-1} \frac{1}{n} f\left(x + \frac{k}{n}\right).$$

Prove that $\{f_n(x)\}$ *converges uniformly on any finite intervals.*

Exercise 7.2. *Assume* $f(x)$ *is Riemann integrable in any closed subset of* $\mathbb{R}$*, and*

$$S_n(x) = \sum_{k=1}^{n} \frac{1}{n} f\left(x + \frac{k}{n}\right), n = 1, 2, \cdots,$$

does function sequences $\{S_n\}$ *converges uniformly on any closed subset of* $\mathbb{R}$ *?*

Hint: **First way:** Prove it first for continuous f; uniform continuity will be helpful here. For the full result, let $[a, b]$ be given. If $\varepsilon > 0$, there exists a continuous g on $[a, b+1]$ such that $\displaystyle\int_a^{b+1} |f - g| < \varepsilon$. Use this and Problem 7.1.

Second way: Suppose $[a, b]$ is given. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$\sum_P (M_j - m_j)\Delta x_j < \varepsilon,$$

whenever the partion P of $[a, b+1]$ satisfies $\mu(P) < \delta$. Here $\mu(P)$ is the mesh size of P and m_j, M_j are the inf and sup of f over the jth subinterval determined by P.

Suppose $1/n < \delta$. Let $x \in [a, b]$ and let $x_k = x + k/n$. Then $\{x_k : k = 0, \ldots, n\}$ can be extended to a partition P of $[a, b+1]$ with $\mu(P) < \delta$. Thus

$$\begin{aligned} \left| S_n(x) - \int_x^{x+1} f(t)\,dt \right| &= \left| \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} (f(x + k/n) - f(t))dt \right| \\ &\le \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} |f(x + k/n) - f(t)|dt \\ &\le \sum_{k=1}^{n} (M_k - m_k) \cdot \frac{1}{n} \le \sum_P (M_j - m_j)\Delta x_j < \varepsilon. \end{aligned}$$

□

Problem 7.3. *Suppose that* $f_1(x)$ *is integrable on* $[a,b]$*, and let*

$$f_{n+1}(x) = \int_a^x f_n(t)\ \mathrm{d}t, \quad n = 1, 2, \cdots.$$

Prove that $\{f_n(x)\}$ *converges uniformly to* 0.

Problem 7.4. *Suppose that* $f \in C(-\infty, +\infty)$*, and* $|f(x)| < |x|$*,* $\forall\, x \neq 0$*. Define* $f_1(x) = f(x)$*,* $f_2(x) = f(f_1(x))$*,* $\cdots$*,* $f_n(x) = f(f_{n-1}(x)), \cdots$*. Porve that* $\{f_n(x)\}$ *converges uniformly on* $[-A, A]$.

Problem 7.5. *Suppose that there exists* $M > 0$ *such that* $|f_0(x)| \leq M$ *and*

$$\sum_{n=0}^{m} |f_n(x) - f_{n+1}(x)| \leq M, \quad m = 0, 1, 2, \cdots.$$

Prove that if $\sum_{n=0}^{\infty} b_n$ *converges, then* $\sum_{n=0}^{\infty} b_n f_n(x)$ *converges uniformly.*

Exercise 7.6. *Suppose* $f(x)$ *is positive monotone increasing function over* $[0, \infty)$*, and* $f(x) \in C^1[0, +\infty)$*. Prove that if* $\int_0^{+\infty} \frac{e^x}{f(x) + f'(x)}\,\mathrm{d}x$ *is convergent, then* $\int_0^{+\infty} \frac{1}{f(x)}\,\mathrm{d}x$ *is convergent.*

Hint: For each $n \in \{0, 1, 2, \cdots\}$, let

$$E_n = \{x \in [n, n+1] : f'(x) \geq 2f(n+1)\}.$$

Also, let $|E_n|$ denotes the length of E_n.

We firstly show that $|E_n| \leq \frac{1}{2}$. Indeed, if $|E_n| > \frac{1}{2}$, then there is

$$f(n+1) \geq f(n) + \int_{E_n} f'(x)\,\mathrm{d}x > f(n) + f(n+1),$$

a contradiction. Then

$$\begin{aligned}\int_n^{n+1} \frac{\mathrm{d}x}{f(x) + f'(x)} &\geq \int_{[n,n+1]\setminus E_n} \frac{\mathrm{d}x}{f(x) + f'(x)} \\ &\geq \int_{[n,n+1]\setminus E_n} \frac{\mathrm{d}x}{3f(n+1)} \\ &\geq \frac{1}{6f(n+1)}.\end{aligned}$$

Hence

$$\frac{1}{f(n+1)} \leq 6\int_n^{n+1} \frac{\mathrm{d}x}{f(x)+f'(x)}.$$

Therefore

$$\int_0^\infty \frac{\mathrm{d}x}{f(x)} \leq \sum_{n=0}^\infty \frac{1}{f(n)} \leq \frac{1}{f(0)} + 6\int_0^\infty \frac{\mathrm{d}x}{f(x)+f'(x)}.$$

Now by the assumption, the right-hand side is finite, and therefore $\displaystyle\int_0^\infty \frac{\mathrm{d}x}{f(x)}$ converges.

□

Exercise 7.7. *Determine whether the following series converges:*

$$\sum_{n=2}^\infty \frac{(-1)^n}{\sqrt{n}-(-1)^{[\sqrt{n}]}},$$

where

$$[x] = \max\{k \in \mathbb{Z} : k \leq x\}.$$

Hint: First I write

$$\frac{(-1)^n}{\sqrt{n}-(-1)^{[\sqrt{n}]}} = \frac{(-1)^n(\sqrt{n}+(-1)^{[\sqrt{n}]})}{n-1} = \frac{(-1)^n\sqrt{n}}{n-1} + \frac{(-1)^{n+[\sqrt{n}]}}{n-1}.$$

It is trivial that $\displaystyle\sum_{n=2}^\infty \frac{(-1)^n\sqrt{n}}{n-1}$ converges. Now I claim that the series $\displaystyle\sum_{n=2}^\infty \frac{(-1)^{n+[\sqrt{n}]}}{n-1}$ converges. Consider the sequence $n+[\sqrt{n}]$, this sequence is (starting from $n=1$):

$$2, 3, 4, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21\ldots$$

we see that the numbers do not show up in the sequence are $5, 11, 19, 29, \ldots$, i.e. numbers of the form m^2+m-1, and these are all odd numbers (Prove this!).

Define

$$a_n = \begin{cases} \frac{(-1)^{n+[\sqrt{n}]}}{n-1}, & n \text{ is not a complete squre,} \\ 0, & \text{otherwise.} \end{cases}$$

Then by alternating series test $\displaystyle\sum_{n=2}^\infty a_n$ converges. On the other hand, define

$$b_n = \begin{cases} \frac{(-1)^{n+[\sqrt{n}]}}{n-1} = \frac{1}{n-1}, & n \text{ is a complete squre,} \\ 0, & \text{otherwise.} \end{cases}$$

then the series $\sum_{n=2}^\infty b_n$ is definitely convergent.

