HOMEWORK

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1. Monday, March 22

We need the following basic tools.

[A] **(Co-area formula)** Given a Lipschitz function $\varrho : X \to \mathbb{R}$ with Lipschitz constant 1, then

$$\mathrm{Vol}_{n-1}(z \cap \{\varrho = t\}) \leq E_n \left.\frac{d}{dr}\right|_{r=t} \mathrm{Vol}_n(z \cap \{\varrho \leq r\}) \tag{1.1}$$

for almost every $r \in \mathbb{R}$.

[B] **(Cone inequality)** If X is a Banach space, $x_0 \in X$, then the (n+1)-chain c obtained by joining each point in z by by the geodesic line (respectively, straight line if X is a Banach space) with x_0 satisfies $\partial c = z$ and

$$\mathrm{Vol}_{n+1}(c) \leq D_n R \, \mathrm{Vol}_n(z)\,, \tag{1.2}$$

where R is the smallest number such that z is contained in the ball of radius R around x_0 .

[C] **(Lower density estimate)** For almost every $x \in Supp(z)$,

$$\liminf_{r \to 0^+} \frac{1}{r^n} \mathrm{Vol}_n(z \cap B(x, r)) > A_n\,. \tag{1.3}$$

Lemma 1.1 (5r-Lemma)**.** *If (X, d) is a metric space. Let $\mathscr{C}$ be a collection of balls of radius less than or equal to $R < \infty$. Then there exists a subfamilies $\mathscr{F}$ such that*

1. $\forall\ B,\ B' \in \mathscr{F}\,,\ B \cap B' = \emptyset\,,$
2. $\bigcup_{B \in \mathscr{C}} B \subset \bigcup_{B' \in \mathscr{F}} 5B'\,,$
3. *If $B \in \mathscr{C}$, $B \cap B' \neq \emptyset$ for some $B' \in \mathscr{F}$, then* $\mathrm{rad}(B') \geq \frac{1}{2}\,\mathrm{rad}(B)$.

Proof. Let $\mathscr{P}$ be the collection of families of balls such that if $\mathscr{G} \in \mathscr{P}$, then $\mathscr{G}$ is disjoint and $\forall\ B \in \mathscr{C}$, if $B \cap B' \neq \emptyset$ for some $B' \subset \mathscr{G}$, then $\mathrm{rad}(B') \geq \frac{1}{2}\,\mathrm{rad}(B)$.

Step 1. $\mathscr{P}$ is not empty. Let $B_0 \in \mathscr{C}$ such that $\mathrm{rad}(B_0) > \frac{1}{2} R$, then $\{B_0\} \in \mathscr{P}$.

Step 2. The partial order in $\mathscr{P}$ is defined by inclusion, then for any $\{\mathscr{G}_1 \subset \mathscr{G}_2 \subset \cdots \subset \mathscr{G}_s \subset \cdots\} \subset \mathscr{P}$, we know $\bigcup_{i=1}^{\infty} \mathscr{G}_i$ is an upper bound.

Step 3. By Zorn's lemma, there exists a max element in $\mathscr{P}$, defined by $\mathscr{F}$. Clearly, $\mathscr{F}$ satisfies (1) and (3).

Next, we claim that $\forall\ B \in \mathscr{C},\ \exists\ B' \in \mathscr{F}$, such that $B \cap B' \neq \emptyset$. In fact, suppose the contrary, if $\exists\ B \in \mathscr{C}$, such that $\forall\ B' \in \mathscr{P},\ B \cap B' = \emptyset$, then we add B' to $\mathscr{F}$ to get a contradiction to maximality of $\mathscr{F}$.

