# Exercises II

June 11, 2022

1. (1) Let $a_i, x \in \mathbb{R}$ with $a_n \neq 0$. Suppose $R > 0$ such that all real roots of $f(x) = a_0 + a_1 x + \cdots a_n x^n = 0$ are contained in $(-R, R)$. Compute the degree $deg(f, [-R, R], 0)$. We can also consider $f$ as a continuous map from $S^1 = \mathbb{R} \cup \{\infty\} = \mathbb{R}P^1$ into itself. Compute $deg(f, S^1)$.

(2) Let $a_i, z \in \mathbb{C}$ with $a_n \neq 0$. Suppose $R > 0$ such that all complex roots of $f(z) = a_0 + a_1 z + \cdots a_n z^n = 0$ are contained in $D(R) = \{z \in \mathbb{C} | |z| < R\}$. Compute the degree $deg(f, D(R), 0)$. We can also consider $f$ as a continuous map from $S^2 = \mathbb{C} \cup \{\infty\} = \mathbb{C}P^1$, the Riemannian sphere, into itself. Compute $deg(f, S^2)$.

2. Let $B = \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n | \sum_1^n |x_i|^2 \leq 1\}$. Assume that $f : \overline{B} \to \mathbb{R}$ be a $C^2$ function such that

$$\nabla f(x) \cdot x = \sum_1^n \frac{\partial f}{\partial x_i} x_i \neq 0, \quad x \in \partial B.$$

Determine the degree

$$deg(\nabla f, B, 0), \quad \nabla f = (\frac{\partial f}{\partial x_,} \cdots, \frac{\partial f}{\partial x_n}) : B \to \mathbb{R}^n.$$

3. Let

$$Q = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_3 = 0, x_1^2 + x_2^2 \leq 1\}, \partial Q = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_3 = 0, x_1^2 + x_2^2 = 1\},$$

$$S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 = 0, (x_2 - 1)^2 + x_3^2 = 1\}, \quad \phi : Q \to \mathbb{R}^3$$

be continuous with $\phi(x) = x$, $x \in \partial Q$. Prove $\phi(Q) \cap S \neq \emptyset$.

4. Let $\lambda \in \mathbb{R}$. Consider the equation

$$\begin{aligned} -u'' &= \lambda f(u), \quad x \in [0, \pi] \\ u'(0) &= u'(\pi) = 0, \end{aligned} \tag{0.1}$$

$f(u) = u + \cos u - 1$. Prove: $(\lambda, 0)$ is a bifurcation point iff $\lambda = n^2$, $n$ is a non-negative integer.

5. Let $X = l^2$ be the real Hilbert space with the standard inner product, and

$$Ax = (x_1, \cdots, k^{-1}x_k, \cdots) : X \to X,$$
$$T(x) = (x_1^2, \cdots, k^{-2}x_k^2, \cdots) : X \to X.$$

Prove: (1) $f(x) = Ax + T(x)$ is a compact map from $X$ to $X$, (2) compute the Leray-Schauder degree $deg(id - f, B, 0)$, where $B = \{x \in X | \|x\| \leq \frac{1}{2}\}$.

6. Let $\Omega \subset \mathbb{R}^n$ be a bounded regular domain. Given some conditions on $f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that the equation

$$\begin{aligned} -\triangle u &= f(x, u, \nabla u), \quad x \in \Omega \\ u(x) &= 0, \quad x \in \partial\Omega \end{aligned} \tag{0.2}$$

possesses a solution $u \in C^{2,\gamma}$.

7. Let $\Omega \subset \mathbb{R}^n$ be a bounded regular domain with $n \geq 3$ and $2 < q < \frac{2n}{n-2}$. Suppose $u \in H_0^1(\omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla \phi dx = \int_\Omega |u|^{q-2} u \phi dx, \quad \forall \phi \in H_0^1(\Omega).$$

Prove that $u$ is $C^2$ via the $L^p$ and $C^\alpha$ estimate of $-\triangle$.

8. Let $\Omega \subset \mathbb{R}^n$ be a bounded regular domain and $X = W_0^{1,p}(\Omega)$ with $p > 1$, $f : \Omega \times \mathbb{R}$ be continuous satisfying

$$|f(x, u)| \leq C(1 + |u|^\alpha), \quad (x, u) \in \Omega \times \mathbb{R},$$

$\alpha < \frac{n+2}{n-2}$, $F(x, u) = \int_0^u f(x, s) ds$.

(1) Prove the functional

$$I(u) = \frac{1}{p} \int_\Omega |\nabla u|^p dx - \int_\Omega F(x, u(x)) dx$$

is w.s.l.s.c. in $X$; (2) compute the Euler-Lagrange equation of $I(u)$.

9. Let $S^2 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 | \sum_1^3 |x_i|^2 = 1\}$, $A, B \in S^2$,

$$M = \{\gamma(t) : [0, 1] \to S^2, \gamma(0) = A, \gamma(1) = B, \gamma \in H^1([0, 1], \mathbb{R}^3)\},$$

$$E(\gamma) = \frac{1}{2} \int_0^1 |\dot{\gamma}(t)|^2 dt = \frac{1}{2} \int_0^1 (\dot{\gamma}_1(t)^2 + \dot{\gamma}_2(t)^2 + \dot{\gamma}_3(t)^2) dt,$$

$\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$. Find a $\gamma_0 \in M$ such that $E(\gamma_0) = \inf_{\gamma \in M} E(\gamma)$ via solving the Euler-Lagrange equation.