Observe that

$$\frac{(-1)^{n+[\sqrt{n}]}}{n-1} = a_n + b_n, \quad \forall n \in \mathbb{N},$$

therefore the series $\sum_{n=2}^{\infty} \frac{(-1)^{n+[\sqrt{n}]}}{n-1}$ is convergent. Hence the original series is convergent.

$\square$

Exercise 7.8. *Let $f(x)$ be a function with positive values and with continuous derivative on $[0, +\infty)$. Suppose a and b real numbers. We know that the following integral converges:*

$$\int_0^{+\infty} \frac{\sqrt{a^2 + b^2 (f'(x))^2}}{f(x)}\, dx < +\infty.$$

Prove that $a = 0$ or $b = 0$.

Hint: If there is a sequence $\{x_n\}$ such that $\lim_{n\to\infty} x_n = \infty$ and $\lim_{n\to\infty} f(x_n) = \infty$ then for any n, one has

$$\int_0^{\infty} \frac{\sqrt{a^2 + b^2 (f'(x))^2}}{f(x)}\, dx \geq |b| \int_0^{x_n} \frac{f'(x)}{f(x)} dx = |b| \ln \frac{f(x_n)}{f(0)}.$$

Hence $b = 0$. If there is not the sequence as above, that is f is bounded, then there is $M > 0$ such that $\frac{1}{f} \geq M$. From this, it is easy to see that $a = 0$. $\square$

Remark 7.9. Exercise 7.8 implies that

$$\int_0^{+\infty} \frac{\sqrt{1 + (f'(x))^2}}{f(x)}\, dx = +\infty.$$

Exercise 7.10. *Determine whether the following series converges:*

$$\sum_{n=1}^{\infty} \frac{\sin n \cdot \sin n^2}{n}.$$

Hint: **Approach 1: Telescoping Sum**

$$\begin{aligned}\sum_{k=1}^{n} \sin(k) \sin\left(k^2\right) &= \frac{1}{2} \sum_{k=1}^{n} (\cos(k(k-1)) - \cos(k(k+1))) \\ &= \frac{1 - \cos(n(n+1))}{2}.\end{aligned}$$

Thus, the partial sums are bounded, by Dirichlet's Test we know the series is convergent.

Approach 2: Summation by Parts

$$
\begin{aligned}
\sum_{k=1}^{\infty} \frac{\sin(k)\sin\left(k^2\right)}{k} &= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\sin(k)\sin\left(k^2\right)}{k} \\
&= \lim_{n\to\infty} \frac{1}{2} \sum_{k=1}^{n} \frac{\cos(k(k-1)) - \cos(k(k+1))}{k} \\
&= \lim_{n\to\infty} \left(\frac{1}{2} - \frac{\cos(n(n+1))}{2n} - \frac{1}{2} \sum_{k=1}^{n-1} \frac{\cos(k(k+1))}{k(k+1)} \right) \\
&= \frac{1}{2} - \frac{1}{2} \sum_{k=1}^{\infty} \frac{\cos(k(k+1))}{k(k+1)},
\end{aligned}
$$

and the last sum converges by comparison to

$$
\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1.
$$

$\square$

8. Week 10 (4.26)

Problem 8.1. *Suppose that* $a_n \geq 0$, $\sum_{n=1}^{\infty} a_n$ *converges, and*

$$b_m = \sum_{n=1}^{\infty} \left(1 + \frac{1}{n^m}\right)^n a_n.$$

Prove that R, *the radius of convergence of the power series* $\sum_{m=1}^{\infty} b_m x^m$, *satisfying* $1/e \leq R \leq 1$.

Problem 8.2. *Suppose that* $\{a_n\}$ *satisfying* $\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} = 1$. *Denote that* $S_n = \sum_{k=0}^{n} a_k$. *Prove that*

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|S_n|} = 1.$$

Problem 8.3. *Suppose that* $a_n \geq 0$, $\sum_{n=1}^{\infty} a_n$ *diverges, and*

$$\lim_{n\to\infty} \frac{a_n}{a_1 + a_2 + \cdots + a_n} = 0.$$

Prove that

$$\overline{\lim_{n\to\infty}} \sqrt[n]{a_n} = 1.$$

Problem 8.4. *Expand* $f(x) = \sin^3 x$ *as a power series at* $x = 0$, *and find the domain of convergence.*

Exercise 8.5. *Find the Taylor series of* $\dfrac{x \sin\alpha}{1 - 2x\cos\alpha + x^2}$ $(|x| < 1)$.

Hint. Suppose that

$$\frac{x \sin\alpha}{1 - 2x\cos\alpha + x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

Then

$$\begin{aligned} x\sin\alpha &= (1-2x\cos\alpha+x^2)\sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots - (2a_0\cos\alpha)x \\ &\quad - (2a_1\cos\alpha)x^2 - (2a_2\cos\alpha)x^3 + \cdots + a_0 x^2 + a_1 x^3 + \cdots . \end{aligned}$$

Comparing the coefficients, we have $a_0 = 0$, $a_1 = \sin\alpha$, $a_2 = \sin 2\alpha$, $\cdots$, $a_n = \sin n\alpha$, $\cdots$. □

Problem 8.6. *Compute the integral* $\displaystyle\int_0^1 \frac{\ln x}{1-x^2}\,\mathrm{d}x$.

Exercise 8.7. *Prove that*

$$e^x + e^{-x} \le 2e^{x^2/2}, \quad x \in \mathbb{R}.$$

***Hint*.** Since

$$e^x + e^{-x} = 2\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!},$$

and

$$2e^{x^2/2} = 2\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!!}.$$

□

Problem 8.8. *Suppose that for* $-1 < x < 1$, *there is*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad \lim_{n\to\infty} n a_n = 0.$$

Prove that if $\displaystyle\lim_{x\to 1-0} f(x) = S$, *then* $\displaystyle\sum_{n=0}^{\infty} a_n = S$.

Problem 8.9. *Suppose that* $\{a_n\}$, $\{b_n\}$ *satisfying* $a_n > 0$, *and series* $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ *converges in* $|x| < 1$, *diverges at* $x = 1$. *Assume that* $\displaystyle\lim_{n\to\infty} \frac{b_n}{a_n} = A$, $0 \le A < +\infty$. *Prove* *that*

$$\lim_{x\to 1-0} \frac{\sum\limits_{n=0}^{\infty} b_n x^n}{\sum\limits_{n=0}^{\infty} a_n x^n} = A.$$

Exercise 8.10. *Suppose that* $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$, *and its radius of convergence is* $R = +\infty$. *Let*

$$f_n(x) = \sum_{k=0}^{n} a_k x^k.$$

Prove that when $n \to \infty$,

$$f(f_n(x)) \rightrightarrows f(f(x)), \quad (a \le x \le b).$$

Hint. $f_n \rightrightarrows f$ and f is uniformly continuous. $\square$

Problem 9.1 (Interchanging the Order of Summation)**.** *If* $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|a_{jk}|<\infty$, *then*

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}a_{jk}=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}a_{jk}.$$