Finally, since $\mathrm{rad}(B') \geq \frac{1}{2}\,\mathrm{rad}(B)$, we know that $B \subset 5B'$, that is (2). Then, we are done. □

Proposition 1.2. *Let X be a metric space and let z be an n-cycle in $X, n \geq 2$. There exist finitely many pairwise disjoint closed balls $B_i \subset X, i = 1, \ldots, k$, with the following properties:*

(1) *The volume of z contained in each ball is "not too small":*

$$\mathrm{Vol}_n\left(z \cap B_1\right) \geq 4^{-n} A_n \varepsilon \operatorname{diam}\left(B_1\right)^n ;$$

(2) *The restriction $z \cap B_i$ is an n -chain whose boundary has "small" volume:*

$$\mathrm{Vol}_{n-1}\left(\partial\left(z \cap B_i\right)\right) \leq E_n\left(A_n \varepsilon\right)^{\frac{1}{n}} n \mathrm{Vol}_n\left(z \cap B_i\right)^{\frac{n-1}{n}} ;$$

(3) *An essential part of the volume of z is contained in the union of these balls:*

$$\mathrm{Vol}_n\left(z \cap \bigcup_{i=1}^k B_i\right) \geq \frac{1}{5^n} \mathrm{Vol}_n(z).$$

Proof. For $x \in \operatorname{supp} z$ and $r \geq 0$ define $V(x, r) := \mathrm{Vol}_n(z \cap B(x, r))$ and

$$r_0(x) := \max \left\{r \geq 0: \frac{V(x, r)}{r^n} \geq A_n \varepsilon\right\}.$$

Note that $0 < r_0(x) < \infty$ for almost every $x \in \operatorname{supp} z$ by the lower density estimate [C]; moreover,

$$V\left(x, 5 r_0(x)\right) < 5^n A_n \varepsilon r_0(x)^n = 5^n V\left(x, r_0(x)\right).$$

By the Vitali $5r$ -covering lemma there exist finitely many points $x_1, \ldots, x_k \in \operatorname{supp} z$ such that the balls $B\left(x_1, 2 r_0\left(x_1\right)\right)$ are pairwise disjoint, the balls $B\left(x_1, 5 r_0\left(x_1\right)\right)$ cover $\operatorname{supp} z$ and

$$\mathrm{Vol}_n\left(z \cap \bigcup_{i=1}^k B\left(x_i, r_0\left(x_i\right)\right)\right) \geq \frac{1}{5^n} \mathrm{Vol}_n(z).$$

Fix $i \in\{1, \ldots, k\}$. We claim that by the definition of $r_0\left(x_i\right)$ there exists a positive measure set of points $r \in\left(r_0\left(x_i\right), 2 r_0\left(x_i\right)\right)$ with

$$\frac{d}{d r} V\left(x_i, r\right) < \left(A_n \varepsilon\right)^{\frac{1}{n}} n V\left(x_i, r\right)^{\frac{n-1}{n}} .$$

In fact, the above inequality equivalents to

$$\frac{d}{d r}\left(V^{\frac{1}{n}}\left(x_i, r\right)\right) < \left(A_n \varepsilon\right)^{\frac{1}{n}} .$$

Suppose to the contrary, we know $\frac{d}{dr}\left(V^{\frac{1}{n}}\left(x_i, r\right)\right) \geq \left(A_n \varepsilon\right)^{\frac{1}{n}}$ for a.e. $r \in\left(r_0\left(x_i\right), 2 r_0\left(x_i\right)\right)$, then

$$\int_{r_0}^r \frac{d}{d r}\left(V^{\frac{1}{n}}\left(x_i, r\right)\right) d t \geq \int_{r_0}^r\left(A_n \varepsilon\right)^{\frac{1}{n}} d t,$$

and note $A_n \varepsilon r_0(x)^n = V\left(x, r_0(x)\right)$, we have

$$V\left(x_i, r\right) \geq A_n \varepsilon r^n,$$

contradiction.

