Exercises I

April 5, 2022

1. Let $\|\cdot\|$ be a a norm on $\mathbb{R}^N$ and $f(x) = \|x\|^2 : \mathbb{R}^N \to \mathbb{R}$. Suppose that f is C^2 near $x = 0$. Prove there is an inner product $(\cdot,\cdot)$ on $\mathbb{R}^N$ such that $\|x\|^2 = (x,x)$, $x \in \mathbb{R}^N$.

2. Let $X = L^p(\Omega)$. Compute the Gateaux and Fréchet derivatives of the functional $f(u) = \int_\Omega |u|^p dx : X \to \mathbb{R}$ for $p > 1$ and the sub-differential $\partial f(0)$ if $p = 1$.

3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $\phi(x,\xi)$, $\frac{\partial \phi(x,\xi)}{\partial \xi} : \Omega \times \mathbb{R} \to \mathbb{R}$ be Carathéodory functions satisfying

$$|\frac{\partial \phi(x,\xi)}{\partial \xi}| \leq b(x) + a|\xi|^r, x \in \Omega, \xi \in \mathbb{R},$$

$a > 0$ be a constant, $b \in L^{\frac{2n}{n+2}}(\Omega), 1 \leq r \leq \frac{n+2}{n-2}$. Prove the functional

$$f(u) = \int_\Omega \phi(x,u(x))dx, \quad H^1(\Omega) \to \mathbb{R}$$

is F-differentiable and

$$< f'(u), h >= \int_\Omega \frac{\partial \phi(x,\xi)}{\partial \xi}(x,u(x))h(x)dx, \quad h \in H^1(\Omega).$$

4. Let X, Y be Banach spaces and $t \to A(t) : [0,1] \to L(X,Y)$ be continuous. Suppose that for all $t \in [0,1]$, $A(t)$ is a Fredholm operator from X to Y, prove the Fredholm index $ind(A(t))$ is independent of $t \in [0,1]$.

5. Let X, Y, Z be Banach spaces and $C : X \times Y \to X$, $A : X \to Z$, $B : Y \to Z$ be bounded linear operators satisfying

(i) $C(x,y) = Ax + By$ for all $(x,y) \in X \times Y$, (ii) C is surjective and B is Fredholm.

Prove the projection map $P : X \times Y \to X, P(x,y) = x$ restricted to $ker(C)$ is a Fredholm map from $ker(C)$ to X and $ind(P|_{ker(C)}) = ind(B)$.

6. Consider

$$u'' + \lambda u = 0,$$
$$u'(0) = u'(\pi) = 0.$$

Prove: (1) if $\lambda \neq k^2, k \in \mathbb{N}$, the equation has only trivial solution $u = 0$ if u is near 0, (2) if $\lambda = k^2$, the equation has a family of solutions $(\lambda_k(s), u_k(s))$ which is C^1 in $s \in (-\delta, \delta)$ for some $\delta > 0$ and $u_k(0) = 0, \lambda_k(0) = k^2,, u_k(s) \neq 0, \lambda_k(s) > k^2$ for $s \neq 0$.

7. Let A be a symmetric $n \times n$ matrix and $f(x) = (Ax, x)$, $x \in S^{n-1} = \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n | x_1^2 + \cdots + x_n^2 = 1\}$, $y = (y_1, \cdots, y_n) \in S^{n-1}$ be a critical point of $f : S^{n-1} \to \mathbb{R}$: $\exists \lambda \in \mathbb{R}$ such that $Ay = \lambda y$. Prove (1) if $y_n > 0$, then this is equivalent that $(y_1, \cdots, y_{n-1}) \in \mathbb{R}^{n-1}$ is a critical point of $\tilde{f}(x_1, \cdots, x_{n-1}) = f(x_1, \cdots, x_{n-1}, \sqrt{1 - x_1^2 - \cdots - x_{n-1}^2}) : U \to \mathbb{R}$ with $U = \{x = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1} | x_1^2 + \cdots + x_{n-1}^2 < 1\}$, (2) λ is a simple eigenvalue of A iff the matrix $\frac{\partial^2 \tilde{f}}{\partial x_i \partial x_j}$ is nondegenerate at $(y_1, \cdots, y_{n-1})$.