Proof. The two double sums in the problem really mean

$$\begin{aligned}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}a_{jk}&=\lim_{n\to\infty}\sum_{j=1}^{n}\left[\sum_{k=1}^{\infty}a_{jk}\right]=\lim_{n\to\infty}\sum_{j=1}^{n}\left[\lim_{m\to\infty}\sum_{k=1}^{m}a_{jk}\right]\\&=\lim_{n\to\infty}\lim_{m\to\infty}\sum_{j=1}^{n}\sum_{k=1}^{m}a_{jk},\\\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}a_{jk}&=\lim_{m\to\infty}\sum_{k=1}^{m}\left[\sum_{j=1}^{\infty}a_{jk}\right]=\lim_{m\to\infty}\sum_{k=1}^{m}\left[\lim_{n\to\infty}\sum_{j=1}^{n}a_{jk}\right]\\&=\lim_{m\to\infty}\lim_{n\to\infty}\sum_{j=1}^{n}\sum_{k=1}^{m}a_{jk}.\end{aligned}$$

That all of these limits exist is part of the conclusion of the probelm. □

Remark 9.2. *The hypothesis* $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|a_{jk}|<\infty$ *really means that for each* $j\in\mathbb{N}$,

$$\sum_{k=1}^{\infty}|a_{jk}|=M_j<\infty\quad\text{and}\quad\sum_{j=1}^{\infty}M_j<\infty.$$

Problem 9.3 (11.14)**.** *Suppose that* $f(x)$ *can be expanded as a power series in a neighborhood of* $x=0$*, and the sequence* $\{f^{(n)}(0)\}$ *is bounded. Prove that* $f(x)$ *is the restriction of a smooth function, which is defined on* $\mathbb{R}$.

Problem 9.4 (11.15)**.** *Suppose that* $f(x)$ *is continuous on* $[0,1]$*, satisfying*

$$\int_0^1 f(x)x^n\,\mathrm{d}x=0,\quad n=0,1,2,\cdots.$$

Prove that $f(x)\equiv 0$ *on* $[0,1]$.

Problem 9.5. *Suppose that $f \in R[0,1]$, satisfying*

$$\int_0^1 f(x)x^n \, \mathrm{d}x = 0, \quad n = 0, 1, 2, \cdots .$$

Prove that $f(x) = 0$ at every continuous point.

Proof. For $\forall \varepsilon > 0$, there exists a $g \in C[0,1]$ such that

$$\int_0^1 |f(x) - g(x)| \, \mathrm{d}x < \varepsilon.$$

By Weierstrass theorem, we know there is a polynomial P such that $|P(x) - g(x)| < \varepsilon$. Note that $|f| \le M$ since f is integrable. Then

$$\begin{aligned}\int_0^1 f^2(x) \, \mathrm{d}x &= \int_0^1 [f(f - g + g - P) + Pf] \, \mathrm{d}x \\ &= \int_0^1 [f(f - g + g - P)] \, \mathrm{d}x \\ &\le M \left[\int_0^1 |f - g| + |g - P| \, \mathrm{d}x \right] \le 2M\varepsilon.\end{aligned}$$

Hence $\int_0^1 f^2(x) \, \mathrm{d}x = 0$, which means that $f(x) = 0$ at every continuous point. □

Problem 9.6.

(1) *Suppose that $f(x) \in C[-1,1]$, satisfying*

$$\int_{-1}^1 x^{2n} f(x) \, \mathrm{d}x = 0, \quad n = 0, 1, 2, \cdots .$$

Prove that $f(x)$ is an odd function.

(2) *Suppose that $f(x) \in C[-1,1]$, satisfying*

$$\int_{-1}^1 x^{2n+1} f(x) \, \mathrm{d}x = 0, \quad n = 0, 1, 2, \cdots .$$

Prove that $f(x)$ is an even function.

Exercise 9.7. *Suppose that $f(x)$ is continuous on $[0,1]$, satisfying*

$$\int_0^1 f(x)x^n \, \mathrm{d}x = 0, \quad n \ge n_0.$$

Prove that $f(x) \equiv 0$ on $[0,1]$.

Hint. Note that for any polynomial $P(x)$, there is

$$\int_0^1 x^{n_0} f(x) P(x) \, \mathrm{d}x = 0.$$

$\square$

Problem 9.8 (11.17)**.** *Suppose that $f(x)$ can be approximated by polynomials on a infinte interval. Prove that $f(x)$ must be a polynomial.*

Problem 9.9 (11.20)**.** *Suppose that $f(x)$ is continuous on $[0,1]$. For any $n \in \mathbb{N}$, we define*

$$B_n(f,x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}.$$

Prove that $B_n(f,x) \rightrightarrows f(x)$ $(x \in [0,1])$.

Problem 9.10 (Lebesgue)**.** *Any continuous function on a interval must have primitive functions.*

Proof. Suppose that $f \in C[a,b]$. By the Weierstrass approximation theorem, we know that there exist polynomials $\{P_n(x)\}$ on $[a,b]$ such that $P_n(x) \rightrightarrows f(x)$, $n \to \infty$.

For every $P_n(x)$, there is a polynomial $Q_n(x)$ such that $Q_n' = P_n$ on $[a,b]$. And we may always assume $Q_n(a) = 0$.

First, we show that $\{Q_n(x)\}$ is uniformly convergent on $[a,b]$. Indeed, since $\{P_n(x)\}$ converges uniformly, we know for $\forall\, \varepsilon > 0$, there exists $N \in \mathbb{N}$, such that for $\forall\, n \geq N$ and $p \in \mathbb{N}$, there is

$$|P_{n+p}(x) - P_n(x)| < \varepsilon.$$

Then by Lagrange's theorem, we have

$$\begin{aligned} |Q_{n+p}(x) - Q_n(x)| &= |[Q_{n+p}(x) - Q_n(x)] - [Q_{n+p}(a) - Q_n(a)]| \\ &= |P_{n+p}(\xi) - P_n(\xi)| < \varepsilon. \end{aligned}$$

Hence, the Cauchy theorem tells us that $\{Q_n(x)\}$ converges uniformly on $[a,b]$. Denote $F(x) := \lim\limits_{n\to\infty} Q_n(x)$.

Next, we show that F is differentiable and $F' = f$ on $[a,b]$. To prove this, it suffices to prove for given $x_0 \in [a,b]$ and $\varepsilon > 0$, there exists $\delta > 0$, such that $0 < |h| < \delta$, there is

$$\left| \frac{F(x_0+h) - F(x_0)}{h} - f(x_0) \right| < 3\varepsilon.$$

Note that

$$\left|\frac{F(x_0+h)-F(x_0)}{h}-f(x_0)\right| \le \left|\frac{Q_n(x_0+h)-Q_n(x_0)}{h}-P_n(x_0)\right| + |P_n(x_0)-f(x_0)| + \left|\frac{F(x_0+h)-Q_n(x_0+h)-F(x_0)+Q_n(x_0)}{h}\right|.$$

For $\forall\,\varepsilon>0$, there exists $N\in\mathbb{N}$, such that $\forall\,n\ge N$, there is $|P_n(x_0)-f(x_0)|<\varepsilon$. Set $n=N$, by Lagrange's theorem and $P_N(x)$ is uniformly continuous on $[a,b]$, we have that there exists $\delta>0$, such that for $|h|<\delta$, there is

$$\left|\frac{Q_N(x_0+h)-Q_N(x_0)}{h}-P_N(x_0)\right| = |P_N(x_0+\theta h)-P_N(x_0)|<\varepsilon,$$

where $0<\theta<1$. By Lagrange's theorem, we also have

$$\begin{aligned}&\left|\frac{Q_{N+p}(x_0+h)-Q_N(x_0+h)-Q_{N+p}(x_0)+Q_N(x_0)}{h}\right|\\ \le&|P_{N+p}(x_0+\theta h)-P_N(x_0+\theta h)|<\frac{\varepsilon}{2}.\end{aligned}$$