Therefore, by the coarea inequality,

$$\begin{aligned}
\mathrm{Vol}_{n-1}\left(\partial\left(z \cap B\left(x_i, r\right)\right)\right. & = \mathrm{Vol}_{n-1}\left(z \cap\left\{x \in X \middle| d\left(x, x_1\right)=r\right\}\right) \\
& < E_n\left(A_n \varepsilon\right)^{\frac{1}{n}} n \mathrm{Vol}_n\left(z \cap B\left(x_i, r\right)\right)^{\frac{n-1}{n}} .
\end{aligned}$$

Choose an r such that the above inequality holds and set $B_i := B\left(x_i, r\right)$. The so-defined B_i clearly satisfy (1), (2) and (3). □

2. Wednesday, March 24

Theorem 2.1 (Gromov)**.** *Let X be an L^∞-space and $n \geq 1$. Then the filling volume of any n-dimensional singular Lipschitz cycle z in X with integer or $\mathbb{Z}_2$ coefficients satisfies*

$$\operatorname{Fillvol}(z) \leq C_n \operatorname{Vol}(z)^{1+\frac{1}{n}}$$

where C_n depends only on n.

Proof. The proof is by induction on n and the case $n = 1$ is trivial, since the diameter of a closed curve is bounded by its length and thus the isoperimetric inequality is a direct consequence of the cone inequality. Suppose now that $n \geq 2$ and that the statement of the theorem holds for $(n-1)$-cycles with some constant $C_{n-1} \geq 1$. Set

$$\varepsilon := \min\left\{\frac{1}{4^{n-1}C_{n-1}^{n-1}A_nE_n^nn^n}, \frac{1}{2}\right\},$$

let z be an n-cycle in X and choose a ball B of finite radius that contains z. Let $B_1, \ldots, B_k$ be balls as in the Proposition 1.2. By the isoperimetric inequality in dimension $n-1$ we can choose for each $i = 1, \cdots, k$ an n -chain c_i satisfying $\partial c_i = \partial\left(z \cap B_i\right)$ and

$$\operatorname{Vol}\left(c_i\right) \leq C_{n-1} \operatorname{Vol}_{n-1}\left(\partial\left(z \cap B_i\right)\right)^{\frac{n}{n-1}} \leq \frac{1}{4} \operatorname{Vol}\left(z \cap B_i\right). \tag{2.1}$$

Here the second inequality follows from (2) of Proposition 1.2 and the definition of ε. We may of course assume that c_i is contained in B_i since otherwise we can project it to B_i via a 1$-$Lipschitz projection $P : X \to B_i$ (and this decreases the volume). If $X = L^\infty(\Omega)$, then $P(f)(y) := \operatorname{sgn}(f(y)) \min\{|f(y)|, 1\}$. Set $\hat{z}_i := (z \cap B_i) - c_i$ and

$$z' = z - \sum_{i=1}^k \hat{z}_i = \left(z \cap \left(\bigcup B_1\right)^c\right) + \sum_{i=1}^k c_i.$$

Observe that these are n-cycles and that, by (2.1),

$$\frac{3}{4} \operatorname{Vol}\left(z \cap B_i\right) \leq \operatorname{Vol}\left(\hat{z}_i\right) \leq \frac{5}{4} \operatorname{Vol}\left(z \cap B_i\right). \tag{2.2}$$

From the Proposition 1.2 and from (2.1),(2.2) we conclude

$$\operatorname{diam}\left(\hat{z}_i\right) \leq \operatorname{diam} B_i \leq \frac{4}{\left(A_n\varepsilon\right)^{\frac{1}{n}}} \operatorname{Vol}\left(z \cap B_i\right)^{\frac{1}{n}} \leq \left(\frac{4^{n+1}}{3A_n\varepsilon}\right)^{\frac{1}{\pi}} \operatorname{Vol}\left(\hat{z}_i\right)^{\frac{1}{n}}, \tag{2.3}$$

and

$$\frac{3}{5}\left[\sum_{t=1}^k \operatorname{Vol}\left(\hat{z}_i\right)\right] + \operatorname{Vol}\left(z'\right) \leq \operatorname{Vol}(z), \tag{2.4}$$

as well as

$$\operatorname{Vol}\left(z'\right) \leq \left(1 - \frac{3}{4}5^{-n}\right) \operatorname{Vol}(z). \tag{2.5}$$

