HOMEWORK

LING WANG 2001110014

1. Wednesday, April 21

Homework

(1) Prove that "If (M^n, g) is a closed Riemaniann manifold with $\pi_1(M)$ exponential growth, then $h_{\mathrm{Vol}}(M^n, g) > 0$ ";
(2) Find an example such that $\forall g : h_{\mathrm{Vol}}(M^n, g) > 0$ but $h_{\mathrm{Vol}}(M^n) = 0$;
(3) $(V^n, \langle\cdot,\cdot\rangle)$ Euclidean space. $T : V^v \longrightarrow V^n$ self-adjoint endomorphism, then

$$\frac{1}{n}\operatorname{tr} T = \frac{1}{\operatorname{Vol}\left(S^{n-1}\right)} \int_{S^{n-1}} \langle v, Tv\rangle dv.$$

Proof of (1). Fix a base point $\tilde{p} \in \tilde{M}$ and $R \geq 0$, let $B(\tilde{p}, R)$ denote the closed ball of center $\tilde{p}$ and radius R in $\tilde{M}$. Let $D := \operatorname{diam}(M^n, g)$, $N := B(\tilde{p}, D)$, $\Gamma = \pi_1(M)$, then $\{\gamma B(\tilde{p}, D)\}_{\gamma\in\Gamma}$ cover $\tilde{M}$, the finite set of generators $S = \{\gamma \in \Gamma \mid$ and $\gamma B \cap B \neq \emptyset\}$ of Γ, the separating distance $v = \min\{d(B, \gamma B) \mid \gamma \in \Gamma, \gamma \notin S\}$, the distance $\lambda = \max\{d(x_0, sx_0) \mid s \in S\}$ as well as the order a of the isotropy subgroup $\Gamma_0 = \{\gamma \in \Gamma \mid \gamma\tilde{p} = \tilde{p}\}$ (observe that Γ_0 is finite and contained in S). Let k be an integer, $k \geq 0$. As the closed balls $B\left(x, \frac{1}{3}v\right)$, for x in the orbit $\Gamma\tilde{p}$, are pairwise disjoint, we have

$$\frac{1}{a} g_S(k) \operatorname{Vol}\left(B\left(\tilde{p}, \frac{1}{3}v\right)\right) \leq \operatorname{Vol}\left(B\left(\tilde{p}, k\lambda + \frac{1}{3}v\right)\right).$$

Since $g_S(k)$ has exponential growth, we know

$$\operatorname{Vol}\left(B\left(\tilde{p}, k\lambda + \frac{1}{3}v\right)\right) \geq c\,\mathrm{e}^{bk}, \quad \text{for some } b, c > 0,$$

then, by definition we have $h_{\mathrm{Vol}}(M^n, g) > 0$. □

Proof of (2). Im sorry that I have not find an example exactly, but there are some results may be help to this question, I show them in the following.

First, John Wilson [1] has announced that exponential growth does not imply uniform exponential growth. More precisely, there exist finitely generated groups with trivial Abelianization which are isomorphic to their permutational wreath product with the alternating group on 31 letters. let $\Gamma \approx \Gamma \operatorname{wr} A_{31}$ be any group of this kind; on the one hand, there exists a sequence $\left(S_n = \{x_n, y_n\}\right)_{n\geq 1}$ of generating sets of Γ, with $x_n^2 = y_n^3 = 1$ for all $n \geq 1$, such that the limit of the corresponding exponential growth rates is 1, in symbols $\lim_{n\to\infty} \omega\left(\Gamma, S_n\right) = 1$; on the other hand, for an appropriate choice of Γ, there exist non-Abelian free subgroups in Γ, so that in particular Γ is of exponential growth.

And another result is due to De la Harpe and Grigorchuk [2], which is proved writes: $h_{\mathrm{alg}}(\Gamma) \geqslant C > 0$, where the constant C depends on Γ. This allows the existence of sequences $(\Gamma_n)_{n\in\mathbb{N}}$ of groups such that $h_{\text{alg}}(\Gamma_n)$ goes to 0, though each Γ_n has uniform exponential growth. Although there is a reference cited, I have failed to find this article on the internet. $\square$

Proof of (3). Let $(e_1,\dots,e_n)$ be orthonormal basis of V and T orthogonal (the existence of such a basis is precisely the spectral theorem). Let $\lambda_k=\langle e_k,Te_k\rangle$ for $k\in\{1,\dots n\}$. For any vector $x=\sum_{k=1}^n x_k e_k$, the squared norm of x is $\langle x,x\rangle = \sum_{k=1}^n x_k^2$ and the quadratic form is given by $\langle x,Tx\rangle=\sum_{k=1}^n \lambda_k {x_k}^2$. Therefore,

$$\int_{S^{n-1}}\langle x,Tx\rangle dx=\sum_{k=1}^n\lambda_k\int_{S^{n-1}}{x_k}^2dx. \tag{1.1}$$

It remains to compute the integrals $I_k=\int_{S^{n-1}}{x_k}^2dx$. This can be done swiftly using a little trick, starting with the observation that any two of these integrals are equal. Indeed, for any $k\neq l$, we can easily find an isometry $\varphi\in O(g)$ such that $x_k^2\circ\varphi=x_l^2$ (namely, the orthogonal reflection through the line spanned by e_k+e_l). Since φ preserves dx, the change of variables theorem ensures that $I_k=I_l$. Since all the integrals I_k are equal, we write $I_k=\frac{1}{n}\sum_{l=1}^n I_l$ for any k. That is $I_k=\frac{1}{n}\int_{S^{n-1}}\left(\sum_{k=1}^n x_k^2\right)dx$. However $\sum_{k=1}^n {x_k}^2=\langle x,x\rangle=1$ for any $x\in S^{n-1}$. This yields $I_k=\frac{1}{n}\int_{S^{n-1}}1dx=\frac{1}{n}\operatorname{vol}\left(S^{n-1}\right)$. Coming (1.1), we obtain the desired result

$$\int_{S^{n-1}}\langle x,Tx\rangle dx=\frac{1}{n}\operatorname{vol}\left(S^{n-1}\right)\sum_{k=1}^n\lambda_k=\frac{1}{n}\operatorname{vol}\left(S^{n-1}\right)\operatorname{tr}T.$$

$\square$

References

[1] J. Wilson, On exponential growth and uniformly exponential growth for groups, Invent. Math. 155 (2004) 287C303

[2] P. De la Harpe, R. Grigorchuk, Limit behaviour of exponential growth rates for finitely generated groups, Monographie de lEnseignement Mathmatique 38 (2001) 351C370.