Then, letting $p\to\infty$, we have

$$\left|\frac{F(x_0+h)-Q_N(x_0+h)-F(x_0)+Q_N(x_0)}{h}\right| \le \frac{\varepsilon}{2},$$

which gives us the desired inequality. □

10. Week 13 (5.17)

Problem 10.1. *Use power series to prove Vandermonde's identity:*

$$\sum_{k=0}^{n} \binom{\alpha}{k} \binom{\beta}{n-k} = \binom{\alpha+\beta}{n}.$$

Consequently, we get

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Problem 10.2. *Suppose that* $\{g_n(x)\}$ *is nonnegative and continuous on* $[0,1]$*, and for every* x^k $(k = 0, 1, 2, \cdots)$,

$$\lim_{n\to\infty} \int_0^1 x^k g_n(x)\ \mathrm{d}x$$

exists. Prove that for any $f \in C[0,1]$,

$$\lim_{n\to\infty} \int_0^1 f(x) g_n(x)\ \mathrm{d}x$$

exists.

Problem 10.3.

(1) *Suppose that* $f \in C[1,+\infty)$, $f(+\infty) = A$*. Prove that for any* $\varepsilon > 0$*, there exists a polynomial* P *such that*

$$\left| f(x) - P\left(\frac{1}{x}\right) \right| < \varepsilon, \quad x \in [1,+\infty).$$

(2) *Suppose that* $f \in C[0,+\infty)$, $f(+\infty) = A$*. Prove that for any* $\varepsilon > 0$*, there exists a polynomial* P *such that*

$$\left| f(x) - P\left(e^{-x}\right) \right| < \varepsilon, \quad x \in (0,+\infty).$$

Problem 10.4 (Riemann's lemma)**.** *Suppose that* $f(x) \in R[a,b]$*, then*

$$\lim_{p\to+\infty} \int_a^b f(x) \sin px\ \mathrm{d}x = 0, \quad \lim_{p\to+\infty} \int_a^b f(x) \cos px\ \mathrm{d}x = 0.$$

Hint. By approximation of Riemann integral and the Weierstrass approximation theorem, we know that for any $\varepsilon > 0$, there is a polynomial P such that

$$\int_a^b |f(x) - P(x)| \, \mathrm{d}x < \varepsilon.$$

By integral by parts, it's easy to see

$$\lim_{p \to +\infty} \int_a^b P(x) \sin px \, \mathrm{d}x = 0, \qquad \lim_{p \to +\infty} \int_a^b P(x) \cos px \, \mathrm{d}x = 0.$$

□

Exercise 10.5. *Suppose that $f \in R[a, b]$. Prove for $\forall\, \varepsilon > 0$, there exist two polynomials $p(x)$ and $P(x)$ on $[a, b]$ satisfying*

(1) $p(x) \le f(x) \le P(x)$, $\forall\, x \in [a, b]$;

(2) $\displaystyle\int_a^b [P(x) - p(x)] \, \mathrm{d}x < \varepsilon$.

Hint. Firstly, show there exist continuous functions $g(x)$ and $h(x)$ on $[a, b]$ satisfying

(1) $g(x) \le f(x) \le h(x)$, $\forall\, x \in [a, b]$;

(2) $\displaystyle\int_a^b [h(x) - g(x)] \, \mathrm{d}x < \varepsilon$.

Then apply the Weierstrass approximation theorem. □

Problem 10.6. *Find the sum of series* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Problem 10.7.

(1) *Find* $\displaystyle S = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$;

(2) *Let* $\displaystyle a_n = 1 - \frac{1}{2} + \cdots + \frac{(-1)^n}{n} - \ln 2$, $n = 1, 2, \cdots$, *find* $\displaystyle\sum_{n=1}^{\infty} a_n$.

Problem 10.8. *Find* $\displaystyle f(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n}$, $\forall\, x \in (0, 2\pi)$.

Remark 10.9. *By Problem 10.8, we know that*

$$\ln \sin \frac{x}{2} = -\ln 2 - \sum_{n=1}^{\infty} \frac{\cos nx}{n}, \quad \forall\, x \in (0, 2\pi).$$

Hence, integrating terms by terms yields

$$\int_0^{\pi/2} \ln \sin x \ \mathrm{d}x = -\frac{\pi}{2} \ln 2.$$

And there is another way to evaluate the integration.

Firstly, we have

$$\int_0^{\frac{\pi}{2}} \ln \sin x \ dx \overset{t=\frac{\pi}{2}-x}{=} \int_0^{\frac{\pi}{2}} \ln \cos t \ dt,$$

and

$$\int_{\frac{\pi}{2}}^{\pi} \ln \sin x \ dx \overset{t=x-\frac{\pi}{2}}{=} \int_0^{\frac{\pi}{2}} \ln \cos t \ dt.$$

Then

$$\begin{aligned}
2\int_0^{\frac{\pi}{2}} \ln \sin x\, dx &= \int_0^{\frac{\pi}{2}} \ln \sin x\, dx + \int_0^{\frac{\pi}{2}} \ln \cos x\, dx \\
&= \int_0^{\frac{\pi}{2}} \ln \sin 2x\, dx - \frac{\pi}{2} \ln 2 \\
&= \frac{1}{2} \int_0^{\pi} \ln \sin x \ dx - \frac{\pi}{2} \ln 2 \\
&= \frac{1}{2} \left(\int_0^{\frac{\pi}{2}} \ln \sin x \ dx + \int_{\frac{\pi}{2}}^{\pi} \ln \sin x \ dx \right) - \frac{\pi}{2} \ln 2 \\
&= \int_0^{\frac{\pi}{2}} \ln \sin x \ dx - \frac{\pi}{2} \ln 2.
\end{aligned}$$

That is

$$\int_0^{\frac{\pi}{2}} \ln \sin x \ dx = -\frac{\pi}{2} \ln 2.$$

Problem 11.1. *Prove that* $\pi = \sum_{n=0}^{\infty} \frac{(n!)^2 2^{n+1}}{(2n+1)!}$.