Let $\hat{c}_i$ sad c' be $(n+1)$-chains with boundaries $\hat{z}_i$ and z', respectively, and which satisfy the cone inequality. The $(n+1)$-chain $c := \hat{c}_1 + \cdots + \hat{c}_k + c'$ has boundary z and satisfies

$$\operatorname{Vol}(c) \leq \operatorname{Vol}\left(\hat{c}_1\right) + \cdots + \operatorname{Vol}\left(\hat{c}_k\right) + \operatorname{Vol}\left(c'\right) \leq C_n \operatorname{Vol}(z)^{1+\frac{1}{n}} + \operatorname{Vol}\left(c'\right), \tag{2.6}$$

for some C_n only depending on n. This is a consequence of (2.3),(2.4) and the fact that for $a_1, \cdots, a_k \geq 0$ sad $\alpha \geq 1$,

$$a_1^\alpha + \cdots + a_k^\alpha \leq (a_1 + \cdots + a_k)^\alpha .$$

Then we denote z' by $z^{(1)}$, and similarly change notations related to z', decomposing $z^{(1)}$ as above, we have another finite balls $\{B_{(1),i}\}_{i=1}^{k_{(1)}}$, such that there exists $z^{(2)}$ satisfies

$$\mathrm{Vol}\left(z^{(2)}\right) \leq \left(1 - \frac{3}{4}5^{-n}\right) \mathrm{Vol}(z^{(1)}) \leq \left(1 - \frac{3}{4}5^{-n}\right)^2 \mathrm{Vol}(z),$$

and let $\hat{c}_i^{(1)}$ sad $c^{(2)}$ be $(n+1)$-chains with boundaries $\hat{z}_i^{(1)}$ and $z^{(2)}$, respectively, and which satisfy the cone inequality. The $(n+1)$-chain $c' = c^{(1)} = \hat{c}_1^{(1)} + \cdots + \hat{c}_{k_{(1)}}^{(1)} + c^{(2)}$ has boundary $z^{(1)}$ and satisfies

$$\begin{aligned} \mathrm{Vol}(c^{(1)}) &\leq \mathrm{Vol}\left(\hat{c}_1^{(1)}\right) + \cdots + \mathrm{Vol}\left(\hat{c}_{k_{(1)}}^{(1)}\right) + \mathrm{Vol}\left(c^{(2)}\right) \\ &\leq C_n \, \mathrm{Vol}(z^{(1)})^{1+\frac{1}{n}} + \mathrm{Vol}\left(c^{(2)}\right), \end{aligned} \tag{2.7}$$

hence, combine (2.5),(2.6),(2.7) we have

$$\mathrm{Vol}(c) \leq C_n \left(1 + \left(1 - \frac{3}{4}5^{-n}\right)^{1+\frac{1}{n}}\right) \mathrm{Vol}(z)^{1+\frac{1}{n}} + \mathrm{Vol}\left(c^{(2)}\right).$$

Thereafter, we set $\lambda := \left(1 - \frac{3}{4}5^{-n}\right)^{1+\frac{1}{n}}$. Clearly, $\lambda < 1$ fixed. Then, repeating the above process m times yields

$$\mathrm{Vol}\left(z^{(m)}\right) \leq \left(1 - \frac{3}{4}5^{-n}\right)^m \mathrm{Vol}(z),$$

and

$$\mathrm{Vol}(c) \leq C_n \left(1 + \lambda + \cdots + \lambda^{m-1}\right) \mathrm{Vol}(z)^{1+\frac{1}{n}} + \mathrm{Vol}\left(c^{(m)}\right).$$

Since $\sum_{k=1}^{\infty} \lambda^k < +\infty$ and cone inequality

$$\mathrm{Vol}\left(c^{(m)}\right) \leq D_n \, \mathrm{diam}\left(z^{(m)}\right) \mathrm{Vol}\left(z^{(m)}\right),$$

we know for large enough m, there is

$$\mathrm{Vol}(c) \leq C_n \, \mathrm{Vol}(z)^{1+\frac{1}{n}},$$

which completes the proof. □