Proof. Note that

$$\frac{(n!)^2 \cdot 2^{n+1}}{(2n+1)!} = \frac{(n!)^2 \cdot 2^{n+1}}{(2n+1)!! \cdot (2n)!!} = \frac{(2n)!!}{(2n+1)!!} \cdot \frac{1}{2^{n-1}} = \frac{1}{2^{n-1}} \int_0^{\pi/2} (\cos x)^{2n+1} \, \mathrm{d}x.$$

Then

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(n!)^2 \cdot 2^{n+1}}{(2n+1)!} &= \sum_{n=0}^{\infty} \frac{1}{2^{n-1}} \int_0^{\pi/2} (\cos x)^{2n+1} \, \mathrm{d}x = \int_0^{\pi/2} (2\cos x) \sum_{n=0}^{\infty} \left(\frac{\cos^2 x}{2} \right)^n \, \mathrm{d}x \\ &= \int_0^{\pi/2} \frac{4 \cos x}{1 - \cos^2 x} \, \mathrm{d}x = 4 \arctan(\sin x)\Big|_0^{\pi/2} = \pi.\end{aligned}$$

□

Problem 11.2. *Suppose that* $f(x) \in C^{\infty}(\mathbb{R})$ *satisfying* $|f^{(k)}(x)| \le M$, $k = 0, 1, 2, \cdots$ *and* $f\left(\frac{1}{2^n}\right) = 0$, $(n = 1, 2, \cdots)$. *Prove that* $f \equiv 0$.

Remark 11.3. $|f^{(k)}(x)| \le M$, $k = 0, 1, 2, \cdots$ *is necessary. For example*

$$f(x) = \begin{cases} e^{-1/x^2} \sin \dfrac{\pi}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Exercise 11.4. *Prove* $f(x)$ *defined in Remark 11.3 is smooth on* $\mathbb{R}$.

Problem 11.5. *Suppose that* $f(x)$ *is a periodic function with period* 2π, *and* $f(x)$ *is bounded on* $(0, 2\pi)$. *Prove that* $b_n \ge 0$ *if* $f(x)$ *is decreasing.*

Proof. By definition, we have

$$\begin{aligned} b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, \mathrm{d}x &= \frac{1}{\pi} \sum_{k=1}^{n} \int_{(k-1)\frac{2\pi}{n}}^{k\frac{2\pi}{n}} f(x) \sin nx \, \mathrm{d}x \\ &= \frac{1}{\pi} \sum_{k=1}^{n} \left[\int_{(k-1)\frac{2\pi}{n}}^{(k-\frac{1}{2})\frac{2\pi}{n}} f(x) \sin nx \, \mathrm{d}x + \int_{(k-\frac{1}{2})\frac{2\pi}{n}}^{k\frac{2\pi}{n}} f(x) \sin nx \, \mathrm{d}x \right] \end{aligned}$$

$$= \frac{1}{\pi} \sum_{k=1}^{n} \left[\int_{(k-1)\frac{2\pi}{n}}^{(k-\frac{1}{2})\frac{2\pi}{n}} f(x) \sin nx \, \mathrm{d}x - \int_{(k-1)\frac{2\pi}{n}}^{(k-\frac{1}{2})\frac{2\pi}{n}} f\left(x + \frac{\pi}{n}\right) \sin nx \, \mathrm{d}x \right]$$
$$= \frac{1}{\pi} \sum_{k=1}^{n} \int_{(k-1)\frac{2\pi}{n}}^{(k-\frac{1}{2})\frac{2\pi}{n}} \left[f(x) - f\left(x + \frac{\pi}{n}\right) \right] \sin nx \, \mathrm{d}x \geq 0.$$

□

Exercise 11.6. *Suppose that $f'(x)$ is bounded on $(0, 2\pi)$. Prove that $a_n \geq 0$ if $f'(x)$ is decreasing.*

Hint. Similar to Problem 11.5. □

Problem 11.7. *Suppose that $f(x)$ is a periodic function with period 2π satisfying*
$$|f(x) - f(y)| \leq L|x - y|^{\alpha} \quad (0 < \alpha \leq 1).$$
Prove that $a_n = O\left(\frac{1}{n^{\alpha}}\right)$, $b_n = O\left(\frac{1}{n^{\alpha}}\right)$.

Proof. Note that
$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, \mathrm{d}x \\ &= \frac{1}{\pi} \int_{-\pi-\frac{\pi}{n}}^{\pi-\frac{\pi}{n}} f\left(x + \frac{\pi}{n}\right) \cos(nx + \pi) \, \mathrm{d}x \\ &= -\frac{1}{\pi} \int_{-\pi}^{\pi} f\left(x + \frac{\pi}{n}\right) \cos nx \, \mathrm{d}x. \end{aligned}$$
Then
$$\begin{aligned} |a_n| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[f(x) - f\left(x + \frac{\pi}{n}\right) \right] \cos nx \, \mathrm{d}x \right| \\ &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f(x) - f\left(x + \frac{\pi}{n}\right) \right| |\cos nx| \, \mathrm{d}x \\ &\leq \frac{1}{2\pi} L \left(\frac{\pi}{n}\right)^{\alpha} \int_{-\pi}^{\pi} |\cos nx| \, \mathrm{d}x \leq L \left(\frac{\pi}{n}\right)^{\alpha}, \end{aligned}$$
which gives us that $a_n = O\left(\frac{1}{n^{\alpha}}\right)$. Similarly, we have $b_n = O\left(\frac{1}{n^{\alpha}}\right)$. □

Problem 11.8. *Find the Fourier expansion of*

$$f(x) = \begin{cases} 0, & -\pi \le x \le 0, \\ \sin x, & 0 < x \le \pi. \end{cases}$$

Problem 11.9. *Use*

$$\sum_{k=1}^{n} \frac{\sin kx}{k} = \sum_{k=1}^{n} \int_0^x \cos kt \, \mathrm{d}t$$

to find

$$S(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n}, \ \forall x \in (0, 2\pi).$$

Proof. Considering

$$\begin{aligned} \sum_{k=1}^{n} \frac{\sin kx}{k} &= \int_0^x \sum_{k=1}^{n} \cos kt \, \mathrm{d}t = -\frac{x}{2} + \int_0^x \frac{\sin(n+1/2)t}{2\sin\frac{t}{2}} \, \mathrm{d}t \\ &= -\frac{x}{2} + \int_0^x \left[\frac{1}{2\sin\frac{t}{2}} - \frac{1}{t}\right] \sin(n+1/2)t \, \mathrm{d}t + \int_0^x \frac{\sin(n+1/2)t}{t} \, \mathrm{d}t. \end{aligned}$$

By Riemann-Lebesgue's lemma, we know that

$$\lim_{n\to\infty} \int_0^x \left[\frac{1}{2\sin\frac{t}{2}} - \frac{1}{t}\right] \sin(n+1/2)t \, \mathrm{d}t = 0.$$

Note that

$$\int_0^x \frac{\sin(n+1/2)t}{t} \, \mathrm{d}t = \int_0^{(n+1/2)x} \frac{\sin t}{t} \, \mathrm{d}t \to \int_0^{\infty} \frac{\sin t}{t} \, \mathrm{d}t = \frac{\pi}{2}, \quad \text{as } n \to \infty.$$

Hence

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n} = \frac{\pi - x}{2}.$$

□

Problem 11.10. *Let*

$$S(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n}, \ \forall x \in (0, 2\pi).$$

Show that

$$\lim_{n\to\infty} \max_{0\le x\le\pi} \{S_n(x) - S(x)\} = \int_0^{\pi} \frac{\sin t}{t} \, \mathrm{d}t - \frac{\pi}{2},$$

where $S_n(x)$ is the partial sum of the previous n terms.

Exercise 11.11 (Hard!). *Find Fourier series of*

$$u(x) = e^{\cos x} \cos \sin x \quad \textit{and} \quad v(x) = e^{\cos x} \sin \sin x.$$

Hint. See solution here. □

Exercise 11.12 (Hard!). *There exists a continuous function whose Fourier series diverges at a point.*

Hint. We describe Fejér example of a continuous function with divergent Fourier series. Fejér example is the even, (2π)-periodic function f defined on $[0, \pi]$ by:

$$f(x) = \sum_{p=1}^{\infty} \frac{1}{p^2} \sin\left[(2^{p^3} + 1)\frac{x}{2}\right].$$

According to Weierstrass M-test, f is continuous. We denote f's Fourier series by

$$\frac{1}{2}a_0 + (a_1 \cos x + b_1 \sin x) + \cdots + (a_n \cos nx + b_n \sin nx) + \cdots.$$

As f is even, the b_n are all vanishing. If we denote for all $m \in \mathbb{N}$:

$$\lambda_{n,m} = \int_0^{\pi} \sin\left[(2m+1)\frac{t}{2}\right] \cos nt \; dt \quad \text{and} \quad \sigma_{n,m} = \sum_{k=0}^{n} \lambda_{k,m}.$$

We have

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos nt \; dt = \frac{2}{\pi}\int_0^{\pi} f(t)\cos nt \; dt \\
&= \frac{2}{\pi}\int_0^{\pi} \left(\sum_{p=1}^{\infty} \frac{1}{p^2} \sin\left[(2^{p^3}+1)\frac{x}{2}\right]\right) \cos nt \; dt \\
&= \frac{2}{\pi}\sum_{p=1}^{\infty} \frac{1}{p^2}\int_0^{\pi} \sin\left[(2^{p^3}+1)\frac{x}{2}\right] \cos nt \; dt \\
&= \frac{2}{\pi}\sum_{p=1}^{\infty} \frac{1}{p^2}\lambda_{n,2^{p^3-1}}.
\end{aligned}$$

We now introduce for all $n \in \mathbb{N}$:

$$S_n = \frac{\pi}{2}\sum_{k=0}^{n} a_k = \sum_{p=1}^{\infty}\sum_{k=0}^{n} \frac{1}{p^2}\lambda_{k,2^{p^3-1}} = \sum_{p=1}^{\infty} \frac{1}{p^2}\sigma_{n,2^{p^3-1}}.$$

We will prove below that for all $n, m \in \mathbb{N}$, we have $\sigma_{m,m} \geq \frac{1}{2} \ln m$ and $\sigma_{n,m} \geq 0$. Indeed,

$$\begin{aligned}\lambda_{n,m} &= \frac{1}{2} \int_0^\pi \left[\sin\left(\frac{2m+1}{2} + n\right) t + \sin\left(\frac{2m+1}{2} - n\right) t \right] \ dt \\ &= \frac{1}{2}\left(\frac{1}{m+n+1/2} + \frac{1}{m-n+1/2}\right) = \frac{m+1/2}{(m+1/2)^2 - n^2}.\end{aligned}$$

Therefore for $n \leq m$ we get $\lambda_{n,m} \geq 0$ and $\sigma_{q,m} \geq 0$ for $q \leq m$. While for $q \geq m$:

$$\begin{aligned}2\sigma_{q,m} &= \sum_{k=0}^{q} \left(\frac{1}{m+k+1/2} + \frac{1}{m-k+1/2}\right) \\ &= \sum_{i=m}^{q+m} \frac{1}{i+1/2} + \sum_{i=m}^{m-q} \frac{1}{i+1/2} \\ &= \sum_{i=m}^{q+m} \frac{1}{i+1/2} + \frac{1}{m+1/2} + \sum_{i=m-q}^{m-1} \frac{1}{i+1/2} \\ &= \frac{1}{m+1/2} + \sum_{i=q-m}^{q+m} \frac{1}{i+1/2} \geq 0.\end{aligned}$$

Hence for $q = m$, we have

$$\begin{aligned}2\sigma_{m,m} &= \frac{1}{m+1/2} + \sum_{i=0}^{2m} \frac{1}{i+1/2} \\ &\geq \sum_{i=0}^{2m} \int_{i+1/2}^{i+3/2} \frac{dt}{t} = \int_{1/2}^{2m+3/2} \frac{dt}{t} \\ &= \ln(4m+3) \geq \ln m,\end{aligned}$$

i.e.

$$\sigma_{m,m} \geq \frac{1}{2} \ln m.$$

Then, we get

$$S_{2^{p^3-1}} \geq \frac{1}{p^2} \sigma_{2^{p^3-1}, 2^{p^3-1}} \geq \frac{1}{2p^2} \ln(2^{p^3-1}) = \frac{p^3-1}{2p^2} \ln 2.$$

As the right hand side diverges to ∞, we can conclude that $\{S_n\}$ diverges and consequently that the Fourier series of f diverges at 0. $\square$

Problem 12.1. *Suppose that* $b_n \searrow 0$, $n \to 0$, *and the series* $\displaystyle\sum_{n=1}^{\infty} \frac{b_n}{n}$ *converges. Then* $\displaystyle f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ *is integrable and absolutely integrable on* $[-\pi, \pi]$.

Proof. It suffices to prove $\displaystyle\int_0^{\pi} |f(x)| \, \mathrm{d}x$ converges. Note that

$$\int_{\pi/(n+1)}^{\pi} |f(x)| \, \mathrm{d}x = \sum_{k=1}^{n} \int_{\pi/(k+1)}^{\pi/k} |f(x)| \, \mathrm{d}x.$$

For $\pi/(k+1) \le x \le \pi/k$, we have $|f(x)| \le \left| \sum_{i=1}^{k} b_i \sin ix \right| + \left| \sum_{i=k+1}^{\infty} b_i \sin ix \right|$. Denote $S_k = b_1 + b_2 + \cdots + b_k$, we know $\left| \sum_{i=1}^{k} b_i \sin ix \right| \le S_k$. For the second term, we know

$$\left| \sum_{i=k+1}^{\infty} b_i \sin ix \right| \le \frac{b_{k+1}}{|\sin x/2|} \le \frac{b_{k+1}}{|x/\pi|} \le (k+1) b_{k+1} \le (k+1) b_k.$$

Hence

$$\int_{\pi/(k+1)}^{\pi/k} |f(x)| \, \mathrm{d}x \le [S_k + (k+1) b_k] \frac{\pi}{k(k+1)} = \pi \left[\frac{S_k}{k(k+1)} + \frac{b_k}{k} \right].$$

Then

$$\int_{\pi/(n+1)}^{\pi} |f(x)| \, \mathrm{d}x \le \pi \sum_{k=1}^{n} \frac{S_k}{k(k+1)} + \pi \sum_{k=1}^{n} \frac{b_k}{k}.$$

Note that

$$\sum_{k=1}^{n} \frac{S_k}{k(k+1)} = \sum_{k=1}^{n} \sum_{i=1}^{k} \frac{b_i}{k(k+1)} = \sum_{i=1}^{n} \sum_{k=i}^{n} \frac{b_i}{k(k+1)} = \sum_{i=1}^{n} \frac{b_i}{i} - \frac{S_n}{n+1},$$

which is

$$\sum_{n=1}^{\infty} \frac{S_n}{n(n+1)} = \sum_{n=1}^{\infty} \frac{b_n}{n}.$$

Then

$$\int_{\pi/(n+1)}^{\pi} |f(x)| \, \mathrm{d}x \le 2\pi \sum_{n=1}^{\infty} \frac{b_n}{n}.$$

Hence $\displaystyle\int_0^{\pi} |f(x)| \, \mathrm{d}x$ converges. $\square$

Problem 12.2. *Let*

$$S_n(x) = \sum_{k=1}^{n} \frac{\cos kx}{k}.$$

Prove that $S_n(x) \geq -1$, *and* $\varliminf_{n\to\infty} \min_{0\leq x\leq \pi} \{S_n(x)\} = -\ln 2$.

Problem 12.3. *Suppose* $f(x)$ *is a periodic function with period* 2π, *and* $f(x)$ *is continuous and piecewise smooth on* $[-\pi, \pi]$. a_n, b_n *are its Fourier coefficients. Find the Fourier expansion of the convolution function*

$$F(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) f(x+t) \ \mathrm{d}t,$$

and derive the Parseval identity.

Problem 12.4. *Suppose that* $f(x)$ *is integrable on* $[0, 2\pi]$. *Prove*

$$\frac{1}{2\pi} \int_0^{2\pi} f(x)(\pi - x) \ \mathrm{d}x = \sum_{n=1}^{\infty} \frac{b_n}{n},$$

where

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \ \mathrm{d}x, \quad (n = 1, 2, \cdots).$$

Exercise 12.5 (Hard!)**.** *Suppose* $f(x)$ *is a continuous, periodic function with period* 2π. *Define*

$$V_n(x) = \frac{(2n)!!}{2\pi(2n-1)!!} \int_{-\pi}^{\pi} f(t) \cos^{2n} \frac{t-x}{2} \ \mathrm{d}t.$$

Prove that $V_n \rightrightarrows f(x)$, $n \to \infty$, $\forall\, x \in [-\pi, \pi]$.

Hint. Let $t - x = u$, we have

$$\begin{aligned}
V_n(x) &= \frac{(2n)!!}{2\pi(2n-1)!!} \int_{-\pi-x}^{\pi-x} f(x+u) \cos^{2n} \frac{u}{2} \ \mathrm{d}u \\
&= \frac{(2n)!!}{2\pi(2n-1)!!} \int_{-\pi}^{\pi} f(x+u) \cos^{2n} \frac{u}{2} \ \mathrm{d}u \\
&= \frac{(2n)!!}{2\pi(2n-1)!!} \int_{0}^{\pi} [f(x+u) - f(x-u)] \cos^{2n} \frac{u}{2} \ \mathrm{d}u.
\end{aligned}$$

Note that

$$\int_0^\pi \cos^{2n} \frac{x}{2} \, \mathrm{d}x = \frac{(2n-1)!!}{(2n)!!}\pi.$$

Hence

$$f(x) = \frac{(2n)!!}{2\pi(2n-1)!!} \int_0^\pi 2f(x) \cos^{2n} \frac{u}{2} \, \mathrm{d}u.$$

Then

$$|V_n(x) - f(x)| = \frac{(2n)!!}{2\pi(2n-1)!!} \left| \int_0^\pi [f(x+u) + f(x-u) - 2f(x)] \cos^{2n} \frac{u}{2} \, \mathrm{d}u \right|.$$

Denote $\varphi(x, u) = f(x+u) + f(x-u) - 2f(x)$. Since $f(x)$ is uniformly continuous on $[-\pi, 2\pi]$, we know for $\forall \varepsilon > 0$, $\exists \delta > 0$ $(\delta < \pi)$, such that when $|x' - x''| < \delta$, there is $|f(x') - f(x'')| < \varepsilon/4$. Then, for $x \in [0, \pi]$, $|u| < \delta$, there is

$$|\varphi(x, u)| \le |f(x+u) - f(x)| + |f(x-u) - f(x)| < \frac{\varepsilon}{2}.$$

Hence

$$\begin{aligned}
&\frac{(2n)!!}{2\pi(2n-1)!!} \left| \int_0^\delta [f(x+u) + f(x-u) - 2f(x)] \cos^{2n} \frac{u}{2} \, \mathrm{d}u \right| \\
\le &\frac{(2n)!!}{2\pi(2n-1)!!} \int_0^\delta |\varphi(x,u)| \cos^{2n} \frac{u}{2} \, \mathrm{d}u \\
< &\frac{\varepsilon}{2} \frac{(2n)!!}{2\pi(2n-1)!!} \int_0^\delta \cos^{2n} \frac{u}{2} \, \mathrm{d}u \\
< &\frac{\varepsilon}{2} \frac{(2n)!!}{2\pi(2n-1)!!} \int_0^\pi \cos^{2n} \frac{u}{2} \, \mathrm{d}u = \frac{\varepsilon}{2}.
\end{aligned}$$

Since $f(x)$ is a continuous, periodic function with period 2π, we know that there exists a $M > 0$ such that

$$|\varphi(x, u)| \le M.$$

Thus

$$\begin{aligned}
&\frac{(2n)!!}{2\pi(2n-1)!!} \left| \int_\delta^\pi [f(x+u) + f(x-u) - 2f(x)] \cos^{2n} \frac{u}{2} \, \mathrm{d}u \right| \\
\le &M \frac{(2n)!!}{2\pi(2n-1)!!} \int_\delta^\pi \cos^{2n} \frac{u}{2} \, \mathrm{d}u \\
\le &M \frac{(2n)!!}{2\pi(2n-1)!!} \cos^{2n} \frac{\delta}{2}.
\end{aligned}$$

Denote $q = \cos \frac{\delta}{2}$. $0 < q < 1$ since $0 < \delta < \pi$. Then $\displaystyle\sum_{n=1}^\infty \frac{(2n)!!}{(2n-1)!!} q^{2n}$ converges, which implies

$$\lim_{n \to \infty} \frac{(2n)!!}{(2n-1)!!} q^{2n} = 0.$$

Hence for $\forall\,\varepsilon > 0$, $\exists\, N > 0$, when $n > N$ there is

$$\frac{(2n)!!}{2\pi(2n-1)!!}\left|\int_{\delta}^{\pi}[f(x+u)+f(x-u)-2f(x)]\cos^{2n}\frac{u}{2}\ \mathrm{d}u\right| < \frac{\varepsilon}{2}.$$

Combinig above all, we have

$$\begin{aligned}|V_n(x)-f(x)| &\le \frac{(2n)!!}{2\pi(2n-1)!!}\left|\int_{0}^{\delta}[f(x+u)+f(x-u)-2f(x)]\cos^{2n}\frac{u}{2}\ \mathrm{d}u\right| \\ &\quad + \frac{(2n)!!}{2\pi(2n-1)!!}\left|\int_{\delta}^{\pi}[f(x+u)+f(x-u)-2f(x)]\cos^{2n}\frac{u}{2}\ \mathrm{d}u\right| \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,\end{aligned}$$

i.e. $V_n \rightrightarrows f(x)$, $n\to\infty$, $\forall\, x\in[-\pi,\pi]$. □

Treating a Fourier series as the formal limit on the unit circle (in the complex plane) of

$$u(z)=\sum_{k=0}^{\infty}\hat{f}(k)z^k+\sum_{k=-\infty}^{-1}\hat{f}(k)\bar{z}^{|k|},\quad z=re^{2\pi i\theta},$$

where

$$\hat{f}(k)=\int_{-1/2}^{1/2}f(t)e^{-2\pi ikt}\ \mathrm{d}t.$$

Since $\{\hat{f}(k)\}$ is a bounded sequence, this function is well defined on $|z|<1$. It can be rewritten as

$$u\left(re^{2\pi i\theta}\right)=\sum_{k=-\infty}^{\infty}\hat{f}(k)r^{|k|}e^{2\pi ik\theta}=\int_{-1/2}^{1/2}f(t)P_r(\theta-t)\ \mathrm{d}t,$$

where

$$P_r(t)=\sum_{k=-\infty}^{\infty}r^{|k|}e^{2\pi ikt}=\frac{1-r^2}{1-2r\cos(2\pi t)+r^2}$$

is the Poisson kernel. The Poisson kernel has properties analogous to those of the Fejér kernel:

$$P_r(t)\ge 0,$$

$$\int_0^1 P_r(t)dt=1,$$

$$\lim_{r\to1^-}\int_{\delta<|t|<1/2}P_r(t)dt=0\quad\text{ if }\delta>0.$$

Therefore, we can show:

Problem 12.6. *If f is continuous on $[0,1]$, then*

$$\lim_{r\to1^-}\max_{0\le x\le1}\{P_r*f(x)-f(x)\}=0.$$

Since the function u is harmonic on $|z| < 1$, it is the solution to the Dirichlet problem with continuous boundary condition:

$$\begin{aligned} \Delta u = 0 \qquad & if |z| < 1, \\ u = f \qquad & if |z| = 1. \end{aligned}$$

What's more, we can study the almost everywhere convergence of $P_r * f(x)$.

Exercise 12.7. *Prove* $\displaystyle\sum_{n=2}^{\infty} \frac{\sin nx}{n \log n}$ *is uniformly convergent on* $(0, 2\pi)$.

Hint. In fact, we can show more generally that if $\{a_n\}$ be a decreasing sequence of real numbers such that $n \cdot a_n \to 0$. Then the series $\sum_{n \geqslant 2} a_n \sin(nx)$ is uniformly convergent on $\mathbb{R}$.

Thanks to Abel transform, we can show that the convergence is uniform on $[\delta, 2\pi - \delta]$ for all $\delta > 0$. Since the functions are odd, we only have to prove the uniform convergence on $[0, \delta]$. Put $M_n := \sup\limits_{k \geqslant n} k a_k$, and $R_n(x) = \displaystyle\sum_{k=n}^{\infty} a_k \sin(kx)$. Fix $x \neq 0$ and N such that $\dfrac{1}{N} \leqslant x < \dfrac{1}{N-1}$. Put for $n < N$:

$$A_n(x) = \sum_{k=n}^{N-1} a_k \sin kx \text{ and } B_n(x) := \sum_{k=n}^{+\infty} a_k \sin(kx),$$

and for $n \geq N$, $A_n(x) := 0$.

Since $|\sin t| \leqslant t$ for $t \geq 0$ we have

$$|A_n(x)| \leqslant \sum_{k=n}^{N-1} a_k k x \leqslant M_n x(N-n) \leqslant \frac{N-n}{N-1} M_n,$$

so $|A_n(x)| \leqslant M_n$.

If $N > n$, we have after writing $D_k = \displaystyle\sum_{j=0}^{k} \sin jx$, $|D_k(x)| \leqslant \dfrac{c}{x}$ on $(0, \delta]$ for some constant c. Indeed, we have $|D_k(x)| \leqslant \dfrac{1}{\sqrt{2(1-\cos x)}}$ and $\cos x = 1 - \dfrac{x^2}{2}(1+\xi)$ where $|\xi| \leqslant \dfrac{1}{2}$, so $2(1-\cos x) \geqslant \dfrac{x^2}{2}$ and $|D_k(x) \leqslant \dfrac{\sqrt{2}}{x}$. Therefore

$$|B_n(x)| \leqslant \frac{\sqrt{2}}{x} \sum_{k=N}^{+\infty} (a_k - a_{k+1}) + a_N \frac{\sqrt{2}}{x} = \frac{2\sqrt{2}}{x} a_N \leqslant 2\sqrt{2} N a_N \leqslant 2\sqrt{2} M_n.$$

We get the same bound if $N \leqslant n$. Finally $|R_n(x)| \leqslant (2\sqrt{2}+1) M_n$ for all $0 \leqslant x \leqslant \delta$, so the convergence is uniform on $\mathbb{R}$. $\square$

Remark 12.8. *It's an example of a Fourier series which is uniformly convergent on the real line, but not absolutely convergent at any point of* $(0, 2\pi)$. *Indeed take* $x \in (0, 2\pi)$. *Since* $|\sin(nx)| \geqslant \sin^2(nx)$, *we would have the convergence of* $\sum_{n \geqslant 2} \frac{\sin^2(nx)}{n \log n}$. *We have* $\sin^2(nx) = \frac{1}{-4}(e^{inx} - e^{-inx})^2 = -\frac{1}{4}(e^{2inx} + e^{-2inx} - 2) = \frac{1}{2} - \frac{1}{2}\cos(2nx)$ *and an Abel transform shows that the series* $\sum_{n \geqslant 2} \frac{\cos(2nx)}{n \log n}$ *is convergent. So the series* $\sum_{n \geqslant 2} \frac{1}{n \log n}$ *would be convergent, which is not the case as the integral test shows.*

Exercise 12.9. *Show* $\sum_{n \geq 2} \frac{\cos nx}{\ln n}$ *is a Fourier series for some integrable function.*

Hint. The series converges pointwise to an even function f on $[-\pi, \pi] \setminus \{0\}$. By the Dirichlet test, the series is uniformly convergent on any interval $[\delta, \pi]$ where $0 < \delta < \pi$. Furthermore, we have f continuous on $[\delta, \pi]$. Thus, we can integrate termwise to obtain

$$\int_\delta^\pi f(x)\, dx = \sum_{n=2}^\infty \frac{1}{\ln n} \int_\delta^\pi \cos nx\, dx = -\sum_{n=2}^\infty \frac{\sin n\delta}{n \ln n}.$$

By Exercise 12.7, the series on the RHS converges uniformly on $[0, \pi]$ since the coefficients $b_n = 1/(n \ln n)$ are monotonically decreasing and satisfy $nb_n \to 0$ as $n \to \infty$. Therefore, we can interchange the limit as $\delta \to 0$ with the sum to obtain

$$\int_0^\pi f(x)\, dx = -\sum_{n=2}^\infty \lim_{\delta \to 0} \frac{\sin n\delta}{n \ln n} = 0.$$

This proves that f is integrable on $[0, \pi]$ as well as $[-\pi, \pi]$ since it is even. By a similar argument we can show that

$$\frac{2}{\pi} \int_0^\pi f(x) \cos nx\, dx = \frac{1}{\ln n}.$$

Therefore, this is a Fourier series for f. ☐

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